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MATHEMATICS.

I.

STUDY AND DIFFICULTIES OF MATHEMATICS.
ARITHMETIC AND ALGEBRA.

EXAMPLES OF THE PROCESSES OF ARITHMETIC AND ALGEBRA.

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ON THE STUDY AND DIFFICULTIES

OF

MATHEMATICS.

CHAPTER I.

Introductory Remarks on the Nature and Objects of Mathematics.

THE object of this Treatise is—1. To point out to the student of Mathematics, who has not the advantage of a tutor, the course of study which it is most advisable that he should follow, the extent to which he should pursue one part of the science before he commences another, and to direct him as to the sort of applications which he should make. 2. To treat fully of the various points which involve difficulties and which are apt to be misunderstood by beginners, and to describe at length the nature without going into the routine of the operations which have been already discussed in the Treatises of Arithmetic, Algebra, and Geometry, published by this Society.

No person commences the study of mathematics without soon discovering that it is of a very different nature from those to which he has been accustomed. The pursuits to which the mind is usually directed before entering on the sciences of algebra or geometry, are such as languages and history, &c. Of these, neither appears to have any affinity with mathematics; yet, in order to see the difference which exists between these studies, for instance, history and geometry, it will be useful to ask how we come by knowledge in each: suppose, for example, we feel certain of a fact related in history, such as the murder of Cæsar, whence did we derive the certainty? how came we to feel sure of the general truth of the circumstances of the narrative? The ready answer to this question will be, that we have not absolute certainty upon this point; but that we have the relation of historians, men of credit, who lived and published their accounts in the very time of which they write; that succeeding ages have received those accounts as true, and that succeeding historians have backed them with a mass of circumstantial evidence which makes it the most improbable thing in the world that the account, or any material part of it, should be false. This is perfectly correct, nor can there be the slightest objection to believing the whole narration upon such grounds; nay, our minds are so constituted, that, upon our knowledge of these arguments, we cannot help believing, in spite of ourselves. But this brings us to the point to which we wish to come; we believe that Cæsar was assassinated by Brutus and his friends, not because there is any absurdity in supposing the contrary, since every one must allow that there is just a possibility that the event never happened: not because we can show that it must necessarily have been that, at a particular day, at a particular place, a successful adventurer must have been murdered in the manner described, but because our evidence of the fact is such, that, if we apply the notions of evidence which every-day experience justifies us in entertaining, we feel that the improbability of the contrary compels us to take refuge in the belief of the fact; and, if we allow that there is still a possibility of its falsehood, it is because this supposition does not involve absolute absurdity, but only extreme improbability.

In mathematics the case is wholly different. It is true that the facts asserted in these sciences are of a nature totally distinct from those of history; so much so, that a comparison of the evidence of the two may almost excite a smile. But if it be remembered that acute reasoners, in every branch of learning, have acknowledged the use, we might almost say the necessity, of a mathematical education, it must be admitted that the points of connexion between these pursuits and others are worth attending to. They are the more so, because there is a mistake into which several have fallen, and have deceived others, and perhaps themselves, by clothing some false reasoning in what they called a mathematical dress, imagining that, by the application of mathematical symbols to their subject, they secured mathematical argument. This

could not have happened if they had possessed a knowledge of the bounds within which the empire of mathematics is contained. That empire is sufficiently wide, and might have been better known, had the time which has been wasted in aggressions upon the domains of others, been spent in exploring the immense tracts which are yet untrodden.

We have said that the nature of mathematical demonstration is totally different from all other, and the difference consists in this—that, instead of showing the contrary of the proposition asserted to be only improbable, it proves it at once to be absurd and impossible. This is done by showing that the contrary of the proposition which is asserted is in direct contradiction to some extremely evident fact, of the truth of which our eyes and hands convince us. In geometry, of the principles alluded to, those which are most commonly used are—

I. If a magnitude be divided into parts, the whole is greater than either of those parts.

II. Two straight lines cannot inclose a space.

III. Through one point only one straight line can be drawn, which never meets another straight line, or which is *parallel* to it.

It is on such principles as these that the whole of geometry is founded, and the demonstration of every proposition consists in proving the contrary of it to be inconsistent with one of these. Thus, in Euclid, Book I., Prop. 4, it is shown that two triangles which have two sides and the included angle respectively equal are equal in all respects, by proving that, if they are not equal, two straight lines will inclose a space, which is impossible. In the Treatise on Geometry, Prop. 4, the same thing is proved in the same way, only the self-evident truth asserted differs in form from that of Euclid, and may be deduced from it, thus—

Two straight lines which pass through the same two points must either inclose a space, or coincide and be one and the same line, but they cannot inclose a space, therefore they must coincide. Either of these propositions being granted, the other follows immediately; it is, therefore, immaterial which of them we use. We shall return to this subject in treating specially of the first principles of geometry.

Such being the nature of mathematical demonstration, what we have before asserted is evident, that our assurance of a geometrical truth is of a nature wholly distinct from that which we can by any means obtain of a fact in history or an asserted truth of metaphysics. In reality, our senses are our first mathematical instructors; they furnish us with notions which we cannot trace any further or represent in any other way than by using single words, which every one understands. Of this nature are the ideas to which we attach the terms number, one, two, three, &c., point, straight line, surface; all of which, let them be ever so much explained, can never be made any clearer than they are already to a child of ten years old. But, besides this, our senses also furnish us with the means of reasoning on the things which we call by these names, in the shape of incontrovertible propositions, such as have been already cited, on which, if any remark is made by the beginner in mathematics, it will probably be, that from such absurd truisms as "the whole is greater than its part," no useful result can possibly be derived, and that we might as well expect to make use of "two and two make four." This observation, which is common enough in the mouths of those who are commencing geometry, is the result of a little pride which does not quite like the humble operation of beginning at the beginning, and is rather shocked at being supposed to want such elementary information. But it is wanted, nevertheless; the lowest steps of a ladder are as useful as the highest. Now, the most common reflexion on the nature of the propositions referred to will convince us of their truth. But they must be presented to the understanding, and reflected on by it, since, simple as they are, it must be a mind of a very superior cast which could by itself embody these axioms, and proceed from them only one step in the road pointed out in any treatise on geometry.

But, although there is no study which presents so simple a beginning as that of geometry, there is none in which difficulties grow more rapidly as we proceed, and what may appear at first rather paradoxical, the more acute the student the more serious will the impediments in the way of his progress appear. This necessarily follows in a science which consists of reasoning from the very commencement, for it is evident that every student will feel a claim to have his objections answered, not by authority, but by argument, and that the intelligent

student will perceive more readily than another the force of an objection and the obscurity arising from an unexplained difficulty, as the greater is the ordinary light the more will occasional darkness be felt. To remove some of these difficulties is the principal object of this Treatise.

We shall now make a few remarks on the advantages to be derived from the study of mathematics, considered both as a discipline for the mind and a key to the attainment of other sciences. It is admitted by all that a finished or even a competent reasoner is not the work of nature alone; the experience of every day makes it evident that education developes faculties which would otherwise never have manifested their existence. It is, therefore, as necessary to *learn to reason* before we can expect to be able to reason, as it is to learn to swim or fence, in order to attain either of those arts. Now, something must be reasoned upon, it matters not much what it is, provided that it can be reasoned upon with certainty. The properties of mind or matter, or the study of languages, mathematics, or natural history, may be chosen for this purpose. Now, of all these, it is desirable to choose the one which admits of the reasoning being verified, that is, in which we can find out by other means, such as measurement and ocular demonstration of all sorts, whether the results are true or not. When the guiding property of the loadstone was first ascertained, and it was necessary to learn how to use this new discovery, and to find out how far it might be relied on, it would have been thought advisable to make many passages between ports that were well known before attempting a voyage of discovery. So it is with our reasoning faculties: it is desirable that their powers should be exerted upon objects of such a nature, that we can tell by other means whether the results which we obtain are true or false, and this before it is safe to trust entirely to reason. Now the mathematics are peculiarly well adapted for this purpose, on the following grounds:—

1. Every term is distinctly explained, and has but one meaning, and it is rarely that two words are employed to mean the same thing.

2. The first principles are self-evident, and, though derived from observation, do not require more of it than has been made by children in general.

3. The demonstration is strictly logical, taking nothing for granted except the self-evident first principles, resting nothing upon probability, and entirely independent of authority and opinion.

4. When the conclusion is attained by reasoning, its truth or falsehood can be ascertained, in geometry by actual measurement, in algebra by common arithmetical calculation. This gives confidence, and is absolutely necessary, if, as was said before, reason is not to be the instructor, but the pupil.

5. There are no words whose meanings are so much alike that the ideas which they stand for may be confounded. Between the meanings of terms there is no distinction, except a total distinction, and all adjectives and adverbs expressing difference of degrees are avoided. Thus it may be necessary to say "A is greater than B;" but it is entirely unimportant whether A is very little or very much greater than B. Any proposition which includes the foregoing assertion will prove its conclusion generally, that is, for all cases in which A is greater than B, whether the difference be great or little. Locke mentions the distinctness of mathematical terms, and says in illustration, "The idea of two is as distinct from the idea of three as the magnitude of the whole earth is from that of a mite. This is not so in other simple modes, in which it is not so easy, nor perhaps possible for us to distinguish between two approaching ideas, which yet are really different; for who will undertake to find a difference between the white of this paper, and that of the next degree to it?"

These are the principal grounds on which, in our opinion, the utility of mathematical studies may be shewn to rest, as a discipline for the reasoning powers. But the habits of mind which these studies have a tendency to form are valuable in the highest degree. The most important of all is the power of concentrating the ideas which a successful study of them increases where it did exist, and creates where it did not. A difficult position, or a new method of passing from one proposition to another, arrests all the attention, and forces the united faculties to use their utmost exertions. The habit of mind thus formed soon extends itself to other pursuits, and is beneficially felt in all the business of life.

As a key to the attainment of other sciences, the use of the mathematics is too well known to make it necessary

that we should dwell on this topic. In fact, there is not in this country any disposition to undervalue them as regards the utility of their applications. But though they are now generally considered as a part, and a necessary one, of a liberal education, the views which are still taken of them as a part of education by a large proportion of the community are still very confined.

The elements of mathematics usually taught are contained in the sciences of arithmetic, algebra, geometry, and trigonometry. We have used these four divisions because they are generally adopted, though, in fact, algebra and geometry are the only two of them which are really distinct. Of these we shall commence with arithmetic, and take the others in succession in the order in which we have arranged them.

Chapter II.

On Arithmetical Notation.

The first ideas of arithmetic, as well as those of other sciences, are derived from early observation. How they come into the mind it is unnecessary to inquire; nor is it possible to define what we mean by number and quantity. They are terms so simple, that is, the ideas which they stand for are so completely the first ideas of our mind, that it is impossible to find others more simple, by which we may explain them. This is what is meant by defining a term; and here we may say a few words on definitions in general, which will apply equally to all sciences.

Definition is the explaining a term by means of others, which are more easily understood, and thereby fixing its meaning, so that it may be distinctly seen what it does imply, as well as what it does not. Great care must be taken that the definition itself is not a tacit assumption of some fact or other which ought to be proved. Thus, when it is said that a square is "a four-sided figure, all whose sides are equal, and all whose angles are right angles," though no more is said than is true of a square, yet more is said than is necessary to define it, because it can be proved that if a four-sided figure have all its sides equal, and one only of its angles a right angle, all the other angles must be right angles also. Therefore, in making the above definition, we do, in fact, affirm that which ought to be proved. Again, the above definition, though redundant in one point, is, strictly speaking, defective in another, for it omits to state whether the sides of the figure are straight lines or curves. It should be, "a square is a four-sided rectilinear figure, all of whose sides are equal, and one of whose angles is a right angle."

As the mathematical sciences owe much, if not all, of the superiority of their demonstrations to the precision with which the terms are defined, it is most essential that the beginner should see clearly in what a good definition consists. We have seen that there are terms which cannot be defined, such as number and quantity. An attempt at a definition would only throw a difficulty in the student's way, which is already done in geometry by the attempts at an explanation of the terms point, straight line, and others, which are to be found in treatises on that subject. A point is defined to be that "which has no parts, and which has no magnitude;" a straight line is that which "lies evenly between its extreme points." Now, let any one ask himself whether he could have guessed what was meant, if, before he began geometry, any one had talked to him of "that which has no parts and which has no magnitude," and "the line which lies evenly between its extreme points," unless he had at the same time mentioned the words "point" and "straight line," which would have removed the difficulty? In this case the explanation is a great deal harder than the term to be explained, which must always happen whenever we are guilty of the absurdity of attempting to make the simplest ideas yet more simple.

A knowledge of our method of reckoning, and of writing down numbers, is taught so early, that the method by which we began is hardly recollected. Few, therefore, reflect upon the very commencement of arithmetic, or upon the simplicity and elegance with which calculations are conducted. We find the method of reckoning by ten in our hands, we hardly know how, and we conclude, so natural and obvious does it seem, that it came with our language, and is a part of it; and that we are not much indebted to instruction for so simple a gift. It has been well observed, that if the whole earth spoke the same language, we should think that the name of any object was not a mere sign *chosen* to represent it, but was a sound which had some real connexion with the thing; and that we should laugh at, and per-

haps persecute, any one who asserted that any other sound would do as well if we chose to think so. We cannot fall into this error, because, as it is, we happen to know that what we call by the sound "horse," the Romans distinguished as well by that of "*equus*," but we commit a similar mistake with regard to our system of numeration, because at present it happens to be received by all civilized nations, and we do not reflect on what was done formerly by almost all the world, and is done still by savages. The following considerations will, perhaps, put this matter on a right footing, and shew that in our ideas of arithmetic we have not altogether rid ourselves of the tendency to attach ideas of mysticism to numbers which has prevailed so extensively in all times.

We know that we have nine signs to stand for the first nine numbers, and one for nothing, or zero. Also, that to represent ten we do not use a new sign, but combine two of the others, and denote it by 10, eleven by 11, and so on. But why was the number *ten* chosen as the limit of our separate symbols—why not nine, eight, or eleven? If we recollect how apt we are to count on the fingers, we shall be at no loss to see the reason. We can imagine our system of numeration formed thus:—A man proceeds to count a number, and to help the memory he puts a finger on the table for each one which he counts. He can thus go as far as ten, after which he must begin again, and by reckoning the fingers a second time he will have counted twenty, and so on. But this is not enough; he must also reckon the number of times which he has done this, and as by counting on the fingers he has divided the things which he is counting into lots of ten each, he may consider each lot as a unit of its kind, just as we say a number of sheep is *one* flock, twenty shillings are *one* pound. Call each lot a *ten*. In this way he can count a ten of tens, which he may call a hundred, a ten of hundreds, or a thousand, and so on. The process of reckoning would then be as follows:—Suppose, to choose an example, a number of faggots is to be counted. They are first tied up in bundles of ten each, until there are not so many as ten left. Suppose there are seven over. We then count the bundles of ten as we counted the single faggots, and tie them up also by tens, forming new bundles of one hundred each with some bundles of ten remaining. Let these last be six in number. We then tie up the bundles of hundreds by tens, making bundles of thousands, and find that there are five bundles of hundreds remaining. Suppose that on attempting to tie up the thousands by tens, we find there are not so many as ten, but only four. The number of faggots is then 4 thousands, 5 hundreds, 6 tens, and 7.

The next question is, how shall we represent this number in a short and convenient manner? It is plain that the way to do this is a *matter of choice*. Suppose, then, that we distinguish the tens by marking their number with one accent, the hundreds with two accents, and the thousands with three. We may then represent this number in any of the following ways:—$76'5''4'''$, $6'75''4'''$, $6'4'''5''7$, $4'''5''6'7$, the whole number of ways being 24. But this is more than we want; one certain method of representing a number is sufficient. The most natural way is to place them in order of magnitude, either putting the largest collection first or the smallest; thus $4'''5''6'7$, or $76'5''4'''$. Of these we choose the first.

In writing down numbers in this way it will soon be apparent that the accents are unnecessary. Since the singly accented figure will always be the second from the right, and so on, the *place* of each number will point out what accents to write over it, and we may therefore consider each figure as deriving a value from the place in which it stands. But here this difficulty occurs. How are we to represent the numbers $3'''3'$, and $4'''2'7$ without accents? If we write them thus, 33 and 427, they will be mistaken for $3'3$ and $4''2'7$. This difficulty will be obviated by placing cyphers so as to bring each number into the place allotted to the sort of collection which it represents; thus, since the trebly accented letters, or thousands, are in the fourth place from the right, and the singly accented letters in the second, the first number may be written 3030, and the second 4027. The cypher, which plays so important a part in arithmetic that it was anciently called the *art of cypher*, or *cyphering*, does not stand for any number in itself, but is merely employed, like blank types in printing, to keep other signs in those places which they must occupy in order to be read rightly. We may now ask what would have been the case if, instead of ten fingers, men had had more or less. For example, by what signs would 4567 have been repre-

sented, if man had nine fingers instead of ten? We may presume that the method would have been the same with the number nine represented by 10 instead of ten, and the omission of the symbol 9. Suppose this number of faggots is to be counted by nines. Tie them up in bundles of nine, and we shall find 4 faggots remaining. Tie these bundles again in bundles of nine, each of which will, therefore, contain eighty-one, and there will be 3 bundles remaining. These tied up in the same way into bundles of nine, each of which contains seven hundred and twenty-nine, will leave 2 odd bundles, and, as there will be only six of them, the process cannot be carried any further. If, then, we represent, by $1'$, a bundle of nine, or *a nine*, by $1''$ a nine of nines, and so on, the number which we write 4567, must be written $6'''\,2''\,3'\,4$. In order to avoid confusion, we will suffer the accents to remain over all numbers which are not reckoned in tens, while those which are so reckoned shall be written in the common way. The following is a comparison of the way in which numbers in the common system are written, and in the one which we have just explained:—

Counting by tens....	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Counting by nines...	1	2	3	4	5	6	7	8	$1'0$	$1'1$	$1'2$	$1'3$	$1'4$	$1'5$

15	16	17	18	19	20	30	40	50	60	70	80	90	100
$1'6$	$1'7$	$1'8$	$2'0$	$2'1$	$2'2$	$3'3$	$4'4$	$5'5$	$6'6$	$7'7$	$8'8$	$1''1'0$	$1''2'1$

We will now write, in the common way, in the tens' system, the process which we went through in order to find how to represent the number 4567 in that of the nines, thus:—

9)4567	
9) 507	— rem. 4.
9) 56	— rem. 3.
9) 6	— rem. 2.
0	— rem. 6.

Representation required, $6'''\,2''\,3'\,4$.

The processes of arithmetic are the same in principle whatever system of numeration is used. To show this, we subjoin a question in each of the first four rules, worked both in the common system, and in that of the nines. There is this difference, that, in the first, the tens must be carried, and in the second the nines.

ADDITION.

636	$7''\,7'\,6$
987	$1'''\,3''\,1'\,6$
403	$4''\,8'\,7$
2026	$2'''\,7''\,0'\,1$

SUBTRACTION.

1384	$1'''\,8''\,0'\,7$
797	$1'''\,0''\,7'\,5$
587	$7''\,2'\,2$

MULTIPLICATION.

297	$3''\,6'\,0$
136	$1''\,6'\,1$
1782	3 6 0
891	2 4 0 0
297	3 6 0
40392	$6''''\,1'''\,3''\,6'\,0$

DIVISION.

$$\begin{array}{l}
633)\ 79125\ (125 \\
\quad\ \ 633 \\
\quad\ \ \overline{\ 1582} \\
\quad\ \ \ 1266 \\
\quad\ \ \overline{\ \ 3165} \\
\quad\ \ \ \ 3165 \\
\quad\ \ \overline{\quad\ \ 0}
\end{array}
\qquad\qquad
\begin{array}{l}
7''\,7'\,3)\ 1^{v}\,3^{iv}\,0^{iii}\,4^{ii}\,7^{i}\,6^{*}\ (1''\,4'\,8 \\
\qquad\quad\ 7\ \ 7\ \ 3 \\
\qquad\quad\ \overline{4\ \ 2\ \ 1\ \ 7} \\
\qquad\quad\ 3\ \ 4\ \ 2\ \ 3 \\
\qquad\qquad\ \overline{6\ \ 8\ \ 4\ \ 6} \\
\qquad\qquad\ 6\ \ 8\ \ 4\ \ 6 \\
\qquad\qquad\quad\ \overline{\ \ 0\ \ }
\end{array}$$

The student should accustom himself to work questions in different systems of numeration, which will give him a clearer insight into the nature of arithmetical processes than he could obtain by any other method. When he uses a system in which numbers are counted by a number greater than ten, he will want some new symbols for figures. For example, in the duodecimal system, where twelve is the number of figures supposed, twelve will be represented by 1′0; there must, therefore, be a distinct sign for ten and eleven, a nine and six reversed, thus, ϱ and ƍ, might be used for these.

Chapter III.

Elementary Rules of Arithmetic.

As soon as the beginner has mastered the notation of arithmetic, he may be made acquainted with the meaning of the algebraical signs $+$, $-$, $\times$, $=$, and also with that for division, or the common way of representing a fraction. There is no difficulty in these signs or in their use. Five minutes' consideration will make the symbol $5+3$ present as clear an idea as the words "5 added to 3." The reason why they usually cause so much embarrassment is, that they are generally deferred until the student commences algebra, when he is often introduced at the same time to the representation of numbers by letters, the distinction of known and unknown quantities, the signs of which we have been speaking, and the use of figures as exponents of letters. Either of these four things is quite sufficient at a time, and there is no time more favourable for beginning to make use of the signs of operation than when the habit of performing the operations commences. The beginner should exercise himself in putting the simplest truths of arithmetic in this new shape, and should write such sentences as the following frequently:—

$$2+7=9,$$
$$6-4=2,$$
$$1+8+4-6=4+2+1,$$
$$2\times2+12\times12=14\times10+2\times2\times2.$$

These will accustom him to the meaning of the signs, just as he was accustomed to the formation of letters by writing copies. As he proceeds through the rules of arithmetic he should take care never to omit connecting each operation with its sign, and should avoid confounding operations together, and considering them as the same, because they produce the same result. Thus, 4×7 does not denote the same operation as 7×4, though the result of both is 28. The first is four multiplied by seven, four taken seven times; the second is seven multiplied by four, seven taken four times; and that $4\times7=7\times4$ is a proposition to be proved, not to be taken for granted. Again, $\frac{1}{7}\times4$ and $\frac{4}{7}$ are marks of distinct operations, though their result is the same, as we shall show in treating of fractions.

The examples which a beginner should choose for practice should be simple, and should not contain very large numbers. The powers of the mind cannot be directed to two things at once: if the complexity of the numbers used requires all the student's attention, he cannot observe the principle of the rule which he is following. Now, at the commencement of his career, a principle is not received and understood by the student as quickly as it is explained by the instructor. He does not, and cannot, generalize at all; he must be taught to do so; and he cannot learn that a particular fact holds good for *all numbers* unless by having it shown that it holds good for *some numbers*, and that for those *some numbers* he may substitute *others*, and use the same demonstration. Until

* To avoid too great a number of accents, Roman numerals are put instead of them; also, to avoid confusion, the accents are omitted after the first line.

he can do this himself he does not understand the principle, and he can never do this except by seeing the rule explained, and trying it himself on small numbers. He may, indeed, and will, believe it on the word of his instructor, but this disposition is to be checked. He must be told that, whatever is not gained by his own thought is not gained to any purpose; that the mathematics are put in his way purposely because they are the only sciences in which he must not trust the authority of any one. The superintendence of these efforts is the real business of an instructor in arithmetic. The merely showing the student a rule by which he is to work, and comparing his answer with a key to the book, printed for the preceptor's private use, to save the trouble which he ought to bestow upon his pupil, is not teaching arithmetic any more than presenting him with a grammar and dictionary is teaching him Latin. When the principle of each rule has been well established by showing its application to some simple examples (and the number of these requisite will vary with the intellect of the student), he may then proceed to more complicated cases, in order to acquire facility in computation. The four first rules may be studied in this way, and these will throw the greatest light on those which succeed.

The student must observe that all operations in arithmetic may be resolved into addition and subtraction; that these additions and subtractions might be made with counters; so that the whole of the rules consist of processes intended to shorten and simplify that which would otherwise be long and complex. For example, multiplication is continued addition of the same number to itself—twelve times seven is twelve sevens added together. Division is a continued subtraction of one number from another; the division of 129 by 3 is a continued subtraction of 3 from 129, in order to see how many threes it contains. All other operations are composed of these four, and are, therefore, the result of additions and subtractions only.

The following principles, which occur so continually in mathematical operations that we are, at length, hardly sensible of their presence, are the foundation of the arithmetical rules :—

I. We do not alter the sum of two numbers by taking away any part of the first, if we annex that part to the second. This may be expressed by signs, in a particular instance, thus :—

$$(20-6)+(32+6)=20+32.$$

II. We do not alter the difference of two numbers by increasing or diminishing one of them, provided we increase or diminish the other as much. This may be expressed thus, in one instance :—

$$(45+7)-(22+7)=45-22.$$
$$(45-8)-(22-8)=45-22.$$

III. If we wish to multiply one number by another, for example 156 by 29, we may break up 156 into any number of parts, multiply each of these parts by 29, and add the results. For example, 156 is made up of 100, 50, and 6. Then

$$156\times29=100\times29+50\times29+6\times29.$$

IV. The same thing may be done with the multiplier instead of the multiplicand. Thus, 29 is made up of 18, 6, and 5. Then

$$156\times29=156\times18+156\times6+156\times5.$$

V. If any two or more numbers be multiplied together, it is indifferent in what order they are multiplied, the result is the same. Thus,

$$10\times6\times4\times3=3\times10\times4\times6=6\times10\times4\times3, \text{\&c.}$$

VI. In dividing one number by another, for example 156 by 12, we may break up the dividend, and divide each of its parts by the divisor, and then add the results. We may part 156 into 72, 60, and 24; this is expressed thus :—

$$\frac{156}{12}=\frac{72}{12}+\frac{60}{12}+\frac{24}{12}$$

The same thing cannot be done with the divisor. It is not true that

$$\frac{156}{12}=\frac{156}{4}+\frac{156}{3}+\frac{156}{5}.$$

The student should discover the reason for himself.

A prime number is one which is not divisible by any other number except 1. When the process of division can be performed, it can be ascertained whether a given number is divisible by any other number, that is, whether it is prime or not. This can be done by dividing it by all the numbers which are less than its half, since it is evident that it cannot be divided into a number of parts, each of which is greater than its half. This process would be laborious when the given number is large; still it may be done, and by this means the number itself may be *reduced to its prime fac-*

tors,* as it is called, that is, it may either be shewn to be a prime number itself or made up by multiplying several prime numbers together. Thus, 306 is 34×9, or $2 \times 17 \times 9$, or $2 \times 17 \times 3 \times 3$, and has for its prime factors 2, 17, and 3, the latter of which is repeated twice in its formation. When this has been done with two numbers, we can then see whether they have any factors in common, and, if that be the case, we can then find what is called their *greatest common measure* or *divisor*, that is, the number made by multiplying all their common factors. It is an evident truth that, if a number can be divided by the product of two others, it can be divided by each of them. If a number can be parted into an exact number of twelves, it can be parted also into a number of sixes, twos, or fours. It is also true that, if a number can be divided by any other number, and the quotient can then be divided by a third number, the original number can be divided by the product of the other two. Thus, 144 is divisible by 2; the quotient, 72, is divisible by 6; and the original number is divisible by 6×2 or 12. It is also true that, if two numbers are prime, their product is divisible by no numbers except themselves. Thus, 17×11 is divisible by no numbers except 17 and 11. Though this is a simple proposition, its proof is not so, and cannot be given to the beginner. From these things it follows that the greatest common measure of two numbers (measure being an old word for divisor) is the product of all the prime factors which the two possess in common. For example, the numbers 90 and 100, which, when reduced to their prime factors, are $2 \times 5 \times 3 \times 3$ and $2 \times 2 \times 5 \times 5$, have the common factors 2 and 5, and are divisible by 2×5, or 10. The quotients are 3×3 and 2×5, or 9 and 10, which have no common factor remaining, and 2×5, or 10, is the greatest common measure of 90 and 100. The same may be shewn in the case of any other numbers. But the method we have mentioned of resolving numbers into their prime factors, being troublesome to apply when the numbers are large, is usually abandoned for another. It happens frequently that a method simple in principle is laborious in practice, and the contrary.

When one number is divided by another, and its quotient and remainder obtained, the dividend may be recovered again by multiplying the quotient and divisor together, and adding the remainder to the product. Thus 171 divided by 27 gives a quotient 6 and a remainder 9, and 171 is made by multiplying 27 by 6, and adding 9 to the product. That is, $171 = 27 \times 6 + 9$. Now, from this equation it is easy to shew that every number which divides 171 and 27 also divides 9, that is, every common measure of 171 and 27 is also a common measure of 27 and 9. We can also shew that 27 and 9 have no common measures which are not common to 171 and 27. Therefore, the common measures of 171 and 27 are those, and no others, which are common to 27 and 9; the greatest common measure of each pair must, therefore, be the same, that is, the greatest common measure of a divisor and dividend is also the greatest common measure of the remainder and divisor. Now take the common process for finding the greatest common measure of two numbers; for example, 360 and 420, which is as follows, and abbreviate the words greatest common measure into their initials g. c. m.:—

```
360 ) 420 ( 1
      360
      ---
       60 ) 360 ( 6
            360
            ---
              0
```

From the theorem above enunciated it appears that

g. c. m. of 420 and 360 is g. c. m. of 60 and 360;

g. c. m. of 60 and 360 is 60;

because 60 divides both 60 and 360, and no number can have a greater measure than itself. Thus may be seen the reason of the common rule for finding the greatest common measure of two numbers.

Every number which can be divided by another without remainder is called a multiple of it. Thus, 12, 18, and 42 are multiples of 6, and the last is a *common multiple* of 6 and 7, because it is divisible both by 6 and 7. The only things which it is necessary to observe on this subject are, 1st, that the product of two numbers is a common multiple of both; 2d, that when the two numbers have a common measure greater than 1, there is a common multiple less than their product; 3d, that when they have no common measure except 1, the

* The *factors* of a number are those numbers by the multiplication of which it is made.

least common multiple is their product. The first of these is evident; the second will appear from an example. Take 10 and 8, which have the common measure 2, since the first is 2×5 and the second 2×4. The product is $2 \times 2 \times 4 \times 5$, but $2 \times 4 \times 5$ is also a common multiple, since it is divisible by 2×4, or 8, and by 2×5, or 10. To find this common multiple we must, therefore, divide the product by the greatest common measure. The third principle cannot be proved in an elementary way, but the student may convince himself of it by any number of examples. He will not, for instance, be able to find a common multiple of 8 and 7 less than 8×7, or 56.

Chapter IV.

Arithmetical Fractions.

When the student has perfected himself in the four rules, together with that for finding the greatest common measure, he should proceed at once to the subject of fractions. This part of arithmetic is usually supposed to present extraordinary difficulties; whereas, the fact is that there is nothing in fractions so difficult, either in principle or practice, as the rule for finding the greatest common measure. We would recommend the student not to attend to the distinctions of proper and improper, pure or mixed fractions, &c. as there is no distinction whatever in the rules, which are common to all these fractions.

When one number, as 56, is to be divided by another, as 8, the process is written thus:—$\frac{56}{8}$. By this we mean that 56 is to be divided into 8 equal parts, and one of these parts is called the quotient. In this case the quotient is 7. But it is equally possible to divide 57 into 8 equal parts; for example, we can divide 57 feet into 8 equal parts, but the eighth part of 57 feet will not be an exact number of feet, since 57 does not contain an exact number of eights; a part of a foot will be contained in the quotient $\frac{57}{8}$, and this quotient is therefore called a fraction, or broken number. If we divide 57 into 56 and 1, and take the eighth part of each of these, whose sum will give the eighth part of the whole, the eighth of 56 feet is 7 feet; the eighth of 1 foot is a fraction, which we write $\frac{1}{8}$, and $\frac{57}{8}$ is $7 + \frac{1}{8}$, which is usually written $7\frac{1}{8}$. Both of these quantities $\frac{57}{8}$, and $7\frac{1}{8}$, are called fractions; the only difference is that, in the second, that part of the quotient which is a whole number is separated from the part which is less than any whole number.

There are two ways in which a fraction may be considered. Let us take, for example, $\frac{5}{8}$. This means that 5 is to be divided into 8 parts, and $\frac{5}{8}$ stands for one of these parts. The same length will be obtained if we divide 1 into 8 parts, and take 5 of them, or find $\frac{1}{8} \times 5$. To prove this let each of the lines drawn below represent $\frac{1}{8}$ of an inch; repeat $\frac{1}{8}$ five times, and repeat the same line eight times.

— — — — —
— — — — —
— — — — —
— — — — —
— — — — —
— — — — —
— — — — —
— — — — —

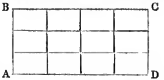
In each column is $\frac{1}{8}$th of an inch repeated 8 times; that is one inch. There are, then, 5 inches in all, since there are five columns. But since there are 8 lines, each line is the eighth of 5 inches, or $\frac{5}{8}$, but each line is also $\frac{1}{8}$th of an inch repeated 5 times, or $\frac{1}{8} \times 5$. Therefore, $\frac{5}{8} = \frac{1}{8} \times 5$; that is, in order to find $\frac{5}{8}$ inches, we may either divide *five inches* into 8 parts, and take *one* of them, or divide *one inch* into 8 parts, and take *five* of them. The symbol $\frac{5}{8}$ is made to stand for both these operations, since they lead to the same result.

The most important property of a fraction is, that if both its numerator and denominator are multiplied by the same number, the value of the fraction is not altered; that is, $\frac{3}{5}$ is the same as $\frac{12}{20}$, or each part is the same when we divide 12 inches into 20 parts, as when we divide 3 inches into 5 parts. Again, we get the same length by dividing 1 inch into 20 parts, and taking 12 of them, which we get by dividing 1 inch into 5 parts and taking 3 of them. This hardly needs demonstration. Taking 12 out of 20 is taking 3 out of 5, since for every 3 which 12 contains, there is a 5 contained in 20. Every fraction, therefore, admits of innumerable alterations in its form, without any alteration in its value. Thus, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$, &c.; $\frac{2}{7} = \frac{4}{14} = \frac{6}{21} = \frac{8}{28}$, &c.

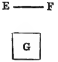
On the same principle it is shewn that

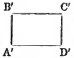
the terms of a fraction may be divided by any number without any alteration of its value. There will now be no difficulty in reducing fractions to a common denominator, in reducing a fraction to its lowest terms; neither in adding nor subtracting fractions, for all of which the rules are given in every book of arithmetic.

We now come to a rule which presents more peculiar difficulties in point of principle than any at which we have yet arrived. If we could at once take the most general view of numbers, and give the beginner the extended notions which he may afterwards attain, the mathematics would present comparatively few impediments. But the constitution of our minds will not permit this. It is by collecting facts and principles, one by one, and thus only, that we arrive at what are called general notions; and we afterwards make comparisons of the facts which we have acquired, and discover analogies and resemblances which, while they bind together the fabric of our knowledge, point out methods of increasing its extent and beauty. In the limited view which we first take of the operations which we are performing, the names which we give are necessarily confined and partial; but when, after additional study and reflection, we recur to our former notions, we soon discover processes so resembling one another, and different rules so linked together, that we feel it would destroy the symmetry of our language if we were to call them by different names. We are then induced to extend the meaniog of our terms, so as to make two rules into one. Also, suppose that when we have discovered and applied a rule, and given the process which it teaches a particular name, we find that this process is only a part of ons more general, which applies to all cases contained under the first, and to others besides. We have only the alternative of inventing a new name, or of extending the meaning of the former one so as to merge the particular process in the more general one of which it is a part. Of this we can give an instance. We began with reasoning upon simple numbers, such as 1, 2, 3, 20, &c. We afterwards divided these into parts, of which we took some number, and which we called fractions, such as $\frac{2}{3}$, $\frac{7}{8}$, $\frac{1}{5}$, &c. Now there is no number which may not be considered as a fraction in as many different ways as we please. Thus 7 is $\frac{14}{2}$ or $\frac{21}{3}$, &c.; 12 is $\frac{144}{12}$, $\frac{72}{6}$, &c. Our new notion of fraction is, then, one which includes all our former ideas of number, and others besides. It is then customary to represent by the word number, not only our first notion of it, but also the extended one, of which the first is only a part. Those to which our first notions applied we call whole numbers, the others fractional numbers, but still the name number is applied both to 2 and $\frac{1}{2}$, to 3 and $\frac{2}{5}$. The rules of which we have spoken is another instance. It is called the multiplication of fractional numbers. Now, if we return to our meaning of the word multiplication, we shall find that the multiplication of one fraction by another appears an absurdity. We multiply a number by taking it several times and adding these together. What, then, is meant by multiplying by a fraction? Still, a rule has been found which, in applying mathematics, it is necessary to use for fractions, in all cases where multiplication would have been used had they been whole numbers. Of this we shall now give a simple example. Take an oblong figure (which is called a rectangle in geometry), such as A B C D, and find the magnitudes of the sides A B and B C in inches. Draw the line



EF equal in length to one inch, and the square G, each of whose sides is one inch. If the lines AB, and BC contain an exact number of inches, the rectangle ABCD contains an exact number of squares, each equal to G, and the number of squares contained is found by multiplying the number of inches in A B by the number of inches in BC. In the present case the number of squares is 3 × 4, or 12. Now, suppose another rectangle A′ B′ C′ D′, of which neither of the sides is an exact number of inches; suppose, for example, that





$A'B'$ is $\frac{2}{3}$ of an inch, and that $B'C'$ is $\frac{5}{7}$ of an inch. We may shew, by reasoning, that we can find how much $A'B'C'D'$ is of G by forming a fraction which has the product of the numerators of $\frac{2}{3}$ and $\frac{5}{7}$ for its numerator, and the product of their denominators for its denominator; that is, that $A'B'C'D'$ contains $\frac{10}{21}$ of G. Here then appears a connexion between the multiplication of whole numbers, and the formation of a fraction whose numerator is the product of two numerators, and its denominator the product of the corresponding denominators. These operations will always come together, that is whenever a question occurs in which, when whole numbers are given, those numbers are to be multiplied together; when fractional numbers are given, it will be necessary, in the same case, to multiply the numerator by the numerator, and the denominator by the denominator, and form the result into a fraction, as above.

This would lead us to suspect some connexion between these two operations, and we shall accordingly find that when whole numbers are formed into fractions, they may be multiplied together by this very rule. Take, for example, the numbers 3 and 4, whose product is 12. The first may be written as $\frac{15}{5}$, and the second as $\frac{8}{2}$. Form a fraction from the product of the numerators and denominators of these, which will be $\frac{120}{10}$, which is 12, the product of 3 and 4.

From these considerations it is customary to call the fraction which is produced from two others in the manner above stated, the *product* of those two fractions, and the process of finding the third fraction, *multiplication*. We shall always find the first meaning of the word multiplication included in the second, in all cases in which the quantities represented as fractions are really whole numbers. The mathematics are not the only branches of knowledge in which it is customary to extend the meaning of established terms. Whenever we pass from that which is simple to that which is complex, we shall see the necessity of carrying our terms with us, and enlarging their meaning, as we enlarge our own ideas. This is the only method of forming a language which shall approach in any degree towards perfection; and more depends upon a well-constructed language in mathematics than in anything else. It is not that an imperfect language would deprive us of the means of demonstration, or cramp the powers of reasoning. The propositions of Euclid upon numbers are as rationally established as any others, although his terms are deficient in analogy, and his notation infinitely inferior to that which we use. It is the progress of discovery which is checked by terms constructed so as to conceal resemblances which exist, and to prevent one result from pointing out another. The higher branches of mathematics date the progress which they have made in the last century and a half, from the time when the genius of Newton, Leibnitz, Descartes, and Hariot turned the attention of the scientific world to the imperfect mechanism of the science. A slight and almost a casual improvement, made by Hariot in algebraical language, has been the foundation of most important branches of the science*. The subject of the last articles is of very great importance, and will often recur to us in explaining the difficulties of algebraical notation.

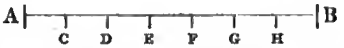
The multiplication of $\frac{5}{6}$ by $\frac{3}{2}$ is equivalent to dividing $\frac{5}{6}$ into 2 parts, and taking three such parts. Because $\frac{5}{6}$ being the same as $\frac{10}{12}$, or 1 divided into 12 parts and 10 of them taken, the half of $\frac{10}{12}$ is 5 of those parts, or $\frac{5}{12}$. Three times this quantity will be 15 of those parts, or $\frac{15}{12}$, which is by our rule the same as what we have called, $\frac{5}{6}$ multiplied by $\frac{3}{2}$. But the same result arises from multiplying $\frac{3}{2}$ by $\frac{5}{6}$, or dividing $\frac{3}{2}$ into 6 parts and taking 5 of them. Therefore, we find that $\frac{3}{2}$ multiplied by $\frac{5}{6}$ is the same as $\frac{5}{6}$ multiplied by $\frac{3}{2}$, or $\frac{3}{2} \times \frac{5}{6} = \frac{5}{6} \times \frac{3}{2}$. This proposition is usually considered as requiring no proof, because it is received very early on the authority of a rule in the elements of arithmetic. But it is not self-evident, for the truth of which we appeal to the beginner himself, and ask him whether he would have seen at once that $\frac{5}{6}$ of an apple divided into 2 parts and 3 of

* The mathematician will be aware that I allude to writing an equation in the form
$$x^2 + ax - b = 0; \text{ instead of}$$
$$x^2 + ax = b.$$

them taken, is the same as $\frac{5}{4}$ of an apple, or one apple and a-half divided into six parts and 5 of them taken.

An extension of the same sort is made of the term division. In dividing one whole number by another, for example, 12 by 2, we endeavour to find how many *twos* must be added together to make 12. In passing from a problem which contains these whole numbers to one which contains fractional quantities, for example $\frac{3}{4}$ and $\frac{2}{5}$, it will be observed that in place of finding how many twos make 12, we shall have to find into how many parts $\frac{2}{5}$ must be divided, and how many of them must be taken, so as to give $\frac{3}{4}$. If we reduce these fractions to a common denominator, in which case they will be $\frac{15}{20}$ and $\frac{8}{20}$; and if we divide the second into 8 equal parts, each of which will be $\frac{1}{20}$, and take 15 of these parts, we shall get $\frac{15}{20}$, or $\frac{3}{4}$. The fraction whose numerator is 15, and whose denominator is 8, or $\frac{15}{8}$, will in these problems take the place of the quotient of the two whole numbers. In the same manner as before, it may be shewn that this process is equivalent to the division of one whole number by another, whenever the fractions are really whole numbers; for example, 3 is $\frac{12}{4}$, and 15 is $\frac{30}{2}$. If this process be applied to $\frac{30}{2}$ and $\frac{12}{4}$, the result is $\frac{120}{24}$, which is 5, or the same as 15 divided by 3. This process is then, by extension, called division: $\frac{15}{8}$ is called the quotient of $\frac{3}{4}$ divided by $\frac{2}{5}$, and is found by multiplying the numerator of the first by the denominator of the second for the numerator of the result, and the denominator of the first by the numerator of the second for the denominator of the result. That this process does give the same result as ordinary division in all cases where ordinary division is applicable, we can easily shew from any two whole numbers, for example, 12 and 2, whose quotient is 6. Now 12 is $\frac{36}{3}$, and 2 is $\frac{10}{5}$, and the rule for what we have called division of fractions will give as the quotient $\frac{180}{30}$, which is 6.

In all fractional investigations, when the beginner meets with a difficulty, he should accustom himself to leave the notation of fractions, and betake himself to their original definition. He should recollect that $\frac{5}{6}$ is 1 divided into 6 parts and five of them taken, or the sixth part of 5, and he should reason upon these suppositions, neglecting all rules until he has established them in his own mind by reflection on particular instances. These instances should not contain large numbers, and it will perhaps assist him if he reasons on some given unit, for example a foot. Let A B be one foot, and divide it into any number of equal parts (7, for example) by the points C, D, E, F, G, and H.



He must then recollect that each of these parts is $\frac{1}{7}$ of a foot; that any two of them together are $\frac{2}{7}$ of a foot; any 3, $\frac{3}{7}$, and so on. He should then accustom himself, without a rule, to solve such questions as the following, by observation of the figure, dividing each part into several equal parts, if necessary; and he may be well assured that he does not understand the nature of fractions until such questions are easy to him.

What is $\frac{1}{4}$ of $\frac{2}{7}$ of a foot? What is $\frac{2}{5}$ of $\frac{1}{3}$ of $\frac{3}{4}$ of a foot? Into how many parts must $\frac{4}{7}$ of a foot be divided, and how many of them must be taken to produce $\frac{14}{15}$ of a foot? What is $\frac{1}{3}+\frac{1}{7}$ of a foot? and so on.

Chapter V.

Decimal Fractions.

It is a disadvantage attending rules received without a knowledge of principles, that a mere difference of language is enough to create a notion in the mind of a student that he is upon a totally different subject. Very few beginners see that in following the rule usually called practice, they are working the same questions as were proposed in compound multiplication;—that the rule of three is only an application of the doctrine of fractions; that the rules known by the name of commission, brokerage, interest, &c., are the same, and so on. No instance, however, is more conspicuous than that of decimal fractions, which are made to form a branch of arithmetic as distinct from ordinary or vulgar fractions as any two parts of the subject whatever. Nevertheless,

there is no single rule in the one which is not substantially the same as the rule corresponding in the other, the difference consisting altogether in a different way of writing the fractions. The beginner will observe that throughout the subject it is continually necessary to reduce fractions to a common denominator: he will see, therefore, the advantage of always using either the same denominator, or a set of denominators, so closely connected as to be very easily reducible to one another. Now of all numbers which can be chosen the most easily manageable are 10, 100, 1000, &c., which are called decimal numbers on account of their connexion with the number ten. All fractions, such as $\frac{75}{100}$, $\frac{333}{1000}$, $\frac{178699}{10}$, which have a decimal number for the denominator, are called decimal fractions. Now a denominator of this sort is known whenever the number of cyphers in it are known; thus a decimal number with 4 cyphers can only be 10,000, or ten thousand. We need not, therefore, write the denominator, provided, in its stead, we put some mark upon the numerator, by which we may know the number of cyphers in the denominator. This mark is for our own selection. The method which is followed is to point off from the numerator as many *figures* as there are *cyphers* in the denominator. Thus $\frac{17334}{1000}$ is represented by 17.334; $\frac{229}{1000}$ thus, .229. We might, had we so pleased, have represented them thus, 17334^3, 229^3; or thus, 17334_3, 229_3, or in any way by which we might choose to agree to recollect that the denominator is 1 followed by 3 cyphers. In the common method this difficulty occurs immediately. What shall be done when there are not as many figures in the numerator as there are cyphers in the denominator? How shall we represent $\frac{88}{10000}$? We must here extend our language a little, and imagine some method by which, without essentially altering the numerator, it may be made to shew the number of cyphers in the denominator. Something of the sort has already been done in representing a number of tens, hundreds, or thousands, &c.; for 5 thousands were represented by 5000, in which, by the assistance of cyphers, the 5 is made to stand in the place allotted to thousands. If, in the present instance, we place cyphers at the beginning of the numerator, until the number of figures and cyphers together is equal to the number of cyphers in the denominator, and place a point before the first cypher, the fraction $\frac{88}{10000}$ will be represented thus, .0088; by which we understand a fraction whose numerator is 88, and whose denominator is a decimal number containing four cyphers.

There is a close connexion between the manner of representing decimal fractions, and the decimal notation for numbers. Take, for example, the fraction 217.3426, or $\frac{2173426}{10000}$. You will recollect that 2173426 is made up of $2000000 + 100000 + 70000 + 3000 + 400 + 20 + 6$. If each of these parts be divided by 10000, and the quotient obtained or the fraction reduced to its lowest terms, the result is as follows:—

$$\frac{2173426}{10000} = 200 + 10 + 7 + \frac{3}{10} + \frac{4}{100} + \frac{2}{1000} + \frac{6}{10000}.$$

We see, then, that in the fraction 217.3426 the first figure 2 counts two hundred; the second figure, 1, ten, and the third 7 units. It appears, then, that all figures on the left of the decimal point are reckoned as ordinary numbers. But on the right of that point we find the figure 3, which counts for $\frac{3}{10}$, 4 which counts $\frac{4}{100}$; 2, or $\frac{2}{1000}$; and 6, or $\frac{6}{10000}$. It appears, therefore, that numbers on the right of the decimal point decrease as they move towards the right, each number being one-tenth of what it would have been had it come one place nearer to the decimal point. The first figure on the right hand of that point is so many tenths of a unit, the second figure so many hundredths of a unit, and so on.

The learner should go through the same investigation with other fractions, and should demonstrate by means of the principles of fractions, generally, such exercises as the following, until he is thoroughly accustomed to this new method of writing fractions:—

$$.68342 = .6 + .08 + .003 + .0004 + .00002$$

$$\text{or } \frac{68342}{100000} = \frac{6}{10} + \frac{8}{100} + \frac{3}{1000} + \frac{4}{10000} + \frac{2}{100000}$$

$$.00012 = .0001 + .00002 = \frac{1}{10000} + \frac{2}{100000}$$

$$163.429 = \frac{163429}{1000} = 163\frac{429}{1000} = \frac{1634}{10} + \frac{29}{1000} = \frac{16342}{100} + \frac{9}{1000}, \text{ \&c.}$$

The rules for addition, subtraction, and multiplication may now be understood. In addition and subtraction, the keeping the decimal points under one another is equivalent to reducing the fractions to a common denominator, as we may shew thus:—Take two fractions, 1.5 and 2.125, or $\frac{15}{10}$ and $\frac{2125}{1000}$, which, reducing the first to the denominator of the second, may be written $\frac{1500}{1000}$ and $\frac{2125}{1000}$. If we add the numerators together, we find the sum of the fractions $\frac{3625}{1000}$, or 3.625

$$\begin{array}{rr} 2125 & 2.125 \\ 1500 & 1.5 \\ \hline 3625 & 3.625 \end{array}$$

The learner can now see the connexion of the rule given for the addition of decimal fractions with that for the addition of vulgar fractions. There is the same connexion between the rules of subtraction. The principle of the rule of multiplication is as follows:—If two decimal numbers be multiplied together, the product has as many cyphers as are in both together. Thus $100 \times 1000 = 100000$, $10 \times 100 = 1000$, &c. Therefore the denominator of the product, which is the product of the denominators, has as many cyphers as are in the denominators of both fractions, and since the numerator of the product is the product of the numerators, the point must be placed in that product so as to cut off as many decimal places as are both in the multiplier and multiplicand. Thus

$$\frac{13}{100} \times \frac{12}{10} = \frac{156}{1000}, \text{ or } .13 \times 1.2 = .156;$$

$$\frac{4}{1000} \times \frac{6}{100} = \frac{24}{100000},$$

$$\text{or } .004 \times .06 = .00024, \text{ \&c.}$$

It is a general rule, that wherever the number of figures falls short of what we know ought to be the number of decimals, the deficiency is made up by cyphers.

It may now be asked, whether all fractions can be reduced to decimal fractions? It may be answered that they cannot. It is a principle which is demonstrated in the science of algebra,—that if a number be not divisible by a prime number, no multiplication of that number, by itself, will make it so. Thus 10 not being divisible by 7, neither 10×10, nor $10 \times 10 \times 10$, &c. is divisible by 7. A consequence of this is, that since 5 and 2 are the only prime numbers which will divide 10, no fraction can be converted into a decimal unless its denominator is made up of products, either of 5 or 2, or of both combined, such as 5×2, $5 \times 5 \times 2$, $5 \times 5 \times 5$, 2×2, &c. To shew that this is the case, take any fraction with such a denominator; for example, $\frac{13}{5 \times 5 \times 5}$. Multiply the numerator and denominator by 2, once for every 5, which is contained in the denominator, and the fraction will then become $\frac{13 \times 2 \times 2 \times 2}{5 \times 5 \times 5 \times 2 \times 2 \times 2}$, or $\frac{2 \times 2 \times 2 \times 13}{10 \times 10 \times 10}$, which is $\frac{104}{1000}$, or .104. In a similar way, any fraction whose denominator has no other factors than 2 or 5, can be reduced to a decimal fraction. We first search for such a number as will, when multiplied by the denominator, produce a decimal number, and then multiply both the numerator and denominator by that number.

No fraction which has any other factor in its denominator can be reduced to a decimal fraction exactly. But here it must be observed, that in most parts of mathematical computation, a very small error is not material. In different species of calculations, more or less exactness may be required; but even in the most delicate operations, there is always a limit beyond which accuracy is useless, because it cannot be appreciated. For example, in measuring land for sale, an error of an inch in five hundred yards is not worth avoiding, since even if such an error were committed, it would not make a difference which would be considered as of any consequence, as in all probability the expense of a more accurate measurement would be more than the small quantity of land thereby saved would be worth. But in the measurement of a line for the commencement of a trigonometrical survey, an error of an inch in five hundred yards would be fatal, because the subsequent processes involve calculations of such a nature that this error would be multiplied, and

cause a considerable error in the final result. Still, even in this case, it would be useless to endeavour to avoid an error of one-thousandth part of an inch in five hundred yards; first, because no instruments hitherto made would shew such an error: and secondly, because if they could, no material difference would be made in the result by a correction of it. Again, we know that almost all bodies are lengthened in all directions by heat. For example:—A brass ruler which is a foot in length to-day, while it is cold, will be more than a foot to-morrow if it is warm. The difference, nevertheless, is scarcely, if at all, perceptible to the naked eye, and it would be absurd for a carpenter, in measuring a few feet of mahogany for a table, to attempt to take notice of it; but in the measurement of the base of a survey, which is several miles in length, and takes many days to perform, it is necessary to take this variation into account, as a want of attention to it may produce perceptible errors in the result: nevertheless, any error which has not this effect, it would be useless to avoid even were it possible. We see, therefore, that we may, instead of a fraction, which cannot be reduced to a decimal, substitute a decimal fraction, if we can find one so near to the former, that the error committed by the substitution will not materially affect the result. We will now proceed to shew how to find a series of decimal fractions, which approach nearer and nearer to a given fraction, and also that, in this approximation, we may approach as near as we please to the given fraction without ever being exactly able to reach it.

Take, for example, the fraction $\frac{7}{11}$. If we divide the series of numbers 70, 700, 7000, &c. by 11, we shall obtain the following results:—

$\frac{70}{11}$	gives the quotient	6,	and the remainder	4,	and is	$6\frac{4}{11}$
$\frac{700}{11}$	"	63	"	7		$63\frac{7}{11}$
$\frac{7000}{11}$	"	636	"	4		$636\frac{4}{11}$
$\frac{70000}{11}$	"	6363	"	7		$6363\frac{7}{11}$
&c.		&c.		&c.		

Now observe, that if two numbers do not differ by so much as 1, their tenth parts do not differ by so much as $\frac{1}{10}$, their hundredth parts by so much as $\frac{1}{100}$, their thousandth parts by so much as $\frac{1}{1000}$, and so on; and also remember, that $\frac{7}{11}$ is the tenth part of $\frac{70}{11}$, the hundredth part of $\frac{700}{11}$, and so on. The two following tables will now be apparent:—

$\frac{70}{11}$	does not differ from	6	by so much as	1
$\frac{700}{11}$	"	63	"	1
$\frac{7000}{11}$	"	636	"	1
$\frac{70000}{11}$	"	6363	"	1
&c.		&c.		&c.

Therefore

$\frac{7}{11}$	does not differ from	$\frac{6}{10}$	or .6,	by so much as	$\frac{1}{10}$	or .1
$\frac{7}{11}$	"	$\frac{63}{100}$	" .63	"	$\frac{1}{100}$	" .01
$\frac{7}{11}$	"	$\frac{636}{1000}$	" .636	"	$\frac{1}{1000}$	" .001
$\frac{7}{11}$	"	$\frac{6363}{10000}$	" .6363	"	$\frac{1}{10000}$	" .0001
&c.		&c.			&c.	

We have then a series of decimal fractions, viz.—.6, .63, .636, .6363, .63636, &c. which continually approach more and more near to $\frac{7}{11}$, and therefore in any calculation in which the fraction $\frac{7}{11}$ appears, any one of these may be substituted for it, which is sufficiently near to suit the purpose for which the calculation is intended. For some purposes .636 would be a sufficient approximation; for others .63636363 would be necessary. Nothing but practice can shew how far the approximation should be carried in each case.

The division of one decimal fraction by another is performed as follows:—Suppose it required to divide 6.42 by 1.213. The first of these is $\frac{642}{100}$, and the second $\frac{1213}{1000}$. The quotient of these by the ordinary rule is $\frac{642000}{121300}$, or $\frac{6420}{1213}$.

This fraction must now be reduced to a decimal on the principles of the last article, by the rule usually given, either exactly, or by approximation, according to the nature of the factors in the denominator.

When the decimal fraction corresponding to a common fraction cannot be exactly found, it always happens that the series of decimals which approximates to it, contains the same number repeated again and again. Thus, in the example which we chose, $\frac{7}{11}$ is more and more nearly represented by the fractions .6, .63, .636, .6363, &c., and if we carried the process on without end, we should find a decimal fraction consisting entirely of repetitions of the figures 63 after the decimal point. Thus, in finding $\frac{1}{7}$, the figures which are repeated in the numerator are 142857. This is what is commonly called a circulating decimal, and rules are given in books of arithmetic for reducing them to common fractions. We would recommend to the beginner to omit all notice of these fractions, as they are of no practical use, and cannot be thoroughly understood without some knowledge of algebra. It is sufficient for the student to know, that he can always either reduce a common fraction to a decimal, or find a decimal near enough to it for his purpose, though the calculation in which he is engaged requires a degree of accuracy which the finest microscope will not appreciate. But in using approximate decimals there is one remark of importance, the necessity for which occurs continually.

Suppose that the fraction 2.143876 has been obtained, and that it is more than sufficiently accurate for the calculation in which it is to be employed. Suppose that for the object proposed it is enough that each quantity employed should be a decimal fraction of three places only, the quantity 2.143876 is made up of 2.143, as far as three places of decimals are concerned, which at first sight might appear to be what we ought to use, instead of 2.143876. But this is not the number which will in this case give the utmost accuracy which three places of decimals will admit of; the common usages of life will guide us in this case. Suppose a regiment consists of 876 men, we should express this in what we call round numbers, which in this case would be done by saying how many hundred men there are, leaving out of consideration the number 76, which is not so great as 100; but in doing this we shall be nearer the truth if we say that the regiment consists of 900 men instead of 800, because 900 is nearer to 876 than 800. In the same manner, it will be nearer the truth to write 2.144 instead of 2.143, if we wish to express 2.143876 as nearly as possible by three places of decimals, since it will be found by subtraction that the first of these is nearer to the third than the second. Had the fraction been 2.14326, it would have been best expressed in three places by 2.143; had it been 2.1435, it would have been equally well expressed either by 2.143 or 2.144, both being equally near the truth; but 2.14351 is a little more nearly expressed by 2.144 than by 2.143.

We have now gone through the leading principles of arithmetical calculation, considered as a part of general Mathematics. With respect to the commercial rules, usually considered as the grand object of an arithmetical education, it is not within the scope of this treatise to enter upon their consideration. The mathematical student, if he is sufficiently well versed in their routine for the purposes of common life, may postpone their consideration until he shall have become familiar with algebraical operations, when he will find no difficulty in understanding the principles or practice of any of them. He should, before commencing the study of algebra, carefully review what he has learnt in arithmetic, particularly the reasonings which he has met with, and the use of the signs which have been introduced. Algebra is at first only arithmetic under another name, and with more general symbols, nor will any reasoning be presented to the student which he has not already met with in establishing the rules of arithmetic. His progress in the former science depends most materially, if not altogether, upon the manner in which he has attended to the latter; on which account the detail into which we have entered on some things which to an intelligent person are almost self-evident, must not be deemed superfluous.

When the student is well acquainted with the principles and practice of arithmetic, and not before, he should commence the study of algebra. It is usual to begin algebra and geometry together, and if the student has sufficient time, it is the best plan which he can adopt. Indeed, we see no reason why the elements of geometry should not precede

those of algebra, and be studied together with arithmetic. In this case the student should read that part of these treatises which relates to geometry, first. It is hardly necessary to say that though we have adopted one particular order, yet the student may reverse or alter that order so as to suit the arrangement of his own studies.

We now proceed to the first principles of algebra, and the elucidation of the difficulties which are found from experience to be most perplexing to the beginner. We suppose him to be well acquainted with what has been previously laid down in this treatise, particularly with the meaning of the signs $+$, $-$, $\times$, and the sign of division.

Chapter VI.

Algebraical Notation and Principles.

Whenever any idea is constantly recurring, the best thing which can be done for the perfection of language, and consequent advancement of knowledge, is to shorten as much as possible the sign which is used to stand for that idea. All that we have accomplished hitherto has been owing to the short and expressive language which we have used to represent numbers, and the operations which are performed upon them. The first step was to write simple signs for the first numbers, instead of words at full length, such as 8 and 7, instead of eight and seven. The next was to give these signs an additional meaning, according to the manner in which they were connected with one another; thus 187 was made to represent one hundred added to eight tens added to seven. The next was to give by new signs directions when to perform the operations of addition, subtraction, multiplication, and division; thus $5+8$ was made to represent 8 added to 5, and so on. With these signs reasonings were made, and truths discovered which are common to all numbers; not at once for every number, but by taking some example, by reasoning upon it, and by producing a result; this result led to a rule which was declared to be a rule which held equally good for all numbers, because the reasoning which produced it might have been applied to any other example as well as to the one which was chosen. In this way we produced some results, and might have produced many more; the following is an instance:—half the sum of two numbers added to half their difference, gives the greater of the two numbers. For example, take 16 and 10, half their sum is 13, half their difference is 3; if we add 13 and 3 we get 16, the greater of the two numbers. We might satisfy ourselves of the truth of this same proposition for any other numbers, such as 27 and 8, 15 and 19, and so on. If we then make use of signs, we find the following truths:—

$$\frac{16+10}{2}+\frac{16-10}{2}=16.$$

$$\frac{27+8}{2}+\frac{27-8}{2}=27.$$

$$\frac{15+9}{2}+\frac{15-9}{2}=15, \text{ and so on.}$$

If, then, we choose any two numbers, and call them the first and second numbers, and call that the first number which is the greater of the two, we have the following:—

$$\frac{\text{First N}^\circ+\text{second N}^\circ}{2}+\frac{\text{First N}^\circ-\text{second N}^\circ}{2}=\text{First N}^\circ.$$

In this way we might express anything which is true of all numbers, by writing first N°, second N°, &c., for the different numbers which enter into our proposition, and we might afterwards suppose the first N°, the second N°, &c. to be any which we please. In this way we might write down the following assertion, which we should find to be always true:—

$$(\text{First N}^\circ+\text{second N}^\circ)\text{ multiplied by }(\text{First N}^\circ-\text{second N}^\circ)$$
$$=\text{First N}^\circ\times\text{first N}^\circ-\text{Second N}^\circ\times\text{second N}^\circ.$$

When any sentence expresses that two numbers or collections of numbers are equal to one another, it is called an *equation*, thus $7+5=12$ is an equation, and the sentences which have been just written down are equations.

Now the next question is, could we not avoid the trouble of writing first N°, second N°, &c. so frequently? This is done by putting letters of the alphabet to stand for these numbers. Suppose, for example, we let x stand for the first number, and y for the second, the two assertions already made will then be written thus:—

$$\frac{x+y}{2}+\frac{x-y}{2}=x.$$

$$(x+y)\times(x-y)=x\times x-y\times y.$$

By the use of letters we are thus enabled to write sentences which say something of all numbers, with a very small part only of the time and trouble necessary for writing the same thing at full length. We now proceed to enumerate the various symbols which are used.

1. The letters of the alphabet are used to stand for numbers, and whenever a letter is used it means either that any number may be used instead of that letter, or that the number which the letter stands for is not known, and that the letter supplies its place in all the reasonings until it is known.

2. The sign $+$ is used for addition, as in arithmetic. Thus $x+z$ is the sum of the numbers represented by x and z. The following equations are sufficiently evident:—

$$x+y+z=x+z+y=y+z+x.$$

If $a=b$, then $a+c=b+c$, $a+c+d=b+c+d$, &c.

3. The sign $-$ is used for subtraction, as in arithmetic. The following equations will show its use:—

$$x+a-b-c+e=x+a+e-b-c=a-c+e-b+x.$$

If $a=b$, $a-c=b-c$, $a-c+d=b-c+d$, &c.

4. The sign $\times$ is used for multiplication as in arithmetic, but when two numbers represented by letters are multiplied together it is useless, since $a\times b$ can be represented by putting a and b together thus, $a\,b$. Also $a\times b\times c$ is represented by $a\,b\,c$; $a\times a\times a$, for the present we represent thus, $a\,a\,a$. When two numbers are multiplied together, it is necessary to keep the sign $\times$; otherwise 7×5 or 35 would be mistaken for 75. It is, however, usual to place a point between two numbers which are to be multiplied together; thus $7\times 5\times 3$ is written $7\,.\,5\,.\,3$. But this point may sometimes be mistaken for the decimal point: this will, however, be avoided by always writing the decimal point at the head of the figure, viz. by writing $\frac{23461}{100}$ thus, 234·61.

5. Division is written, as in arithmetic; thus, $\frac{a}{b}$ signifies that the number represented by a is to be divided by the number represented by b.

6. All collections of numbers are called expressions; thus, $a+b$, $a+b-c$, $a\,a+b\,b-d$, are algebraical expressions.

7. When two expressions are to be multiplied together, it is indicated by placing them side by side, and inclosing each of them in brackets. Thus, if $a+b+c$ is to be multiplied by $d+e+f$, the product is written in any of the following ways:—

$$(a+b+c)\,(d+e+f),$$

$$[a+b+c]\,[d+e+f],$$

$$\overline{a+b+c}\,.\,\overline{d+e+f},$$

$$\overline{a+b+c|}\,.\,\overline{d+e+f|}$$

8. That a is greater than b is written thus, $a>b$.

9. That a is less than b is written thus, $a<b$.

10. When there is a product in which all the factors are the same, such as $xxxxx$, which means that five equal numbers, each of which is represented by x, are multiplied together, the letter is only written once, and above it is written the number of times which it occurs, thus, $xxxxx$ is written x^5. The following table should be carefully studied by the student:—

$x\times x$ . . . or xx is written x^2, and is called the square, or second power of x.
$x\times x\times x$ . . . xxx . . . x^3, ... cube, or third power of x.
$x\times x\times x\times x$ . . $xxxx$ . . . x^4, ... fourth power of x.
$x\times x\times x\times x\times x$ $xxxxx$ . . . x^5, ... fifth power of x.
&c. &c. &c. &c.

There is no point which is so likely to create confusion in the ideas of a beginner as the likeness between such expressions as $4x$ and x^4. On this account it would be better for him to omit using the latter expression, and to put $xxxx$ in its place until he has acquired some little facility in the operations of algebra. If he does not pursue this course he must recollect that the 4, in

these two expressions, has different names and meanings. In $4x$ it is called a *coefficient*, in x^4 an *exponent* or *index*. The difference of meaning will be apparent from the following tables :—

$$\begin{array}{ll} 2x=x+x & x^2=x\times x=xx, \\ 3x=x+x+x & x^3=x\times x\times x \text{ or } xxx, \\ 4x=x+x+x+x & x^4=x\times x\times x\times x \text{ or } xxxx, \\ \&c. & \&c. \end{array}$$

$$\begin{array}{lll} \text{If } x=3 & 2x=6 & x^2=9, \\ & 3x=9 & x^3=27, \\ & 4x=12 & x^4=81, \\ & \&c. & \&c. \end{array}$$

The beginner should frequently *write* for himself such expressions as the following :—

$$4a^2b^2=aaabb+aaabb+aaabb+aaabb$$

$$5a^4x=aaaax+aaaax+aaaax+aaaax+aaaax$$

$$9a^2b^3+4ab^4=9aabbb+4abbbb$$

$$\frac{a^2+b^2}{a^2-b^2}=\frac{aa+bb}{aa-bb}=\frac{aa}{aa-bb}+\frac{bb}{aa-bb}=\frac{aa-cc}{aa-bb}+\frac{bb+cc}{aa-bb}$$

$$\frac{a^3-b^3}{a^2-b^2}=\frac{aaa-bbb}{aa-bb}=\frac{aa+ab+bb}{a+b}$$

With many such expressions every book on algebra will furnish him, and he should then satisfy himself of their truth by putting some numbers at pleasure instead of the letters, and making the results agree with one another. Thus, to try the expression $\frac{a^3-b^3}{a-b}=a^2+ab+b^2$, or, which is the same, $\frac{aaa-bbb}{a-b}=aa+ab+bb$. Let a stand for 6 and b stand for 4, then, if this expression be true, $\frac{6.6.6-4.4.4}{6-4}=6.6+6.4+4.4$, which is correct, since each of these expressions is found, by calculation, to be 76.

The student should then exercise himself in the solution of such questions as the following: — What is $a^2+b^2-\frac{ab}{a+b}+\frac{a}{a-b}-a$, I. when a stands for 6, and b for 5, II. when a stands for 13, and b for 2, and so on. He should stop here until he has, by these means, made the signs familiar to his eye and their meaning to his mind: nor should he proceed to any further algebraical operations until he can readily find the value of any algebraical expression, when he knows the numbers which the letters stand for. He cannot, at this period of his course, write too many algebraical expressions, and he must particularly avoid slurring over the sense of what he has before him, and must write and rewrite each expression until the meaning of the several parts forces itself upon his memory at first sight, without even the necessity of putting it in words. It is the neglecting to do this which renders the operations of algebra so tedious to the beginner. He usually proceeds to the addition, subtraction, &c. of symbols, of the meaning of which he has but an imperfect idea, and which have been newly introduced to him in such numbers that perpetual confusion is the consequence. We cannot, therefore, use too many arguments to induce him not to mind the drudgery of reducing algebraical expressions into figures. This is the connecting link between the new science and arithmetic, and, unless that link be well fastened, the knowledge which he has previously acquired in arithmetic will help him but little in acquiring algebra.

The order of the terms of any algebraical expression may be changed without changing the value of that expression. This needs no proof, and the following are examples of the change :—

$$\begin{aligned} & a+b+ab+c+ac-d-e-de-f \\ =\ & a-d+b-e+ab-de+c-f+ac \\ =\ & a+b-d-e-de-f+ac+c+ab \\ =\ & ab+ac-de+a+b+c-e-f-d \end{aligned}$$

When the first term changes its place, as in the fourth of these expressions, the sign $+$ is put before it, and should, properly speaking, be written wherever there is no sign, to indicate that the term in question increases the result of the rest, but it is usually omitted. The

negative sign is often written before the first term, as in the expression $-a+b$: but it must be recollected that this is written on the supposition that a is subtracted from what comes after it.

When an expression is written in brackets, with some sign before it, such as $a-(b-c)$, it is understood that the expression in brackets is to be considered as one quantity, and that its result or total is to be connected with the rest by the sign which precedes the brackets. In this example it is the *difference* of b and c which is to be subtracted from a. If $a=12$, $b=6$, and $c=4$, this is 10. In the expression $a-b$ made by subtracting b from a, too much has been subtracted by the quantity c, since it is not b, but $b-c$, which must be subtracted from a. In order, therefore, to make $a-(b-c)$ c must be added to $a-b$, which gives $a-b+c$. Therefore,

$$a-(b-c)=a-b+c$$

Similarly $$a+b-(c+d-e-f)=a+b-c-d+e+f$$

$$(ax^2-bx+c)-(dx^2-ex+f)=ax^2-bx+c-dx^2+ex-f.$$

When the positive sign is written before an expression in brackets, the brackets may be omitted altogether, unless they serve to show that the expression in question is multiplied by some other. Thus, instead of $(a+b+c)+(d+e+f)$, we may write $a+b+c+d+e+f$, which is, in fact, only saying that two wholes may be added together by adding together all the parts of which they are composed. But the expression $a+(b+c)(d+e)$ must not be written thus: $a+b+c(d+e)$, since the first expresses that $(b+c)$ must be multiplied by $(d+e)$ and the product added to a, and the second that c must be multiplied by $(d+e)$ and the product added to $a+b$. If a, b, c, d, and e, stand for 1, 2, 3, 4, and 5, the first is 46 and the second 30.

When two or more quantities have exactly the same letters repeated the same number of times, such as $4a^2b^3$, and $6a^2b^3$, they may be reduced into one by merely adding the coefficients and retaining the same letters. Thus, $2a+3a$ is $5a$, $6bc-4bc$ is $2bc$, $3(x+y)+2(x+y)$ is $5(x+y)$. These things are evident, but beginners are very liable to carry this farther than they ought, and to attempt to reduce expressions which do not admit of reduction. For example, they will say that $3b+b^2$ is $4b$ or $4b^2$, neither of which is true, except when b stands for 1. The expression $3b+b^2$, or $3b+bb$, cannot be made more simple until we know what b stands for. The following table will, perhaps, be of service.

$6\,a^2b^3+3\,a^3b^2$ is not $9\,a^5b^5$
$6\,a^3-4\,a^2$ is not $2\,a$
$2\,b\,a+3b$ is not $5\,ab$

Such are the mistakes which beginners almost universally make, mostly for want of a moment's consideration. They attempt to reduce quantities which cannot be reduced, which they do by adding the exponents of letters as well as their coefficients, or by collecting several terms into one, and leaving out the signs of addition and subtraction. The beginner cannot too often repeat to himself that two terms can never be made into one, unless both have the same letters, each letter being repeated the same number of times in both, that is, having the same index in both. When this is the case, the expressions may be reduced by adding or subtracting the coefficients according to the sign, and affixing the common letters with their indices. When there is no coefficient, as in the expression a^2b, the quantity represented by a^2b being only taken once, 1 is called the coefficient. Thus,

$$3\,ab+4\,ab+6\,ab-ab-7\,ab=5\,ab$$
$$6\,xy^2+3\,xy^2-5\,xy^2+xy^2\quad=5\,xy^2$$

The student must also recollect that he is not at liberty to change an index from one letter to another, as by so doing he changes the quantity represented. Thus a^4b and ab^4 are quantities totally distinct, the first representing *aaaab* and the second *abbbb*. The difference in all the cases which we have mentioned will be made more clear, by placing numbers at pleasure instead of letters in the expressions, and calculating their values; but, in conclusion, the following remark must be attended to. If it were asserted that the expression $\dfrac{a^2+b^2}{a+b}$ is the same as $a+b-\dfrac{2\,ab}{2\,a-b}$, and we wish to proceed to see whether this is always the case or no, if we commence accidentally by supposing b to stand for 2 and a for 4, we shall find that the first is the same as the second, each being $3\frac{1}{3}$. But we must not conclude from this that they are always the same, at least until we

have tried whether they are so, when other numbers are substituted for a and b. If we place 6 and 8 instead of a and b, we shall find that the two expressions are not equal, and therefore we must conclude that they are not always the same. Thus in the expressions $3x - 4$ and $2x + 8$, if x stand for 12, these are the same, but if it stands for any other number they are not the same.

Chapter VII.

Elementary Rules of Algebra.

The student should be very well acquainted with the principles and notation hitherto laid down before he proceeds to the algebraical rules for addition and subtraction. He should then take some simple examples of each, and proceed to find the sum and difference by reasoning as follows. Suppose it required to add $c - d$ to $a - b$. The direction to do this may either be written in the common way thus:

$$\begin{array}{r} a - b \\ c - d \\ \hline \text{Add} \end{array}$$

or more properly thus: Find $(a - b) + (c - d)$.

If we add c to a, or find $a + c$, we have too much; first, because it is not a which is to be increased by $c - d$ but $a - b$; this quantity must therefore be decreased by b on this account, or must become $a + c - b$; but this is still too great, because it is not c which was to be added but $c - d$; it must therefore be decreased by d on this account, or must become $a + c - b - d$ or $a - b + c - d$. From a few reasonings of this sort the rule may be deduced; and not till then should an example be chosen so complicated as to make the student lose sight for one moment of his demonstration. The process of subtraction we have already performed, and from a few examples of this method the rule may be deduced.

The multiplication of a by $c - d$ is performed thus: a is to be taken $c - d$ times. Take it first c times or find ac. This is too great, because a has been taken too many times by d. From ac we must therefore subtract d times a, or ad, and the result is that

$$a(c - d) = ac - ad.$$

This may be verified from arithmetic, in which the same process is shewn to be correct; and this whether the numbers a, c, and d are whole or fractional. For example, it will be found that $6(14 - 9)$ or 6×5 is the same as $6 \times 14 - 6 \times 9$, or as $84 - 54$. Also that $\frac{2}{3}\left(\frac{1}{7} - \frac{2}{15}\right)$, or $\frac{2}{3} \times \frac{1}{105}$ is the same as $\frac{2}{3} \times \frac{1}{7} - \frac{2}{3} \times \frac{2}{15}$, or as $\frac{2}{21} - \frac{4}{45}$. Upon similar reasoning the following equations may be proved:

$$a(b + c - d) = ab + ac - ad.$$
$$(p + pq - ar)xz = pxz - pqxz - arxz.$$
$$(a^2 + 2b^2)b^2, \text{ or } (aa + 2bb)bb = aabb + 2bbbb.$$
$$= a^2b^2 + 2b^4.$$

Also when a multiplication has been performed, the process may be reversed and the factors of it may be given. Thus, on observing the expression $ab - ac + a^2$, or $ab - ac + aa$, we see that in its formation every term has been multiplied by a; that is, it has been made by multiplying $b - c + a$ by a, or a by $b - c + a$. There will now be no difficulty in perceiving that

$$ac + ad + bc + bd = a(c + d) + b(c + d) = (a + b)(c + d)$$
$$a^2 - ab^2 + 2abc - dc + 3a = a(a - b^2 + 3) + c(2ab - d)$$

It is proved in arithmetic that if numbers, whether whole or fractional, are multiplied together, the product remains the same when the order in which they are multiplied is changed. Thus $6 \times 4 \times 3 = 3 \times 6 \times 4 = 4 \times 6 \times 3$, &c., and $\frac{2}{3} \times \frac{4}{5} = \frac{4}{5} \times \frac{2}{3}$, &c. Also, that a part of the multiplication may be made, and the partial product substituted instead of the factors which produced it, thus, $3 \times 4 \times 5 \times 6$ is $12 \times 5 \times 6$, or $15 \times 4 \times 6$, or 90×4. From these rules two complicated single terms may be multiplied together, and the product represented in the most simple manner which the case admits of. Thus if it be required to multiply

$6a^3b^4c$, which is $6aaabbbbc$

by $12a^2b^3c^3d$, which is $12aabbbcccd$,

the product is written thus: $6aaabbbbc\ 12aabbbcccd$, which multiplication may be performed in the following order: $6 \times 12aaaaabbbbbbbccccd$, which is represented by $72a^5b^7c^4d$. A few examples of this sort will establish the rule for the multiplication of such quantities which is usually given in the treatises on Algebra.

It is to be recollected that in every algebraical formula which is true of all numbers, any algebraical expression

may be substituted for one of the letters, provided care is taken to make the substitution wherever that letter occurs thus from the formula:

$$a^2 - b^2 = (a + b)(a - b)$$

we may deduce the following by making substitutions for a. If this formula be always true, it is true when a is equal to $p + q$, that is, it is true if $p + q$ be put instead of a wherever that letter occurs in the formula. Therefore,

$$\overline{p+q}^2 - b^2 = (p+q+b)(p+q-b).$$

Similarly, $\overline{b+m}^2 - b^2 = (2b+m)m$

$$\overline{x+y}^2 - \overline{x-y}^2 = \overline{x+y+x-y}\ \overline{x+y-(x-y)}$$

$$= 4xy, \text{ and so on.}$$

We have already established the formula,

$$(p - q)a = ap - aq.$$

Instead of a let us put $r - s$, and this formula becomes

$$(p - q)(r - s) = \overline{r-s}\,p - \overline{r-s}\,q.$$

But $\overline{r-s}\,p = pr - ps$, and $\overline{r-s}\,q = qr - qs$.

Therefore $\overline{p-q}\ \overline{r-s} = pr - ps - (qr - qs)$

$$= pr - ps - qr + qs$$

By reasoning in the same way we may prove that

$$\overline{p-q}\ \overline{r+s} = pr + ps - qr - qs.$$

A few examples of this sort will establish what is called the rule of signs in multiplication; viz. that a term of the multiplicand multiplied by a term of the multiplier has the sign + before it if the terms have the same sign, and − if they have different signs. But here the student must avoid using an incorrect mode of expression, which is very common, viz. the saying that + multiplied by + gives +; − multiplied by + gives −; and so on. He must recollect that the signs + and − are not quantities, but *directions* to add and subtract, and that, as has been well said by one of the most luminous writers on algebra in our language, we might as well say, that take away multiplied by take away gives add, as that − multiplied by − gives +*.

The only way in which the student should accustom himself to state this rule is the following: "In multiplying two algebraical expressions, multiply each term of the one by each term of the other, and wherever two terms are preceded by the same sign put + before the product of the two; when the signs are different put the sign − before their product."

If the student should meet with an equation in which positive and negative signs stand by themselves, such as

$$+ab \times -c = -abc,$$

let him, for the present, reject the example in which it occurs, and defer the consideration of such equations until he has read the explanation of them to which we shall soon come. Above all, he must reject the definition still sometimes given of the quantity $-a$, that it is less than nothing. It is astonishing that the human intellect should ever have tolerated such an absurdity as the idea of a quantity less than nothing; above all, that the notion should have outlived the belief in judicial astrology and the existence of witches, either of which is ten thousand times more possible.

These remarks do not apply to such an expression as $-b + a$, which we sometimes write instead of $a - b$, as long as it is recollected that the one is merely used to stand for the other, and for the present a must be considered as greater than b.

In writing algebraical expressions, we have seen that various arrangements may be adopted. Thus $ax^2 - bx + c$ may be written as $c + ax^2 - bx$, or $-bx + c + ax^2$. Of these three the first is generally chosen, because the highest power of x is written first; the highest but one comes next; and last of all the term which contains no power of x. When written in this way the expression is said to be arranged in

* Frend, Principles of Algebra. The author of this treatise is far from agreeing with the work which he has quoted in the rejection of the isolated negative sign which prevails throughout it, but fully concurs in what is there said of the methods then in use for explaining the difficulties of the negative sign.

descending powers of x; had it been written thus, $c - bx + ax^2$, it would have been arranged in ascending powers of x; in either case it is said to be arranged in powers of x, which is called the principal letter. It is usual to arrange all expressions which occur in the same question in powers of the same letter, and practice must dictate the most convenient arrangement. Time and trouble is saved by this operation, as will be evident from multiplying two unarranged expressions together, and afterwards doing the same with the same expressions properly arranged.

In multiplying two arranged expressions together, while collecting such terms into one as will admit of it, it will always be evident that the first and last of all the products contain powers of the principal letter which are found in no other part, and stand in the product unaltered by combination with any other terms, while in the intermediate products there are often two or more which contain the same power of the principal letter, and can be reduced into one. This will be evident in the following examples:

$$\begin{array}{ll}
\text{Multiply} \ldots & x^6 - 3x^5 + x^4 \\
\text{By} \ldots & \underline{x^4 - 2x^2 + x} \\
\text{The product is} . & x^{10} - 3x^9 + x^8 \\
& \qquad\qquad - 2x^8 + 6x^7 - 2x^6 \\
& \qquad\qquad\qquad + x^7 \quad - 3x^6 + x^5 \\
\hline
\text{Or} \ldots\ldots & x^{10} - 3x^9 - x^8 + 7x^7 - 5x^6 + x^5.
\end{array}$$

$$\begin{array}{ll}
\text{Multiply} \ldots & ax^3 + bx^2 + cx \\
\text{By} \ldots & \underline{dx^2 + ex + f} \\
\text{The product is} & adx^5 + bdx^4 + cdx^3 \\
& \qquad + aex^4 + bex^3 + cex^2 \\
& \qquad\qquad afx^3 + bfx^2 + cfx \\
\hline
\text{Or} \ldots\ldots & adx^5 + (bd+ae)x^4 + (cd+be+af)x^3 + (ce+bf)x^2 + cfx.
\end{array}$$

It is plain from the rule of multiplication, that the highest power of x in a product must be formed by multiplying the highest power in one factor by the highest power in the other, or when the two factors have been arranged in descending powers, the *first* power in one by the first power in the other. Also, that the lowest power of x, or should it so happen, the term in which there is no power of x, is made by multiplying the last terms in each factor. These being the highest and lowest, there can be no other such power, consequently neither of these terms can coalesce with any other, as is the case in the intermediate products. This remark will be of most convenient application in division, to which we now come.

Division is in all respects the reverse of multiplication. In dividing a by b we find the answer to this question;—if a be divided into b equal parts, what is the magnitude of each of those parts? The quotient is, from the definition of a fraction, the same as the fraction $\frac{a}{b}$, and all that remains is to see whether that fraction can be represented by a simple algebraical expression without fractions or not; just as in arithmetic the division of 200 by 26 is the reduction of the fraction $\frac{200}{26}$ to a whole number, if possible. But we must here observe that a distinction must be drawn between algebraical and arithmetical fractions. For example, $\frac{a+b}{a-b}$ is an algebraical fraction, that is, there is no expression without fractions which is always equal to $\frac{a+b}{a-b}$. But it does not follow from this that the number which $\frac{a+b}{a-b}$ represents, is always an arithmetical fraction; the contrary may be shown. Let a stand for 12, and b for 6, then $\frac{a+b}{a-b}$ is 3. Again, a^2+ab is a quantity which does not contain algebraical fractions, but it by no means follows that it may not represent an arithmetical fraction. To show that it may, let $a=\frac{1}{2}$ and $b=2$, then $a^2+ab=1\frac{1}{4}$ or $\frac{5}{4}$. Other examples will clear up this point if any doubt yet exist in the mind of the student. Nevertheless, the following propositions of arithmetic and algebra, which only differ in this, that "*whole number*" in the arithmetical proposition is replaced by

"*simple expression*" * in the algebraical one, connect the two subjects and render those demonstrations which are in arithmetic confined to whole numbers, equally true in algebra as far as regards simple expressions :—

The sum, difference, or product of two whole numbers, is a whole number.

One number is said to be a measure of another when the quotient of the two is a whole number.

The greatest common measure of two whole numbers is the greatest whole number which measures both, and is the product of all the prime numbers which will measure both.

When one number measures two others, it measures their sum, difference, and product.

In the division of one number by another, the remainder is measured by any number which measures the dividend and divisor.

A fraction is not altered by multiplying or dividing both its numerator and denominator by the same quantity.

The sum, difference, or product of two simple expressions is a simple expression.

One expression is said to be a measure of another when the quotient of the two is a simple expression.

The greatest common measure of two expressions is the common measure which has the highest exponents and coefficients, and is the product of all prime simple expressions which measure both.

When one expression measures two others, it measures their sum, difference, and product.

In the division of one expression by another, the remainder is measured by any expression which measures the dividend and divisor.

A fractional expression is not altered by multiplying or dividing both its numerator and denominator by the same expression.

In the term simple expression are included those quantities which contain arithmetical fractions, provided there is no algebraical quantity, or quantity represented by letters in the denominator; thus $\frac{1}{4}\, ab+\frac{1}{2}$ is called a simple expression. We now proceed to the division of one simple expression by another, and we will take first the case where neither quantity contains more than one term. For example, what is $42\, a^4 b^3 c$ divided by $6\, a^2\, bc$? that is, what quantity must be multiplied by $6\, a^2\, bc$, in order to produce $42\, a^4\, b^3\, c$. This last expression written at length, is $42\, aaaa\, bbb\, c$, and 42 is 6×7. We can then separate this expression into the product of two others, one of which shall be $6\, a^2\, bc$, or $6\, aa\, bc$; it will then be $6\, aa\, bc \times 7\, aa\, bb$, and it is $7\, aabb$ which must be multiplied by $6\, aa\, bc$ in order to produce $42\, a^4\, b^3\, c$. A few examples worked in this way, will lead the student to the rule usually given in all cases but one, to which we now come. We have represented cc, ccc, $cccc$, &c. by c^2, c^3, c^4, &c., and have called them the second, third, fourth, &c. powers of c. The extension of this rule would lead us to represent c by c^1, and call it the first power of c. Again, we have represented $c+c$, $c+c+c$, $c+c+c+c$, &c. by $2c$, $3c$, $4c$, and have called 2, 3, 4, &c. the coefficients of c. The extension of this rule would lead us to write c thus, $1c$, or, rather, if we attend to the last remark, $1c^1$. This instance leads us to observe the gradual progress of our language. We begin with the quantity c by itself; we proceed in our course, shortening by new signs the more complicated combinations of c, and the original quantity c forces itself anew upon our attention as a part of the series,

$$c,\ 2c,\ 3c,\ 4c, \text{ \&c., and}$$
$$c,\ c^2,\ c^3,\ c^4, \text{ \&c.}$$

in each of which, except the first, there is a distinct figure, which is called a coefficient or exponent, according to its situation. We then deduce rules in which the terms coefficient or exponent occur, but which, of course, cannot apply to the first term in each series, because, as yet, it has neither coefficient nor exponent. Among such rules are the following :—

To add two terms of the first series, add the coefficients, and affix to the sum the letter c. Thus $4c+3c=7c$.

To multiply two terms of the second series, add the exponents, and make this sum the exponent of c. Thus $c^4 \times c^3 = c^7$.

To divide a term of the second series by one which comes before it, subtract the exponent of the divisor from the exponent of the dividend, and make this difference the exponent of c. Thus, $\frac{c^7}{c^4} = c^3$.

* By a simple expression is meant one which does not contain the principal letter in the denominator of any fraction.

These rules are intelligible for all terms of the series except the first, to which, nevertheless, they will apply if we agree that $1c^1$ shall represent c, as will be evident by applying either of them to find $4c+c$, $c^4\times c$, or $\frac{c^4}{c}$. We therefore *agree* that $1c^1$ shall stand for c, and although c is not written thus, it must be remembered that c is to be considered as having the coefficient 1 and the exponent 1, which is an amendment and enlargement of our algebraical language, derived from experience. It may be said that this is all superfluous, because, if c^2 stand for cc, and c^3 for ccc, what can c^1 stand for but c? But it must be recollected that, since the symbol c^1 has not yet received a meaning, we are at liberty to make it stand for anything which we please, for example, for $\frac{1+c}{c}$, or $c-c^2$, or any other. If we did this, there would, indeed, be a great violation of analogy, that is, what c^1 stands for would not be as like that which c^2 has been made to stand for, as the meaning of c^3 is to that of c^4; but, nevertheless, we should not be led to any incorrect results as long as we remembered to make c^1 always stand for the same thing. These remarks are here introduced in order to show the manner in which analogy is followed in extending the language of algebra, and to prove that, after a certain period, we may rather be said to discover new symbols than to make them. The immense importance of this branch of the subject makes it necessary that it should be fully and early understood by all who intend to pursue their mathematical studies to any depth. To illustrate it still further, we subjoin another instance, which has not been noticed in its proper place.

The signs + and — were first used to connect one quantity with others, and to show what arithmetical operations were performed on other quantities by means of the first. But the first quantity on which we begin the operation is not preceded by any sign, not being considered as added or subtracted to any previous one. Rules were afterwards deduced for the addition and subtraction of the total result of several expressions in which these signs occur, as follows:

To add two expressions, form a third, which has all the quantities in the first two, with the same signs.

To subtract one expression from another, change the sign of each term of the subtrahend, and proceed as in the last rule.

The only terms in which these rules do not apply are those which have no sign, viz. the first of each. But they will apply to those terms, and will produce correct results, if we place the sign + before each of them. We are thus led to see that an algebraical term which has no sign is equivalent in all operations to one which is preceded by the sign +. We, therefore, consider this sign as prefixed, though it is not always written, and thus we are furnished with a method of containing under one rule that which would otherwise require two.

From these considerations the following appears to be the best and most natural course of proceeding in the invention of additional symbols. When a rule has been discovered which is not quite general, and which only fails in its application to a few instances, annex such additional symbols to those already in use, or change and modify these so as to make the rule applicable in all cases, provided always this can be done without making the same symbol stand for two different things, and without any violation of analogy. If the rule itself, by its application to any case, should produce a new symbol hitherto unexplained, it is a sign that the rule has been applied to a case which was never intended to fall under it when it was made. For the solution of this case we must have recourse to first principles, but when, by these means, the result has been found, it will be best to agree that the new symbol furnished by the rule shall stand for the result furnished by the principle, by which means the generality of the rule will be attained and the analogy of language will not be injured. Of this the following is a remarkable instance:—

To divide c^8 by c^5 the rule tells us to subtract 5 from 8, and make the result the exponent of c, which gives the quotient c^3. If we *apply the same rule* to divide c^6 by c^6, since 6 subtracted from 6 leaves 0, the result is c^0, a new symbol, to which we have attached no meaning. The fact is that the rule was formed from observation of different powers of c, and was never intended to apply to the division of a power of c by the same power. If we apply the common principles to the division of c^6 by c^6, the result is 1. We,

therefore, agree that c^0 shall stand for 1, and the least inspection will show that this agreement does not affect the truth of any result derived from the rule. If, in the solution of any problem, the symbol c^0 should appear, we must consider it is a sign that we have, in the course of the investigation, divided a power of c by itself by the common rule, without remarking that the quotient is 1. We must, therefore, replace c^0 by 1, but it is entirely indifferent at what stage of the process this is done.

Several extensions might be noticed, which are made almost intuitively, to which these observations will apply. Such, for example, is the multiplication and division of any number by 1, which is not contemplated in the definition of these operations. Such is also the continual use of 0 as a quantity, the addition and subtraction of it from other quantities, and the multiplication of it by others, neither of which were contemplated when these operations were first thought of.

We now proceed to the principles on which more complicated divisions are performed. The question proposed in division, and the manner of answering it, may be explained in the following manner. Let A be an expression which is to be divided by H, and let Q be the quotient of the two. By the meaning of division, if there be no remainder $A = QH$, since the quotient is the expression which must multiply the divisor, in order to produce the dividend. Now let the quotient be made up of different terms, a, b, c, &c., let it be $a + b - c + d$. That is, let

$$A = QH \tag{1}$$

$$Q = a + b - c + d \tag{2}$$

By putting, instead of Q in (1), that which is equal to it in (2), we find

$$A = (a + b - c + d)\,H = aH + bH - cH + dH \tag{3}$$

Now suppose that we can by any method find the term a of the quotient, that is, that we can by trial or otherwise find one term of the quotient. In (3), when the term a is found, since H is known, the term aH is found. Now if two quantities are equal, and from them we subtract the same quantity, the remainders will be equal. Subtract aH from the equal quantities A and $aH + bH - cH + dH$, and we shall find

$$A - aH = bH - cH + dH = (b - c + d)\,H. \tag{4}$$

If, then, we multiply the term of the quotient found by the divisor, and subtract the product from the dividend, and call the remainder B; then

$$B = (b - c + d)\,H. \tag{5}$$

That is, if B be made a dividend, and H still continue the divisor, the quotient is $b - c + d$, or all the first quotient, except the part of it which we have found. We then proceed in the same manner with this new dividend, that is, we find b and also bH, and subtract it from B, and let $B - bH$ be represented by C, which gives by the process which has just been explained

$$C = (-c + d)\,H = -cH + dH. \tag{6}$$

We now come to a negative term of the quotient. Let us suppose that we have found c, and that its sign in the quotient is $-$. If two quantities are equal, and we add the same quantity to both, the sums are equal. Let us therefore add cH to both the equal quantities in (6), and the equation will become

$$C + cH = dH; \tag{7}$$

or if we denote $C + cH$ by D, this is

$$D = dH.$$

There is only one term of the quotient remaining, and if that can be found the process is finished. But as we cannot know when we have come to the last term, we must continue the same process, that is, subtract dH from D, in doing which we shall find that dH is equal to D, or that the remainder is nothing. This indicates that the quotient is now exhausted and that the process is finished.

We will now apply this to an example in which the quotient is of the same form as that in the last process, namely, consisting of four terms, the third of which has the negative sign. This is the division of

$$x^4 - y^4 - 3x^2y^2 + x^3y + 2xy^3 \text{ by } x - y.$$

Arrange the first quantity in descending powers of x which will make it stand thus—

$$x^4 + x^3y - 3x^2y^2 + 2xy^3 - y^4 \quad \text{(A)}$$

One term of the quotient can be found immediately, for since it has been shewn that the term containing the highest power of x in a product, is made up of nothing but the product of the terms containing the highest powers of x which occur in the multiplier and multiplicand, and considering that the expression (A) is the product of $x - y$ and the quotient, we shall recover the highest power of x in the quotient by dividing x^4, the highest power of x in (A), by x, its highest

power in $x-y$. This division gives x^3 as the first term of the quotient. The following is the common process, and with each line is put the corresponding step of the process above explained, of which this is an example:—

		(H)	(A)	(a)
		$x-y)$	$x^4+x^3y-3x^2y^2+2xy^3-y^4$	$(x^3$
(a H)	Subtract		$\underline{x^4-x^3y}$	
				(b)
(B)	Second dividend . .		$2x^3y-3x^2y^2+2xy^3-y^4$	$(+2x^2y$
(b H)	Subtract . .		$\underline{2x^3y-2x^2y^2}$	
				(c)
(C)	Third dividend		$-x^2y^2+2xy^3-y^4$	$(-xy^2$
(c H)	Subtract		$\underline{-x^2y^2+xy^3}$	
				(d)
(D).	Fourth dividend		xy^3-y^4	$(+y^3$
(d H)	Subtract		$\underline{xy^3-y^4}$	
			0	

The whole quotient is therefore $x^3+2x^2y-xy^2+y^3$.

The second and following terms of the quotient are determined in exactly the same manner as the first. In fact this process is not the finding of a quotient directly from the divisor and dividend, but one term is first found, and by means of that term another dividend is obtained, which only differs from the first in having one term less in the quotient, viz. that which was first found. From this second dividend one term of its quotient is found, and so on until we obtain a dividend whose quotient has only one term, the finding of which finishes the process. It is usual also to neglect all the terms of the first dividend, except those which are immediately wanted, taking down the others one by one as they become necessary. This is a very good method in practice, but should be avoided in explaining the principle, since the first subtraction is made from the whole dividend, though the operation may only affect the form of some part of it.

If the student will now read attentively what has been said on the greatest common measure of two numbers, and then examine the connexion of whole numbers in arithmetic and simple expressions in algebra, with which we commenced the subject of division, he will see that the greatest algebraical common measure of two expressions may be found in exactly the same manner as the same operation is performed in arithmetic. He must also recollect that the greatest common measure of two expressions A and B is not altered by multiplying or dividing either of them, A for example, by any quantity, provided that quantity has no measure in common with B. For example, the greatest common measure of a^2-x^2 and ba^3-bx^3 is the same with that of $2a^2-2x^2$ and a^3-x^3, since though a new measure is now introduced into the first, and taken away from the second, nothing is introduced or taken away which is common to both. The same observation applies to arithmetic also. For example, take the numbers 162 and 180. We may, without altering their greatest common measure, multiply the first by 7 and the second by 11, &c. The rule for finding the greatest common measure should be practised with great attention hy all who intend to proceed beyond the usual stage in algebra. To others it is not of the same importance, as the necessity for it never occurs in the lower branches of the science.

In proceeding to the subject of fractions, it must be observed that, in the same manner as in arithmetic, when there is a remainder which cannot be further divided by the divisor, that is, where the dividend is so reduced that no simple term multiplied by the first term of the divisor will give the first term of the remainder, as in the case where the divisor is a^2x+bx^2 and the remainder $ax+b$; in this case a fraction must be added to the quotient, whose numerator is this remainder, and whose denominator is the divisor. Thus in dividing a^4+b^4 by $a+b$, the quotient is $a^3-a^2b+ab^2+b^3$, and the remainder $2b^4$, whence

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3+\frac{2b^4}{a+b}$$

The arithmetical rules for the addition, &c. of fractions hold equally good when the numerators and denominators are themselves fractions. Thus $\frac{\frac{3}{2}}{\frac{2}{7}}$ and $\frac{\frac{1}{3}}{\frac{3}{4}}$ are added, &c., exactly in the same way as $\frac{2}{5}$ and $\frac{3}{7}$, the sum of the second being

$$\frac{7 \times 2 + 5 \times 3}{5 \times 7}$$

and that of the first

$$\frac{\frac{3}{2} \times \frac{3}{4} + \frac{2}{7} \times \frac{1}{3}}{\frac{2}{7} \times \frac{3}{2}}$$

The rules for the addition, &c. of algebraic fractions are exactly the same as in arithmetic; for both the numerator and denominator of every algebraic fraction stands either for a whole number or a fraction, and therefore the fraction itself is either of the same form as $\frac{6}{7}$ or $\frac{\frac{2}{3}}{\frac{1}{3}}$. Nevertheless the student should attend to some examples of each operation upon algebraic fractions, by way of practice in the previous operations. As the subject is not one which presents any peculiar difficulties, we shall now pass on to the subject of equations, concluding this article with a list of formulæ which it is highly desirable that the student should commit to memory before proceeding to any other part of the subject.

$$(a + b) + (a - b) = 2a \qquad (1)$$

$$(a + b) - (a - b) = 2b \qquad (2)$$

$$a - (a - b) = b \qquad (3)$$

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad (4)$$

$$(a - b)^2 = a^2 - 2ab + b^2 \qquad (5)$$

$$(2ax + b)^2 = 4a^2x^2 + 4abx + b^2 \qquad (6)$$

$$(a + b)(a - b) = a^2 - b^2 \qquad (7)$$

$$\left.\begin{array}{l}(x + a)(x + b) = x^2 + (a + b)x + ab \\ (x - a)(x - b) = x^2 - (a + b)x + ab\end{array}\right\} \qquad (8)$$

$$\frac{a}{b} = \frac{ma}{mb} \qquad (9)$$

$$a + \frac{c}{d} = \frac{ad+c}{d} \quad a - \frac{c}{d} = \frac{ad - c}{d} \qquad (10)$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \qquad (11)$$

$$\frac{a}{b} \times c = \frac{ac}{b} = \frac{a}{\frac{b}{c}}, \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad (12)$$

$$\frac{\frac{a}{b}}{c} = \frac{a}{bc} = \frac{\frac{a}{c}}{b} \qquad (13)$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc} = \frac{\frac{a}{c}}{\frac{b}{d}} \qquad (14)$$

$$\frac{1}{\frac{a}{b}} = \frac{b}{a} \qquad (15)$$

Chapter VIII.

Equations of the First Degree.

We have already defined an equation, and have come to many equations of different sorts. But all of them had this character, that they did not depend upon the particular number which any letter stood for, but were equally true, whatever numbers might be put in place of the letters. For example, in the equation

$$\frac{a^2 - 1}{a + 1} = a - 1$$

the truth of the assertion made in this algebraical sentence is the same, whether a be considered as representing 1, 2, $2\frac{1}{2}$, &c., or any other number or fraction whatever. The second side of this equation is, in fact, the result of the operation pointed out on the first side. On the first side you are directed to

divide $a^2 - 1$ by $a + 1$; the second side shews you the result of that division. An equation of this description is called an *identical* equation, because, in fact, its two sides are but different ways of writing down the same number. This will be more clearly seen in the identical equations

$$a + a = 2a, \ 7a - 3a + b = 4a - 3b + 4b, \text{ and } \frac{a}{b} \times b = a.$$

The whole of the formulæ at the end of the last article are examples of identical equations. There is not one of them which is not true for all values which can be given to the letters which enter into them, provided only that whatever a letter stands for in one part of an equation, it stands for the same in all the other parts.

If we take, now, such an equation as $a+1=8$, we have an equation which is no longer true for every value which can be given to its algebraic quantities. It is evident that the only number which a can represent consistently with this equation is 7, as any other supposition involves absurdity. This is a new species of equation, which can only exist in some particular case, which particular case can be found from the equation itself. The solution of every problem leads to such an equation, as will be shown hereafter, and, in the elements of algebra, this latter species of equation is of most importance. In order to distinguish them from identical equations, they are called *equations of condition*, because they cannot be true when the letters contained in them stand for any number whatever, and their very existence makes a condition which the letters contained must fulfil. The solution of an equation of condition is the process of finding what number the letter must stand for in order that the equation may be true. Every such solution is a process of reasoning, which, setting out with supposing the truth of the equation, proceeds by self-evident steps, making use of the common rules of arithmetic and algebra. We shall return to the subject of the solution of equations of condition, after showing, in a few instances, how we come to them in the solution of problems. In equations of condition, the quantity whose value is determined by the equation is usually represented by one of the last letters of the alphabet, and all others by some of the first. This distinction is necessary only for the beginner; in time he must learn to drop it, and consider any letter as standing for a quantity known or unknown, according to the conditions of the problem.

In reducing problems to algebraical equations no general rule can be given. The problem is some property of a number expressed in words by which that number is to be found, and this property must be written down as an equation in the most convenient way. As examples of this, the reduction of the following problems into equations is given:—

I. What number is that to which, if 56 be added, the result will be 200 diminished by twice that number?

Let x stand for the number which is to be found.

Then $x+56=200-2x$.

If, instead of 56, 200, and 2, any other given numbers, a, b, and c, are made use of in the same manner, the equation which determines x is

$$x+a=b-cx.$$

II. Two couriers set out from the same place, the second of whom goes three miles an hour, and the first two. The first has been gone four hours, when the second is sent after him. How long will it be before he overtakes him?

Let x be the number of hours which the second must travel to overtake the first. At the time when this event takes place, the first has been gone $x+4$ hours, and will have travelled $(x+4)\,2$, or $2x+8$ miles. The second has been gone x hours, and will have travelled $3x$ miles. And, when the second overtakes the first, they have travelled exactly the same distance, and, therefore,

$$3x=2x+8.$$

If, instead of these numbers, the first goes a miles an hour, the second b, and c hours elapse before the second is sent after the first,

$$bx=ax+ac.$$

Four men, A, B, C, and D, built a ship which cost 2607*l.*, of which B paid twice as much as A, C paid as much as A and B, and D as much as C and B. What did each pay?

Suppose that A paid x pounds,
then B paid $2x$ . . .
C paid $x+2x$ or $3x$ . . .
D paid $2x+3x$ or $5x$ . . .

All together paid $x+2x+3x+5x$, or $11x$, therefore

$$11x=2607.$$

There are two cocks, from the first of which a cistern is filled in 12 hours, and the second in 15. How long would they be in filling it if both were opened together?

Let x be the number of hours which would elapse before it was filled. Then, since the first cock fills the cistern in 12 hours, in one hour it fills $\frac{1}{12}$ of it, in two hours $\frac{2}{12}$, &c., and in x hours $\frac{x}{12}$. Similarly, in x hours, the second cock fills $\frac{x}{15}$ of the cistern. When the two have exactly filled the cistern, the sum of these fractions must represent a whole or 1, and, therefore,

$$\frac{x}{12}+\frac{x}{15}=1.$$

If the times in which the two can fill the cistern are a and b hours, the equation becomes

$$\frac{x}{a}+\frac{x}{b}=1.$$

A person bought 8 yards of cloth for 3*l.* 2*s.*, giving 9*s.* a yard for some of it and 7*s.* a yard for the rest; how much of each sort did he buy?

Let x be the number of yards at 7*s.* Then $7x$ is the number of shillings they cost. Also $8-x$ is the number of yards at 9*s.*, and $\overline{8-x}\,9$, or $72-9x$, is the number of shillings they cost. And the sum of these, or $7x+72-9x$, is the whole price, which is 3*l.* 2*s.*, or 62 shillings, and, therefore,

$$7x+72-9x=62.$$

These examples will be sufficient to show the method of reducing a problem to an equation. Assuming a letter to stand for the unknown quantity, by means of this letter the same quantity must be found in two different forms, and these must be connected by the sign of equality. However the reduction into equations of such problems as are usually given in the treatises on algebra rarely occurs in the applications of mathematics. The process is a useful exercise of ingenuity, but no student need give a great deal of time to it. Above all, let no one suppose, because he finds himself unable to reduce to equations the conundrums with which such books are usually filled, that, therefore, he is not made for the study of mathematics, and should give it up. His future progress depends in no degree upon the facility with which he discovers the equations of problems; we mean as far as power of comprehending the subsequent sciences is concerned. He may never, perhaps, make any considerable step for himself, but, without doing this, he may derive all the benefits which the study of mathematics can afford, and even apply them extensively. There is nothing which discourages beginners more than the difficulty of reducing problems to equations, and yet, as respects its utility, if there be anything in the elements which may be dispensed with, it is this. We do not wish to depreciate its utility as an exercise for the mind, or to hinder all from attempting to conquer the difficulties which present themselves; but to remind every one that, if he can read and understand all that is set before him, the essential benefit derived from mathematical studies will be gained, even though he should never make one step for himself in the solution of any problem.

We return now to the solution of equations of condition. Of these there are various classes. Equations of the first degree, commonly called simple equations, are those which contain only the first power of the unknown quantity. Of this class are all the equations to which we have hitherto come in the solution of problems. The principle by which they are solved is, that two equal quantities may be increased or diminished, multiplied, or divided by any quantity, and the results will be the same. In algebraical language, if $a=b$ $a+c=b+c$, $a-c=b-c$ $ac=bc$ and $\frac{a}{c}=\frac{b}{c}$. In every elementary book it is stated that any quantity may be removed from one side of the equation to the other, provided its sign be changed. This is nothing but an application of the principle just stated, as may be shown thus:—Let $a+b-c=d$, add c to both quantities, then

$a+b-c+c=d+c$ or $a+b=d+c$.

Again subtract b from both quantities, then $a+b-c-b=d-b$, or $a-c=d-b$. Without always repeating the principle, it is derived from observation, that its effect is to remove quantities from one side of an equation to another, changing their sign at the same time. But the beginner should not use this rule until he is perfectly familiar with the manner of using the principle. He should, until he has mastered a good many examples, continue the operation at full length, instead of using the rule, which is an abridgment of it. In fact it would be

better, and not more prolix, to abandon the received phraseology, and in the example just cited, instead of saying "bring the term b to the other side of the equation," to say "subtract b from both sides," and instead of saying "bring c to the other side of the equation," to say "add c to both sides."

Suppose we have the fractions $\frac{3}{4}$, $\frac{1}{7}$, and $\frac{5}{14}$. If we multiply them all by the product of the denominators $4 \times 7 \times 14$, or 392, all the products will be whole numbers. They will be $\frac{3 \times 392}{4}$, $\frac{1 \times 392}{7}$, and $\frac{5 \times 392}{4}$, and since 392 is measured by 4, 3×392 is also measured by 4, and $\frac{3 \times 392}{4}$ is a whole number, and so on. But any common multiple of 4, 7, and 14 will serve as well. The least common multiple will therefore be the most convenient to use for this purpose. The least common multiple of 4, 7, and 14, is 28, and if the three fractions be multiplied by 28, the results will be whole numbers. The same also applies to algebraic fractions. Thus $\frac{a}{b}$, $\frac{c}{de}$, and $\frac{e}{bdf}$, will become simple expressions, if they are multiplied by $b \times de \times bdf$, or $b^2 d^2 ef$. But the most simple common multiple of b, de, and bdf, is $bdef$, which should be used in preference to $b^2 d^2 ef$.

This being premised, we can now reduce any equation which contains fractions to one which does not. For example, take the equation

$$\frac{x}{3} + \frac{2x}{5} = \frac{7}{10} - \frac{3-2x}{6}.$$

If we multiply both these equal quantities by any other, the results will be equal. We choose, then, the least quantity, which will convert all the fractions into simple quantities, that is, the least common multiple of the denominators 3, 5, 10, and 6, which is 30. If we multiply both equal quantities by 30, the equation becomes

$$\frac{30x}{3} + \frac{60x}{5} = \frac{210}{10} - \frac{30(3-2x)}{6}. \qquad (1)$$

But $\frac{30x}{3}$ is $\frac{30}{3} \times x$, or $10x$, $\frac{60x}{5}$ is $\frac{60}{5} \times x$, or $12x$, &c.; so that

$$\text{we have } 10x + 12x = 21 - 5(3-2x), \qquad (2)$$
$$\text{or} \quad 10x + 12x = 21 - (15 - 10x), \qquad (3)$$
$$\text{or} \quad 10x + 12x = 21 - 15 + 10x. \qquad (4)$$

Beginners very commonly mistake this process, and forget that the sign of subtraction, when it is written before a fraction, implies that the whole result of the fraction is to be subtracted from the rest. As long as the denominator remains, there is no need to signify this by putting the numerator between brackets, but when the denominator is taken away, unless this be done, the sign of subtraction belongs to the first term of the numerator only, and not to the whole expression. The way to avoid this mistake would be to place in brackets the numerators of all fractions which have the negative sign before them, and not to remove those brackets until the operation of subtraction has been performed, as is done in equation (4).

The following operations will afford exercise to the student, sufficient, perhaps, to enable him to avoid this error:—

$$a + \frac{b-c+d-e}{f} = \frac{af+b-c+d-e}{f}$$

$$a - \frac{b-c+d-e}{f} = \frac{af-b+c-d+e}{f}$$

$$a + b + \frac{\overline{a-b}|^2}{a+b} = \frac{2a^2+2b^2}{a+b}$$

$$a + b - \frac{\overline{a-b}|^2}{a+b} = \frac{4ab}{a+b}$$

We can now proceed with the solution of the equation. Taking up the equation (4) which we have deduced from it, subtract $10x$ from both sides, which gives $10x+12x-10x=21-15$, or $12x=6$: divide these equal quantities by 12, which gives $\frac{12x}{12} = \frac{6}{12}$, or $x=\frac{1}{2}$. This is the only value which x can have so as to make the given equation true, or, as it is called, to *satisfy* the equation. If, instead of x we substitute $\frac{1}{2}$, we shall find that—

$$\frac{\frac{1}{2}}{3} + \frac{2 \times \frac{1}{2}}{5} = \frac{7}{10} - \frac{3-2 \times \frac{1}{2}}{6}, \text{ or } \frac{1}{6} + \frac{1}{5} = \frac{7}{10} - \frac{2}{6};$$

this we find to be true, since $\frac{1}{6}+\frac{1}{5}$ is $\frac{11}{30}$, and $\frac{7}{10}-\frac{2}{6}=\frac{22}{60}$, and $\frac{11}{30}=\frac{22}{60}$. In these equations of the first degree there is one unknown quantity, and all the others are known. These known quantities may be represented by letters, and, as we have said, the first letters of the alphabet are commonly used for that purpose. We will now take an equation of exactly the same form as the last, putting letters in place of numbers:—

$$\frac{x}{a}+\frac{b\,x}{c}=\frac{d}{e}-\frac{f-g\,x}{h}.$$

The solution of this equation is as follows: multiply both quantities hy $aceh$, the most simple multiple of the denominators, it then becomes,—

$$\frac{acehx}{a}+\frac{abcehx}{c}=\frac{acdeh}{e}-\frac{aceh\,(f-gx)}{h},$$

$$\text{or, } cehx+abehx=acdh-ace\,(f-gx),$$

$$\text{or, } cehx+abehx=acdh-acef+acegx.$$

Subtract $acegx$ from both sides, and it becomes

$$cehx+abehx-acegx=acdh-acef$$

$$\text{or } \overline{ceh+abeh-aceg}\,x=acdh-acef.$$

Divide both sides by $ceh+abeh-aceg$, which gives

$$x=\frac{acdh-acef}{ceh+abeh-aceg}.$$

The steps of the process in the second case are exactly the same as in the first; the same reasoning establishes them both, and the same errors are to be avoided in each. If from this we wish to find the solution of the equation first given, we must substitute 3 for a, 2 for b, 5 for c, 7 for d, 10 for e, 3 for f, 2 for g, and 6 for h, which gives for the value of x,

$$\frac{3\times5\times7\times6-3\times5\times10\times3}{5\times10\times6+3\times2\times10\times6-3\times5\times10\times2},\text{ or }\frac{3\times5\times12}{3\times2\times10\times6},\text{ or }\frac{180}{360},$$

which is $\frac{1}{2}$, the same as before.

If in one equation there are two unknown quantities, the condition is not sufficient to fix the values of the two quantities; it connects them, nevertheless, so that if one can be found the other can be found also. For example, the equation $x+y=8$ admits of an infinite number of solutions, for take x to represent any whole number or fraction less than 8, and let y represent what x wants of 8, and this equation is satisfied. If we have another equation of condition existing between the same quantities, for example, $3\,x-2\,y=4$; this second equation by itself has an infinite number of solutions: to find them, y may be taken at pleasure, and $x=\frac{4+2\,y}{3}$. Of all the solutions of the second equation, one only is a solution of the first; thus there is only one value of x and y which satisfies both the equations, and the finding of these values is the solution of the equation. But there are some particular cases in which every value of x and y which satisfies one of the equations satisfies the other also; this happens whenever one of the equations can be deduced from the other. For example, when $x+y=8$, and $4\,x-29=3-4\,y$, the second of these is the same, as $4\,x+4\,y=3+29$, or $4\,x+4\,y=32$, which necessarily follows from the first equation.

If the solution of a problem should lead to two equations of this sort, it is a sign that the problem admits of an infinite number of solutions, or is what is called an indeterminate problem. The solution of equations of the second degree does not contain any peculiar difficulty; we shall therefore proceed to the consideration of the isolated negative sign.

Chapter IX.

On the Negative Sign, &c.

If we wish to say that 8 is greater than 5 by the number 3, we write this equation $8-5=3$. Also to say that a exceeds b by c, we use the equation $a-b=c$. As long as some numbers whose value we know are subtracted from others equally known, there is no

fear of our attempting to subtract the greater from the less; of our writing $3-8$ for example, instead of $8-3$. But in prosecuting investigations in which letters occur, we are liable, sometimes from inattention, sometimes from ignorance as to which is the greater of two quantities, or from misconception of some of the conditions of a problem, to reverse the quantities in a subtraction, for example to write $a-b$ where b is the greater of two quantities, instead of $b-a$. Had we done this with the sum of two quantities, it would have made no difference, because $a+b$ and $b+a$ are the same, but this is not the case with $a-b$ and $b-a$. For example, $8-3$ is easily understood; 3 can be taken from 8 and the remainder is 5; but $3-8$ is an impossibility, it requires you to take from 3 more than there is in 3, which is absurd. If such an expression as $3-8$ should be the answer to a problem, it would denote either that there was some absurdity inherent in the problem itself, or in the manner of putting it into an equation. Nevertheless, as such answers will occur, the student must be aware what sort of mistakes give rise to them, and in what manner they affect the process of investigation.

We would recommend to the beginner to make experience his only guide in forming his notions of these quantities, that is, to draw his rules from the observation of many results, not from any theory. The difficulties which encompass the theory of the negative sign are explained at best in a manner which would embarrass him: probably he would not see the difficulties themselves; too easy belief has always been the fault of young students in mathematics, and it is a great point gained to get them to start an objection. We shall observe the effect of this error in denoting a subtraction on every species of investigation to which we have hitherto come, and shall deduce rules which the student will recollect are the results of experience, not of abstract reasoning. The extensions to which he will be led have rendered Algebra much more general than it was before, have made it competent to the solution of many, very many questions which it could not have touched, had they not been attended to. They do, in fact, constitute part of the groundwork of modern Algebra, and should be considered by the student who is desirous of making his way into the depths of the science with the highest degree of attention. If he is well practised in the ordinary rules which have hitherto been explained, few difficulties can afterwards embarrass him, except those which arise from some confusion in the notions which he has formed upon this part of the subject.

For brevity's sake we hereafter use this phrase. Where the signs of every term in an expression are changed, it is said to have changed its form. Thus $+a-b$ and $+b-a$ are in different forms, and if a be greater than b, the first is the correct form, and the second incorrect. An extension of a rule is made, by which such a quantity as $3-8$ is written in a different way. Suppose that $+3-8$ is connected with any other number thus, $56+3-8$. This may be written $56+3-(3+5)$, or $56+3-3-5$, or $56-5$. It appears, then, that $+3-8$, connected with any number is the same as -5 connected with that number; from this we say that $+3-8$, or $3-8$ is the same thing as -5, or $3-8=-5$. This is another way of writing the equation $8-3=5$, and indicates equally that 8 is greater than 5 by 3. In the same way, $a-b=-c$ indicates that b is greater than a by the quantity c. If a be nothing, this equation becomes $-b=-c$, which indicates that $b=c$, since if the equation $a-b=-c$ be written in its true form $b-a=c$, and if $a=0$, then $b=c$. We can now understand the following equations:—

$$a-b+c-d=-e$$
$$\text{or } b+d-a-c=e$$
$$2ab-a^2-b^2=-d-e$$
$$\text{or } a^2+b^2-2ab=d+e.$$

We must not commence any operations upon such an equation as $a-b=-c$, until we have satisfied ourselves of the manner in which they should be performed, by reference to the correct form of the equation. This correct form is $b-a=c$. This gives $d+b-a=d+c$, or $d-(a-b)=d+c$. Write instead of $a-b$ its symbol $-c$, and then $d-(-c)=d+c$. Here we have performed an operation with $a-b$, which is no quantity, since a is less than b, but this is done because our present object is, in applying the common rules to such expressions, to watch the results, and exhibit them in their real forms. The first side $a-(-c)$, is in a form in which we can attach no meaning to it, and the second side gives its real form $d+c$,

The meaning of this expression is, that if with $a - b$, which we think to be a quantity, but which is not, since a is less than b, we follow the algebraical rule in subtracting $a - b$ from d, we shall thereby get the same result as if we had added the real quantity $b - a$ to d. If we make use of the form $d - (-c)$, it is because we can use it in such a manner as never to lose sight of its connexion with its real form $d + c$, and because we can establish rules which will lead us to the end of a process without any error, except those which we can correct as certainly at the end as at the beginning.

The rule by which we proceed, and which we shall establish by numerous examples, is, that wherever two like signs come together, the corresponding part of the real form has a positive sign, and wherever two unlike signs come together, the real form has a negative sign. Thus the real form of $d - (-c)$ is $d + c$. Again, take the real form $b - a = c$ of the equation $a - b = -c$, and it follows that $d - (b - a) = d - c$, or $d - b + a = d - c$, or $d + a - b = d - c$, or $d + (a - b) = d - c$. This is $d + (-c) = d - c$, another case in which the rule is verified. Again, multiply together $a - b$ and $m - n$, the product is $am - an - bm + bn$. This is the same product as arises from multiplying $b - a$ by $n - m$, written in a different order. If, then, $b - a = c$, and $n - m = p$, or $a - b = -c$, and $m - n = -p$, we find that $(-c) \times (-p) = cp$. By which result we mean that a mistake, in the form of both $a - b$ and $m - n$, will not produce a mistake in the form of their product, which remains what it would have been had the mistake not been made. Again

$$(n-m)(b-a) = bn - bm - an + am$$
$$(n-m)(a-b) = an - am - bn + bm$$

If the first product be real and equal to P, the second is represented by $-$P. The first is cp, the second is $(-c)\times p$, which gives

$$(-c)\times p = -cp.$$

That is, a mistake in the form of one factor only alters the form of the product. To distinguish the right form from the wrong one, we may prefix $+$ to the first, and $-$ to the second, and we may then recapitulate the results, and add others, which the student will now be able to verify.

The sign $+$ placed before single quantities shows that the form of the quantity is correct; the sign $-$ shows that it has been mistaken or changed.

$$a + (+b) = a + b \qquad a + (-b) = a - b$$
$$a - (+b) = a - b \qquad a - (-b) = a + b$$
$$(+a)\times(+b) = +ab \qquad (+a)\times(-b) = -ab$$
$$(-a)\times(-b) = +ab = (+a)\times(+b)$$

$$\frac{+a}{+b} = +\frac{a}{b}$$
$$\frac{+a}{-b} = -\frac{a}{b} = \frac{-a}{+b}$$
$$\frac{-a}{-b} = +\frac{a}{b}$$

$$-a \times -a \qquad = \qquad +a$$
$$-a \times -a \times -a \qquad = +a^2 \times -a \qquad = -a^3$$
$$-a \times -a \times -a \times -a = -a^3 \times -a \qquad = +a^4$$

&c. &c.

We see, then, that a change in the form of any quantity changes the form of those powers whose exponent is an odd number, but not of those whose exponent is an even number. By these rules we shall be able to tell what changes would be made in an expression by altering the forms of any of its letters. It may be fairly asked whether we are not changing the meaning of the signs $+$ and $-$, in making $+a$ stand for an expression in which we do not alter the signs, and $-a$ for one in which the signs are altered. The change is only in name, for since the rule of addition is, "annex the expressions which are to be added without altering the signs of either," or "annex the expressions with-

out altering the form of either;" the quantity $a+b$, which is the sum of the two expressions a and b, stands for the same as $+a+b$, in which the new notion of the sign $+$ is used, viz. the expressions a and b are annexed with unaltered forms, which is denoted by writing together $+a$ and $+b$. Again, the rule for subtraction is, "change the sign of the subtrahend or expression which is to be subtracted, and annex the result to the other expression," or "change the form of the subtrahend and annex it to the other," which, the expressions being a and b, is written $a-b$, which answers equally well to the second notion of the sign $-$, since $+a-b$ indicates that a and b are to be annexed, the first without, the second with a change of form. These ideas of the signs $+$ and $-$ give, therefore, in practice, the same results as the former ones, and, in future, the two meanings may be used indiscriminately. But when a single term is used, such as $+a$ or $-a$, the last acquired notions of $+$ and $-$ are always understood.

This much being premised, we can see, by numberless instances, that, if the form of a quantity is to be changed, it matters nothing whether it is changed at the beginning of the process, or whether we wait till the end, and then follow the rules abovementioned. This is evident to the more advanced student, from the nature of the rules themselves, but the beginner should satisfy himself of this fact from experience. We now give a proof of this, as far as one expression can prove it, in the solution of the equations,—

$$\frac{a^2}{b}+ax=\frac{a^2x}{b}+a-b$$

$$\text{and } \frac{a^2}{b}-ax=\frac{a^2x}{b}-a-b$$

which two equations only differ in the form in which a appears. For, if the form of a in the first equation be altered, that of $\frac{a^2}{b}$ and $\frac{a^2x}{b}$ is unaltered, $+ax$ becomes $-ax$, and $+a$ becomes $-a$. We now solve the two equations in opposite columns.

$$\frac{a^2}{b}+ax=\frac{a^2x}{b}+a-b$$

$$a^2+abx=a^2x+ab-b^2$$

$$a^2-ab+b^2=a^2x-abx$$

$$=(a^2-ab)x$$

$$x=\frac{a^2-ab+b^2}{a^2-ab}$$

$$\frac{a^2}{b}-ax=\frac{a^2x}{b}-a-b$$

$$a^2-abx=a^2x-ab-b^2$$

$$a^2+ab+b^2=a^2x+abx$$

$$=(a^2+ab)x$$

$$x=\frac{a^2+ab+b^2}{a^2+ab}$$

The only difference between these expressions arises from the different form of a in the two. If, in either of them, $-a$ be put instead of $+a$, and the rules laid down be followed, the other will be produced. We see, then, that a simple alteration of the form of a in the original equation produces no other change in the result, or in any one of the steps which lead to that result, except a simple alteration in the form of a. From this it follows that, having the solution of an equation, we have also the solution of all the equations which can be formed from it, by altering the form of the different known quantities which are contained in it. And, as all problems can be reduced to equations, the solution of one problem will lead us to the solution of others, which differ from the first in producing equations in which some of the known quantities are in different forms. Also, in every identical eqnation, the form of one or more of its quantities may be altered throughout, and the equation will still remain identically true. For example,

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2$$

Change $+b$ into $-b$, and this equation will become

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2,$$

which last, common division will show to be true.

Again, suppose that when a, b, and c are in a given form, which we denote by $+a$, $+b$, and $+c$, the solution of a problem is,

$$x=\frac{b^2-4ac}{a+c-b}.$$

The following table will show the alterations which take place in x when the forms of a, b, and c are changed in different manners, and the verification of it will be an exercise for the student.

Forms of a, b, and c.	Values of x.
$+a, +b, +c$	$\frac{b^2-4ac}{a+c-b}$
$+a, +b, -c$	$\frac{b^2+4ac}{a-c-b}$
$+a, -b, -c$	$\frac{b^2+4ac}{a-c+b}$
$-a, +b, -c$	$-\frac{b^2-4ac}{b+a+c}$
$-a, -b, -c$	$\frac{b^2-4ac}{b-a-c}$

Also, the expression for x may be written in the following different ways, the forms of a, b, and c remaining the same.

$$\frac{b^2-4ac}{a+c-b} - \frac{b^2-4ac}{b-a-c} - \frac{4ac-b^2}{a+c-b} \quad \frac{4ac-b^2}{b-a-c}.$$

We now proceed to apply these principles to the solution of the following problems:—

Two couriers, A and B, in the course of a journey between the towns C and D, are at the same moment of time at A and B. A goes m miles, and B, n miles an hour. At what point between C and D are they together? It is evident that the answer depends upon whether they are going in the same or opposite directions, whether A goes faster or slower than B, and so on. But all these, as we shall see, are included in the same general problem, the difference between them corresponding to the different forms of the letters which we shall have occasion to use. After solving the different cases which present themselves, each upon its own principle, we shall compare the results in order to establish their connexion. Let the distance AB be called a.

Case first.—Suppose that they are going in the same direction from C to D, and that m is greater than n. They will then meet at some point between B and D. Let that point be H, and let AH be called x. Then A travels through AH, or x, in the time during which B travels through BH or $x-a$. But, since A goes m miles an hour, he travels the distance x in $\frac{x}{m}$ hours. Again, B travels the distance $x-a$ in $\frac{x-a}{n}$ hours. These times are the same, and, therefore,

$$\frac{x}{m} = \frac{x-a}{n} \text{ or } x = \frac{ma}{m-n} = \text{AH}$$

$$\text{and } x-a = \frac{na}{m-n} = \text{BH}$$

The time which elapses before they meet is $\frac{x}{m}$ or $\frac{a}{m-n}$.

Case second.—Suppose them now moving in the same direction as before, but let B move faster than A. They never will meet after they come to A and B, since B is continually gaining upon A, but they must have met at some point before reaching A and B. Let that point be H, and, as before, let AH $= x$.

Then since A travels through H A or x in the time during which B travels through H B, or $x+a$, in the same manner as in the last ease, we show that

$$\frac{x}{m} = \frac{x+a}{n} \text{ or } x = \frac{ma}{n-m} = \text{AH}$$

$$\text{and } x + a = \frac{na}{n-m} = \text{BH}$$

The time elapsed is . . $\frac{a}{n-m}$

Case third.—If they are moving from D to C, and if B moves faster than A, the point H is the same as in the last case, since, if having in the last ease arrived at A and B, they move back again at the same rate, they will both arrive at the point H together. The answers in this case are therefore the same as in the last.

Case fourth.—Similarly, if they are moving from D to C, and A moves faster than B, the answers are the same as in the first case, since this is a reverse of the first case, as the third is of the second. We reserve for the present the case in which they move equally fast, as another species of difficulty is involved which has no connexion with the present subject. We shall return to it hereafter.

Case fifth. — Suppose them now moving in contrary directions, viz.: A towards D and B towards C. Whether A moves faster or slower than B, they must now meet somewhere between A and B, as before let them meet in H, and let A H = x.

Then A moves through A H, or x, in the same time as B moves through B H, or $a-x$. Therefore $\frac{x}{m} = \frac{a-x}{n}$, or

$$x = \frac{ma}{m+n}$$

$$a-x = \frac{na}{m+n}$$

The time elapsed is $= \frac{a}{m+n}$

Case sixth.—Let them be moving in contrary direetions, but let A be moving towards C, and B towards D. They will then have met somewhere between A and B, and as this is only the reverse of the last case, just as the fourth is of the first, or the third of the second, the answers are the same. We now exhibit the results of these different cases in a table, stating the circumstances of each ease, and also whether the time of meeting is before or after the instant which finds them at A and B.

Circumstances of the Case.	Direction of the point H.	Value of A H.	Value of B H.	Time of meeting.
1. Both move from C to D, A moves faster than B.	Between B and D.	$\frac{ma}{m-n}$	$\frac{na}{m-n}$	$\frac{a}{m-n}$ after
2. Both move from C to D, A moves slower than B.	Between A and C.	$\frac{ma}{n-m}$	$\frac{na}{n-m}$	$\frac{a}{n-m}$ before.
3. Both move from D to C, A moves slower than B.	Between A and C.	$\frac{ma}{n-m}$	$\frac{na}{n-m}$	$\frac{a}{n-m}$ after.
4. Both move from D to C, A moves faster than B.	Between B and D.	$\frac{ma}{m-n}$	$\frac{na}{m-n}$	$\frac{a}{m-n}$ before.
5. A moves towards D and B towards C.	Between A and B.	$\frac{ma}{m+n}$	$\frac{na}{m+n}$	$\frac{a}{m+n}$ after.
6. A moves towards C and B towards D.	Between A and B.	$\frac{ma}{m+n}$	$\frac{na}{m+n}$	$\frac{a}{m+n}$ before.

Now $\frac{a}{m-n}$ and $\frac{a}{n-m}$ are the same quantity written in different forms, for $n-m$ is $-(m-n)$; and according to the rules $\frac{a}{n-m}$ is $-\frac{a}{m-n}$. Similarly $\frac{ma}{n-m} = -\frac{ma}{m-n}$, and so on. We see also, that in the first and second cases, which differ in this, that A H falls to the right in the first, and to the left in the second, the forms of AH are differ-

ent, there being $\frac{ma}{m-n}$ in the first, and $-\frac{ma}{m-n}$ in the second. Again, in the same cases, in the first of which the time of meeting is *after*, and in the second *before* the moment of being at A and B, we see a difference of form in the value of that time; in the first it is $\frac{a}{m-n}$, and in the second $-\frac{a}{m-n}$, or $\frac{a}{n-m}$. The same remarks apply to the third and fourth examples. Again, in the first and fifth cases, which only differ in this, that B is moving towards D in the first, and in the contrary direction towards C in the fifth, the values of A H, and of the time, may be deduced from the first by changing the form of n, and writing $+n$, instead of $-n$. The expression for B H in the first, if the form of n be likewise changed, becomes $-\frac{na}{m+n}$, which is the valne of B H in the fifth, but in a different form. But we observe, that B H falls to the left of B in the fifth, whereas it fell to the right in the first. Again, in the first and sixth examples, which differ in this, that A moves towards D in the second and towards C in the sixth, the value of A H in the sixth may be deduced from that of A H in the first, by changing the form of m, which change makes A H become $\frac{-ma}{-m-n}$, or $\frac{-ma}{-(m+n)}$, or $\frac{ma}{m+n}$. If we alter the value of the time in the first, in the same manner, it becomes $\frac{a}{-m-n}$, or $-\frac{a}{m+n}$, which is of a different form from that in the sixth; but it must also be observed, that the first is after and the other before the moment when they are at A and B. In the fifth and sixth examples which differ in this, that the direction in which both are going is changed, since in the fifth they move towards one another, and in the sixth away from one another, the values of A H and B H in the one may be deduced from those in the other by a change of form, both in m and n, which gives the same values as before. But if m and n change their forms in the expression for the time, the value in the sixth case is $\frac{a}{-m-n}$, or $-\frac{a}{m+n}$. Also the time in the fifth case is after the moment at which they are at A and B, and in the sixth case it is before. From these comparisons we deduce the following general conclusions:—

1. If we take the first case as a standard, we may, from the values which it gives, deduce those which hold good in all the other cases. If a second case be taken, and it is required to deduce answers to the second case from those of the first, this is done by changing the sign of all those quantities whose directions are opposite in the second case to what they are in the first, and if any answer should appear in a negative form, such as $\frac{ma}{m-n}$, when m is less than n, which may be written thus $-\frac{ma}{n-m}$, it is a sign that the quantity which it represents is different in direction in the first and second cases. If it be a right line measured from a given point in all the cases, such as A H, it is a sign that A H falls on the left in the second case, if it fell on the right in the first case, and the converse. If it be the time elapsed between the moment in which the couriers are at A and B and their meeting, it is a sign that the moment of meeting is before the other, in the second case, if it were after it in the first, and the converse. We see, then, that these six cases can be all contained in one if we apply this rule, and it is indifferent which of the cases is taken as the standard, provided the corresponding alterations are made to determine answers to the rest.

This detail has been entered into, in order that the student may establish, from his own experience, the general principle, which will conclude this part of the subject. Further illustration is contained in the following problem:—

A workman receives a shillings a day for his labour, or a proportion of a shillings for any part of a day which he works. His expenses are b shillings every day, whether he works or no, and after m days he finds that he has gained c shillings. How many days did he work? Let x be that number of days, x being either whole or fractional, then for his work he receives ax shillings, and during the m days his expenditure is

bm shillings, and since his gain is the difference between his receipts and expenditure;—

$$ax - bm = c$$
$$\text{or } x = \frac{bm + c}{a}$$

Now suppose that he had worked so little as to lose c shillings instead of gaining anything. The equation from which x is derived is now

$$bm - ax = c$$

which, when its form is changed, becomes

$$ax - bm = -c,$$

an equation which only differs from the former in having $-c$ written instead of c. The solution of the equation is $x = \frac{bm - c}{a}$ which only differs from the former in having $-c$ instead of $+c$. It appears then that we may alter the solution of a problem which proceeds upon the supposition of a gain into the solution of one which supposes an equal loss, by changing the form of the expression which represents that gain; and also that if the answer to a problem which we have solved upon the supposition of a gain should happen to be negative, suppose it $-c$, we should have proceeded upon the supposition that there is a loss and should in that case have found a loss, c. When such principles as these have been established we have no occasion to correct an erroneous solution by recommencing the whole process, but we may, by means of the form of the answer, set the matter right at the end. The principle is, that a negative solution indicates that the nature of the answer is the very reverse of that which it was supposed to be in the solution; for example, if the solution supposes a line measured in feet in one direction, a negative answer, such as $-c$, indicates that c feet must be measured in the opposite direction; if the answer was thought to be a number of days *after* a certain epoch, the solution shows that it is c days before that epoch; if we supposed that A was to receive a certain number of pounds, it denotes that he is to pay c pounds, and so on. In deducing this principle we have not made any supposition as to what $-c$ is; we have not asserted that it indicates the subtraction of c from 0; we have derived the result from observation only, which taught us first to deduce rules for making that alteration in the result which arises from altering $+c$ into $-c$ at the commencement; and secondly, how to make the solution of one case of a problem serve to determine those of all the others. By observation then the student must acquire his conviction of the truth of these rules, reserving all metaphysical discussion upon such quantities as $+c$ and $-c$ to a later stage, when he will be better prepared to understand the difficulties of the subject. We now proceed to another class of difficulties, which are generally, if possible, as much misconceived by the beginner as the use of the negative sign.

Take any fraction $\frac{a}{b}$. Suppose its numerator to remain the same, but its denominator to decrease, by which means the fraction itself is increased. For example, $\frac{5}{12}$ is greater than $\frac{5}{20}$ or the twelfth part of 5 is greater than its twentieth part. Similarly $\frac{2\frac{1}{3}}{4\frac{1}{8}}$ is greater than $\frac{2\frac{1}{3}}{\frac{27}{2}}$, &c. If, then, b be diminished more and more, the fraction $\frac{a}{b}$ becomes greater and greater, and there is no limit to its possible increase. To show this, suppose that b is a part of a, or that $b = \frac{a}{m}$. Then $\frac{a}{b}$ or $\frac{a}{\frac{a}{m}}$ is m. Now since b may diminish so as to be equal to any part of a, however small, that is, so as to make m any number, however great, $\frac{a}{b}$ which is $= m$ may be any number however great. This diminution of b, and the consequent increase of $\frac{a}{b}$, may be carried on to any extent, which we may state in these words: As the quantity b becomes nearer and nearer to 0, the fraction $\frac{a}{b}$ increases, and in the interval in which b passes from its first magnitude to 0, the fraction $\frac{a}{b}$ passes from its first value through every possible greater number. Now, suppose that the solution of a problem in its most general form is $\frac{a}{b}$, but that in one particular case of that problem b is $=0$. We have, then instead of a solution, $\frac{a}{0}$, a symbol to which we have not hitherto given

a meaning. To take an instance: return to the problem of the two couriers, and suppose that they move in the same direction from C to D (Case first) at the same rate, or that $m = n$. We find that A H $= \frac{ma}{m-n}$ or $\frac{ma}{m-n}$ or $\frac{ma}{0}$. On looking at the equation which produced this result we find that it becomes $\frac{x}{m} = \frac{x-a}{m}$, or $x = x - a$, which is impossible. On looking at the manner in which this equation was formed, we find that it was made on the supposition that A and B are together at some point, which in this case is also impossible, since if they move at the same rate, the same distance which separated them at one moment will separate them at any other, and they will never be together, nor will they ever have been together on the other side of A. The conclusion to be drawn is, that such an equation as $x = \frac{a}{0}$ indicates that the supposition from which x was deduced can never hold good. Nevertheless in the common language of algebra it is said that they meet at an infinite distance, and that $\frac{a}{0}$ is infinite. This phrase is one which in its literal meaning is an absurdity, since there is no such thing as an infinite number, that is a number which is greater than any other, because the mind can set no bounds to the magnitude of the numbers which it can conceive, and whatever number it can imagine, however great, it can imagine the next to it. But as the use of the phrase is very general, the only method is to attach a meaning which shall not involve absurdity or confusion of ideas. The phrase used is this. When $c = b$ $\frac{a}{c-b} = \frac{a}{0}$ and is infinitely great. The student should always recollect that this is an abbreviation of the following sentence. " The fraction $\frac{a}{c-b}$ becomes greater and greater as c approaches more and more near to b; and if c, setting out from a certain value, should change gradually until it becomes equal to b, the fraction $\frac{a}{c-b}$ setting out also from a certain value, will attain any magnitude however great, before c becomes equal to b." That is, before a fraction can assume the form $\frac{a}{0}$, it must increase without limit. The symbol ∞ is used to denote such a fraction, or in general any quantity which increases without limit. The following equation will tend to elucidate the use of this symbol. In the problem of the two couriers, the equation which gave the result $\frac{ma}{0}$ was $\frac{x}{m} = \frac{x-a}{m}$, or $x = x - a$, which is evidently impossible. Nevertheless, the larger x is taken the more near is this equation to the truth, as may be proved by dividing both sides by x, when it becomes $1 = 1 - \frac{a}{x}$, which is never exactly true. But the fraction $\frac{a}{x}$ decreases as x increases, and by taking x sufficiently great may be reduced to any degree of smallness. For example, if it is required that $\frac{a}{x}$ should be as small as $\frac{1}{10000000}$ of a unit, take x as great as $10000000a$, and the fraction becomes $\frac{a}{10000000a}$, or $\frac{1}{10000000}$. But as $\frac{a}{x}$ becomes smaller and smaller, the equation $1 = 1 - \frac{a}{x}$ becomes nearer and nearer the truth, which is expressed by saying that when $1 = 1 - \frac{a}{x}$, or $x = x - a$, the solution is $x = \infty$. In the solution of the problem of the two couriers this does not appear to hold good, since when $m = n$ and $x = \frac{ma}{0}$ the same distance a always separates them, and no travelling will bring them nearer together. To show what is meant by saying that the greater x is, the nearer will it be a solution of the problem, suppose them to have travelled at the same rate to a great dis-

tance from C. They can never come together unless C A becomes equal to CB, or A coincides with B, which never happens, since the distance A B is always the same. But if we suppose that they have met, though an error always will arise from this false supposition, it will become less and less as they travel further and further from C. For example, let C A = 10000000 A B, then the supposing that they have met, or that B and A coincide, or that BA = 0, is an error which involves no more than $\frac{1}{10000000}$ of A C; and though A B is always of the same numerical magnitude, it grows smaller and smaller in comparison with A C, as the latter grows greater and greater.

Let us suppose now that in the problem of the two couriers they move in the same direction at the same rate, as in the case we have just considered, but that moreover they set out from the same point, that is, let $a = 0$. It is now evident that they will always be together, that is, that any value of x whatever is an answer to the question. On looking at the value of A H, or $\frac{ma}{m-n}$, we find the numerator and denominator both equal to 0, and the value of AH appears in the form $\frac{0}{0}$. But from the problem we have found that one value cannot be assigned to A H, since every point of their course is a point where they are together. The solution of the following equation will further elucidate this. Let

$$ax + by = c$$
$$dx + ey = f$$

from which, by the common method of solution, we find

$$x = \frac{ce - bf}{ae - bd},\ y = \frac{af - cd}{ae - bd}.$$

Now, let us suppose that $ce = bf$ and $ae = bd$. Dividing the first of these by the second, we find $\frac{ce}{ae} = \frac{bf}{bd}$, or $\frac{c}{a} = \frac{f}{d}$ or $cd = af$. The values both of x and y in this case assume the form $\frac{0}{0}$; to find the cause of this we must return to the equations. If we divide the first of these by c, and the second by f, we find that

$$\frac{a}{c}x + \frac{b}{c}y = 1$$
$$\frac{d}{f}x + \frac{e}{f}y = 1.$$

But the equations $ce = bf$ and $cd = af$ give us $\frac{b}{c} = \frac{e}{f}$ and $\frac{a}{c} = \frac{d}{f}$, that is, these two are, in fact, one and the same equation repeated, from which, as has been explained before, an infinite number of values of x and y can be found; in fact, any value may be given to x provided y be then found from the equation. We see that in these instances, when the value of any quantity appears in the form $\frac{0}{0}$ that quantity admits of an infinite number of values, and this indicates that the conditions given to determine that quantity are not sufficient. But this is not the only cause of the appearance of a fraction in the form $\frac{0}{0}$. Take the identical equation

$$\frac{a^2 - b^2}{c(a-b)} = \frac{a+b}{c}.$$

When a approaches towards b, $a+b$ approaches towards $2b$, and $a^2 - b^2$ and $a - b$ approach more and more nearly towards 0. If $a = b$ the equation assumes this form:

$$\frac{0}{0} = \frac{2b}{c}.$$

This may be explained thus: if we multiply the numerator and denominator of the fraction $\frac{A}{B}$ by $a - b$ (which does not alter its value) it becomes $\frac{Aa - Ab}{Ba - Bb}$. If in the course of an investigation this has been done when the two quantities a and b are equal to one another, the fraction $\frac{A}{B}$ or $\frac{Aa - Ab}{Ba - Bb}$ will appear in the form $\frac{0}{0}$. But since the result would have been $\frac{A}{B}$ had that multiplication not been performed, this last fraction must be used instead of the unmeaning form

$\frac{0}{0}$. Thus the fraction $\frac{a^2-b^2}{c(a-b)}$ or $\frac{(a+b)(a-b)}{c(a-b)}$ is the fraction $\frac{a+b}{c}$ after its numerator and denominator have been multiplied by $a-b$, and may be used in all cases except that in which $a=b$. When the form $\frac{0}{0}$ occurs, the problem must be carefully examined in order to ascertain the reason.

Chapter X.

Equations of the Second Degree.

Every operation of algebra is connected with another which is exactly opposite to it in its effects. Thus addition and subtraction, multiplication and division, are reverse operations, that is, what is done by the one is undone by the other. Thus $a+b-b$ is a, and $\frac{ab}{b}$ is a. Now in connexion with the raising of powers is a contrary operation called the extraction of roots. The term root is thus explained: We have seen that aa, or a^2, is called the square of a; from which a is called the square root of a^2. As 169 is called the square of 13, 13 is called the square root of 169. The following table will show how this phraseology is carried on.

a is called the square root of a^2,	which is denoted by	$\sqrt{a^2}$
a	cube root of a^3, ...	$\sqrt[3]{a^3}$
a	fourth root of a^4, ...	$\sqrt[4]{a^4}$
a	fifth root of a^5, ...	$\sqrt[5]{a^5}$
&c.	&c.	&c.

If b stand for a^4, $\sqrt[4]{b}$ stands for a, and the foregoing table may be represented thus:

$$\text{If } a^2=b;\ a=\sqrt{b}$$
$$a^3=b \quad a=\sqrt[3]{b},\ \&c.$$

The usual method of proceeding, is to teach the student to extract the square root of any algebraical quantity immediately after the solution of equations of the first degree. We would rather recommend him to omit this rule until he is acquainted with the solution of equations of the second degree, except in the cases to which we now proceed. In arithmetic, it must be observed, that there are comparatively very few numbers of which the square root can be extracted. For example, 7 is not made by the multiplication either of any whole number or fraction by itself. The first is evident; the second cannot be readily proved to the beginner, but he may, by taking a number of instances, satisfy himself of this, that no fraction which is really such, that is whose numerator is not measured by its denominator, will give a whole number when multiplied by itself, thus $\frac{4}{3}\times\frac{4}{3}$ or $\frac{16}{9}$ is not a whole number, and so on. The number 7, therefore, is neither the square of a whole number, or of a fraction, and, properly speaking, has no square root. Nevertheless fractions can be found extremely near to 7, which have square roots, and this degree of nearness may be carried to any extent we please. Thus if required between 7 and $7\frac{1}{1000000000}$ could be found a fraction which has a square root, and the fraction in the last might be decreased to any extent whatever, so that though we cannot find a fraction whose square is 7, we may nevertheless find one whose square is as near to 7 as we please. To take another example, if we multiply 1·4142 by itself the product is 1·99996164, which only differs from 2 by the very small fraction .00003826, so that the square of 1·4142 is very nearly 2, and fractions might be found where squares are still nearer to 2. Let us now suppose the following problem. A man buys a certain number of yards of stuff for two shillings, and the number of yards which he gets is exactly the number of shillings which he gives for a yard. How many yards does he buy? Let x be this number, then $\frac{2}{x}$ is the price of one yard, and $x=\frac{2}{x}$ or $x^2=2$. This, from what we have said, is impossible, that is, there is no exact number of yards, or parts of yards, which will satisfy the conditions; nevertheless 1·4142 yards will nearly do it, 1.4142136 still more nearly, and if the

problem were ever proposed in practice, there would be no difficulty in solving it with sufficient nearness for any purpose. A problem, therefore, whose solution contains a square root which cannot be extracted, may be rendered useful by approximation to the square root.

Equations of the second degree, commonly called quadratic equations, are those in which there is the second power, or square of an unknown quantity: such as $x^2 - 3 = 4x^2 - 15$, $x^2 + 3x = 2x^2 - x - 1$, &c. By transposition of their terms, they may always be reduced to one of the following forms:

$$ax^2 + b = 0$$
$$ax^2 - b = 0$$
$$ax^2 + bx + c = 0$$
$$ax^2 - bx + c = 0$$
$$ax^2 + bx - c = 0$$
$$ax^2 - bx - c = 0$$

For example, the two equations given above, are equivalent to $3x^2 - 12 = 0$, and $x^2 - 4x - 1 = 0$, which agree in form with the second and last. In order to proceed to each of these equations, first take the equation $x^2 = a^2$. This equation is the same as $x^2 - a^2 = 0$, or $(x+a)(x-a) = 0$. Now, in order that the product of two or more quantities may be equal to nothing, it is sufficient that *one* of those quantities be nothing, and therefore a value of x may be derived from either of the following equations:—

$$x - a = 0$$
$$\text{or } x + a = 0$$

the first of which gives $x = a$, and the second $x = -a$. To elucidate this, find x from the following equation:—

$$\overline{3x+a}(a^2+x^2) = (x^2+ax)(a^2+ax+2x^2)$$

develop this equation, and transpose all its terms on one side, when it becomes

$$x^4 - 2a^2x^2 + a^4 + 2a^3x - 2ax^3 = 0$$
$$\text{or } (x^2 - a^2)^2 - 2ax(x^2 - a^2) = 0$$
$$\text{or } (x^2 - a^2)(x^2 - 2ax - a^2) = 0.$$

This last equation is true when $x^2 - a^2 = 0$, or when $x^2 = a^2$, which is true either when $x = +a$, or $x = -a$. If in the original equation $+a$ is substituted instead of x, the result is $4a \times 2a^2 = 2a^2 \times 4a^2$; if $-a$ be substituted instead of x, the result is $0 = 0$, which show that $+a$ and $-a$ are both correct values of x. We have here noticed, for the first time, an equation of condition, which is capable of being solved by more than one value of x. We have found two, and shall find more when we can solve the equation $x^2 - 2ax - a^2 = 0$, or $x^2 - 2ax = a^2$. Every equation of the second degree, if it has one value of x, has a second, of which $x^2 = a^2$ is an instance, where $x = \pm a$, in which by the double sign $\pm$ is meant, that either of them may be used at pleasure. We now proceed to the solution of $ax^2 - bx + c = 0$. In order to understand the nature of this equation, let us suppose that we take for x such a value, that $ax^2 - bx + c$, instead of being equal to 0, is equal to y, that is

$$y = ax^2 - bx + c^* \qquad (1);$$

in which the value of y depends upon the value given to x, and changes when x changes. Let m be one of those quantities which, when substituted instead of x, make $ax^2 - bx + c$ equal to nothing, in which case m is called a root of the equation,

$$ax^2 - bx + c = 0 \qquad (2);$$

and it follows that

$$am^2 - bm + c = 0 \qquad (3).$$

Subtract (3) from (1), the result of which is

$$y = a(x^2 - m^2) - b(x - m)$$
$$= \overline{x - m}\,(a\overline{x+m} - b)$$

Here y is evidently equal to 0, when $x = m$, as we might expect from the supposition which we made; but it is also nothing when $a(x + m) - b = 0$; there is, therefore, another value of x, for which $y = 0$; if we call this n we find it from the equation $a(n + m) - b = 0$,

$$\text{or } n + m = \frac{b}{a} \qquad (4).$$

In (3) substitute for b its value derived from (4), from which $b = a(n + m)$; it then becomes

$$am^2 - am(n + m) + c = 0,$$
$$\text{or } c - amn = 0,$$

$$\text{which gives } mn = \frac{c}{a} \qquad (5).$$

Substitute in (1) the values of b and c derived from (4) and (5), which gives

$$y = ax^2 - a(m+n)x + amn$$
$$= a(x^2 - \overline{m+n}\,x + mn).$$

Now the second factor of this expression arises from multiplying together $\overline{x - m}$

* In the investigations which follow, a, b, and c are considered as having the sign which is marked before them, and no change of form is supposed to take place.

and $\overline{x-n}$; therefore,

$$y = a\,(x-m)\,(x-n) \qquad (6).$$

To take an example of this, let $y = 4x^2 - 5x + 1$. Here when $x = 1$, $y = 4 - 5 + 1 = 0$, and therefore $m = 1$. If we divide $4x^2 - 5x + 1$ by $x - 1$, the quotient (which is without remainder) is $4x - 1$, and therefore

$$y = (x-1)\,(4x-1).$$

This is also nothing when $4x - 1 = 0$, or when x is $\frac{1}{4}$. Therefore $n = \frac{1}{4}$, and $y = 4\,(x-1)\,(x-\frac{1}{4})$, a result coinciding with that of (6). If, therefore, we can find one of the values of x which satisfy the equation $ax^2 - bx + c = 0$, we can find the other and can divide $ax^2 - bx + c$ into the factors a, $x - m$ and $x - n$, or

$$ax^2 - bx + c = a\,(x-m)\,(x-n).$$

If we multiply $x + m$ by $x + n$, the only difference between $\overline{x+m}\,\overline{x+n}$ and $\overline{x-m}\,\overline{x-n}$ is in the sign of the term which contains the first power of x. If, therefore,

$$ax^2 - bx + c = a\,(x-m)\,(x-n),$$

it follows that

$$ax^2 + bx + c = a\,(x+m)\,(x+n).$$

We now take the expression $ax^2 - bx - c$. If there is one value of x which will make this quantity equal to 0, let this be m, and

$$\begin{aligned}
\text{Let } y &= ax^2 - bx - c \\
\text{Then } 0 &= am^2 - bm - c, \\
\text{from which } y &= a\,(x^2 - m^2) - b\,(x - m) \\
&= \overline{x-m}\,(a\,\overline{x+m} - b) \\
&= \overline{x-m}\,(ax + am - b).
\end{aligned}$$

Let $\dfrac{am-b}{a}$ be called n, or let $am - b = an$; then

$$\begin{aligned}
y &= \overline{x-m}\,(ax + an) \\
&= a\,(x-m)\,(x+n).
\end{aligned}$$

As an example, it may be shown that

$$3x^2 - x - 2 = 3\,(x-1)\,(x+\tfrac{2}{3}).$$

Again, with regard to $ax^2 + bx - c$, since $\overline{x+m}\,\overline{x-n}$ only differs from $\overline{x-m}\,\overline{x+n}$ in the sign of the term which contains the first power of x, it is evident that

$$\begin{aligned}
\text{if } ax^2 - bx - c &= a\,(x-m)\,(x+n) \\
ax^2 + bx - c &= a\,(x+m)\,(x-n).
\end{aligned}$$

Results similar to those of the first case may be obtained for all the others, and these results may be arranged in the following way. In the first and third, m is a quantity, which, when substituted for x, makes $y = 0$, and in the second and fourth m and n are the same as in the first and third.

1st.	$y = ax^2 - bx + c = a\,(x-m)\,(x-n)$	$m + n = \dfrac{b}{a}$	$mn = \dfrac{c}{a}$.
2d.	$y = ax^2 + bx + c = a\,(x+m)\,(x+n)$	$m + n = \dfrac{b}{a}$	$mn = \dfrac{c}{a}$.
3d.	$y = ax^2 - bx - c = a\,(x-m)\,(x+n)$	$m - n = \dfrac{b}{a}$	$mn = \dfrac{c}{a}$.
4th.	$y = ax^2 + bx - c = a\,(x+m)\,(x-n)$	$m - n = \dfrac{b}{a}$	$mn = \dfrac{c}{a}$.

We must now inquire in what cases a value can be found for x, which will make $y = 0$ in these different expressions, and in this consists the solution of equations of the second degree.

Let $y = ax^2 - bx + c \qquad (1)$,

and observe that $(2ax - b)^2 = 4a^2x^2 - 4abx + b^2$. Multiply both sides of (1) by $4a$, which gives

$$4ay = 4a^2x^2 - 4abx + 4ac \qquad (2).$$

Add b^2 to the first two terms of the

second side of (2), and subtract it from the third, which will not alter the whole, and this gives

$$4ay = 4a^2x^2 - 4abx + b^2 + 4ac - b^2$$
$$= (2ax - b)^2 + 4ac - b^2 \quad (3).$$

Now it must be recollected that the square of any quantity is positive whether that quantity is positive or negative. This has been already sufficiently explained in saying that a change of the form of any expression does not change the form of its square. Common multiplication shows that $\overline{c-d}|^2$ and $\overline{d-c}|^2$ are the same thing; and, since one of these must be positive, the other must be also positive. Whenever, therefore, we wish to say that a quantity is positive, it can be done by supposing it equal to the square of an algebraical quantity. In equation (3) there are three distinct cases to be considered.

I. When b^2 is greater than $4ac$, that is, when $b^2 - 4ac$ is positive, let $b^2 - 4ac = k^2$, which expresses the condition.

Then $4ay = (2ax - b)^2 - k^2 \quad (4)$

and we determine those values of x, which make $y = 0$, from the equation,

$$(2ax - b)^2 - k^2 = 0.$$

We have already solved such an equation, and we find that

$$2ax - b = \pm k,$$

where either sign may be taken. This shows that y or $ax^2 - bx + c$ is equal to nothing either when

$$\text{instead of } x \text{ is put } \frac{b+k}{2a} = \frac{b+\sqrt{b^2-4ac}}{2a} = m,$$

$$\text{or } \frac{b-k}{2a} = \frac{b-\sqrt{b^2-4ac}}{2a} = n,$$

the second values are formed by putting, instead of k its value $\sqrt{b^2-4ac}$. They are both positive quantities, because k^2 being equal to b^2-4ac is greater than b^2, and therefore k is greater than b, and therefore $\frac{b+k}{2a}$ and $\frac{b-k}{2a}$ are both positive. These are the quantities which we have called m and n in the former investigations, and, therefore,

$$ax^2 - bx + c = a(x-m)(x-n) = a\left(x - \frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x - \frac{b-\sqrt{b^2-4ac}}{2a}\right)$$

actual multiplication of the factors will show that this is an identical equation.

II. When b^2, instead of being greater than $4ac$, is equal to it, or when $b^2 - 4ac = 0$ and $k = 0$. In this case the values of m and n are equal, each being $\frac{b}{2a}$ and

$$y = ax^2 - bx + c = a(x-m)\,\overline{x-n}| = a\left(x - \frac{b}{2a}\right)^2$$

In this case y is said, in algebra, to be a perfect square, since its square root can be extracted, and is $\sqrt{a}\left(x - \frac{b}{2a}\right)$.

Arithmetically speaking, this would not be a perfect square unless a was a number whose square root could be extracted, but in algebra it is usual to call any quantity a perfect square with respect to any letter, which, when reduced, does not contain that letter under the sign $\sqrt{\ }$. This result is one which often occurs, and it must be recollected that when $b^2 - 4ac = 0$, $ax^2 - bx + c$ is a perfect square.

III. When b^2 is less than $4ac$, or when $b^2 - 4ac$ is negative and $4ac - b^2$ positive, let $4ac - b^2 = k^2$, and equation (3) becomes

$$4ay = \overline{2ax - b}|^2 + k^2.$$

In this case no value of x can ever make $y = 0$, for the equation $v^2 + w^2 = 0$ indicates that v^2 is equal to w^2 with a contrary sign, which cannot be, since all squares have the same sign. The values of x are said, in this case, to be impossible, and it indicates that there is something absurd or contradictory in the conditions of a problem which leads to such a result. Having found that whenever

$$ax^2-bx+c=a\,(x-m)\;(x-n),$$

it follows that $ax^2+bx+c=a\,(x+m)\;(x+n)$.

I. We know that when b^2 is greater than $4ac$,

$$ax^2+bx+c=a\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right).$$

II. When $b^2=4ac$, $ax^2+bx+c=a\left(x+\dfrac{b}{2a}\right)^2$, and y is a perfect square.

III. When b^2 is less than $4ac$, ax^2+bx+c cannot be divided into factors.

Now, let $y=ax^2-bx-c$ (1)

As before, $4ay=4a^2x^2-4abx+b^2-4ac-b^2$

$= (2ax-b)^2-(b^2+4ac)$ (2)

Let $b^2+4ac=k^2$. Then

$4ay=(2ax-b)^2-k^2$. (3)

Therefore y is 0 when $\overline{2ax-b}|^2=k^2$, or when $2ax-b=\pm k$;

$$\text{That is, } m=\frac{b+k}{2a}=\frac{b+\sqrt{b^2+4ac}}{2a}$$

$$n=\frac{b-k}{2a}=\frac{b-\sqrt{b^2+4ac}}{2a}$$

Now, because b^2 is less than b^2+4ac, b is less than $\sqrt{b^2+4ac}$, therefore n is a negative quantity. Leaving, for the present, the consideration of the negative quantity, we may decompose (3) into factors by means of the general formula $p^2-q^2=\overline{p-q}\;\overline{p+q}$, which gives

$$4ay=\overline{2ax-b-k}\;\;\overline{2ax-b+k}$$

$$=4a^2\left(x-\frac{k+b}{2a}\right)\left(x+\frac{k-b}{2a}\right)$$

from which y or

$$ax^2-bx-c=a\left(x-\frac{\sqrt{b^2+4ac}+b}{2a}\right)\left(x+\frac{\sqrt{b^2+4ac}-b}{2a}\right)$$

Therefore, from what has been proved before,

$$ax^2+bx-c=a\left(x+\frac{\sqrt{b^2+4ac}+b}{2a}\right)\left(x-\frac{\sqrt{b^2+4ac}-b}{2a}\right).$$

The following are some examples, of the truth of which the student should satisfy himself, both by reference to the one just established, and by actual multiplication:—

$$2x^2-7x+3=2\left(x-\frac{7+\sqrt{49-24}}{4}\right)\left(x-\frac{7-\sqrt{49-24}}{4}\right)$$

$$=2\;(x-3)\;(x-\tfrac{1}{2})$$

$$3x^2-6x+1=3\left(x-\frac{3+\sqrt{6}}{3}\right)\left(x-\frac{3-\sqrt{6}}{3}\right)^*$$

$$5\tfrac{1}{2}\,x^2-22x+22=5\tfrac{1}{2}\;(x-2)^2$$

$$5x^2+9x-7=5\left(x+\frac{\sqrt{221}+9}{10}\right)\left(x-\frac{\sqrt{221}-9}{10}\right).$$

* Recollect that $\sqrt{24}=\sqrt{6\times4}=\sqrt{6}\times\sqrt{4}=2\sqrt{6}$.

If we collect together the different results at which we have arrived, to which species of tabulation the student should take care to accustom himself, we have the following :—

$$ax^2 + bx + c = a\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right) \quad \text{(A)}$$

$$ax^2 - bx + c = a\left(x - \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{b - \sqrt{b^2 - 4ac}}{2a}\right) \quad \text{(B)}$$

$$ax^2 + bx - c = a\left(x + \frac{\sqrt{b^2 + 4ac} + b}{2a}\right)\left(x - \frac{\sqrt{b^2 + 4ac} - b}{2a}\right) \quad \text{(C)}$$

$$ax^2 - bx - c = a\left(x - \frac{\sqrt{b^2 + 4ac} + b}{2a}\right)\left(x + \frac{\sqrt{b^2 + 4ac} - b}{2a}\right) \quad \text{(D)}$$

These four cases may be contained in one, if we apply those rules for the change of signs which we have already established. For example, the first side of (C) is made from that of (A), by changing the sign of c; the second side of (C) is made from that of (A) in the same way. We have also seen the necessity of taking into account the negative quantities which satisfy an equation, as well as the positive ones; if we take these into account, each of the four forms of $ax^2 + bx + c$ can be made equal to nothing by two values of x. For example, in (1), when

$$ax^2 + bx + c = 0$$

$$\text{either } x + \frac{b - \sqrt{b^2 - 4ac}}{2a} = 0$$

$$\text{or } x + \frac{b + \sqrt{b^2 - 4ac}}{2a} = 0$$

If we call the values of x derived from the equations m and n, we find that

$$m = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad n = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{(A')}$$

In the cases marked (B) (C) and (D), the results are

$$m = \frac{b + \sqrt{b^2 - 4ac}}{2a} \qquad n = \frac{b - \sqrt{b^2 - 4ac}}{2a} \quad \text{(B')}$$

$$m = \frac{-b + \sqrt{b^2 + 4ac}}{2a} \qquad n = \frac{-b - \sqrt{b^2 + 4ac}}{2a} \quad \text{(C')}$$

$$m = \frac{b + \sqrt{b^2 + 4ac}}{2a} \qquad n = \frac{b - \sqrt{b^2 + 4ac}}{2a} \quad \text{(D')}$$

and in all the four cases the form of $ax^2 + bx + c$ which is used, is the same as the corresponding form of

$$a(x - m)(x - n)$$

and the following results may be easily obtained. In (A'), both m and n, if they exist at all, are negative. I say, if they exist at all, because it has been shown that if $b^2 - 4ac$ is negative, the quantity $ax^2 + bx + c$ cannot be divided into factors at all, since $\sqrt{b^2 - 4ac}$ is then no algebraical quantity, either positive or negative.

In (B'), both, if they exist at all, are positive.

In (C') there are always real values for m and n, since $b^2 + 4ac$ is always positive; one of these values is positive, and the other negative, and the negative one is numerically the greatest.

In (D') there are also real values of m and n, one positive, and the other negative, of which the positive one is numerically the greatest. Before proceeding any further, we must notice an extension of a phrase which is usually adopted. The words greater and less, as applied to numbers, offer no difficulty, and from them we deduce, that if a be greater than b, $a - c$ is greater than $b - c$, as long as these subtractions are possible, that is, as long as c can be taken both from a and b. This is the only case which was considered when the rule was made, but in extending the

meaning of the word subtraction, and using the symbol -3 to stand for $5-8$, the principle that if a be greater than b, $a-c$ is greater than $b-c$, leads to the following results. Since 6 is greater than 4, $6-12$ is greater than $4-12$, or -6 is greater than -8; again $6-6$ is greater than $4-6$, or 0 is greater than -2. These results, particularly the last, are absurd, as has been noticed, if we continue to mean by the terms greater and less, nothing more than is usually meant by them in arithmetic; but in extending the meaning of one term, we must extend the meaning of all which are connected with it, and we are obliged to apply the terms greater and less in the following way. Of two algebraical quantities with the same or different signs, that one is the greater which, when both are connected with a number numerically greater than either of them, gives the greater result. Thus -6 is said to be greater than -8, because $20-6$ is greater than $20-8$, 0 is greater than -4, because $6+0$ is greater than $6-4$; $+12$ is greater -30, because $40+12$ is greater than $40-30$. Nevertheless -30 is said to be *numerically* greater than $+12$, because the number contained in the first is greater than that in the second. For this reason it was said, that in (C′), the negative quantity was *numerically* greater than the positive, because any positive quantity is in algebra called greater than any negative one, even though the number contained in the first should be less than that in the second. In the same way -14 is said to lie between $+3$ and -20, being less than the first and greater than the second. The advantage of these extensions is the same as that of others; the disadvantage attached to them, which it is not fair to disguise, is that, if used without proper caution, they lead the student into erroneous notions, which some elementary works, far from destroying, confirm, and even render necessary, by adopting these very notions as definitions; as for example, when they say that a negative quantity is one which is less than nothing; as if there could be such a thing, the usual meaning of the word less being considered, and as if the student had an idea of a quantity less than nothing already in his mind, to which it was only necessary to give a name.

The product $\overline{x-m}\;\overline{x-n}$ is positive when $\overline{x-m}$ and $\overline{x-n}$ have the same, and negative when they have different signs. This last can never happen except when x lies between m and n, that is, when x is algebraically greater than the one, and less than the other. The following table will exhibit this, where different products are taken with various signs of m and n, and three values are given to x one after the other, the first of which is less than both m and n, the second between both, and the third greater than both.

Product.	Value of x.	Value of the product with its sign.
$\overline{x-4}\;\overline{x-7}$	$+1$	$+18$
$m=+4$	$+5$	-2
$n=+7$	$+10$	$+18$
$\overline{x+10}\;\overline{x-3}$	-12	$+30$
$m=-10$	-7	-30
$n=+3$	$+4$	$+14$
$\overline{x+2}\;\overline{x+12}$	-13	$+9$
$m=-2$	-6	-24
$n=-12$	-1	$+11$

The student will see the reason of this, and perform a useful exercise in making two or three tables of this description for himself. The result is that $\overline{x-m}$ $\overline{x-n}$ is negative when x lies between m and n, is nothing when x is either equal to m or to n, and positive when x is greater than both, or less than both. Consequently, $a(x-m)(x-n)$ has the same sign as a when x is greater than both m and n, or less than both, and a different sign from a when x lies between both. But whatever may be the signs of a, b, and c, if there are two quantities m and n, which make

$$ax^2+bx+c=a(x-m)(x-n),$$

that is, if the equation $ax^2+bx+c=0$ has real roots, the expression ax^2+bx+c always has the same sign as a for all values of x, except when x lies between these roots.

It only remains to consider those cases in which ax^2+bx+c cannot be decomposed into different factors, which happens whenever b^2-4ac is 0, or negative. In the first case when $b^2-4ac=0$, we have

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2$$

$$ax^2-bx+c=a\left(x-\frac{b}{2a}\right)^2$$

and as these expressions are composed of factors, one of which is a square, and therefore positive, they have always the same sign as the other factor, which is a. When $b^2 - 4ac$ is negative, we have proved that if $y = ax^2 \pm bx + c$, $4ay = (2ax \pm b)^2 + k^2$, where $k^2 = 4ac - b^2$, and therefore $4ay$ being the sum of two squares is always positive, that is, $ax^2 \pm bx + c$ has the same sign as a, whatever may be the value of x. When $c = 0$, the expression becomes $ax^2 + bx$, or $x(ax + b)$, which is nothing either when $x = 0$, or when $ax + b = 0$ and $x = -\frac{b}{a}$; the general expressions for m and n become in this case $\frac{-b + \sqrt{b^2}}{2a}$ and $\frac{-b - \sqrt{b^2}}{2a}$, which give the same results.

When $b = 0$, the expression is reduced to $ax^2 + c = 0$, which is nothing when $x = \pm\sqrt{-\frac{a}{c}}$, which is not possible, except when c and a have different signs. In this case, that is, when the expression assumes the form $ax^2 - c$, it is the same as

$$a\left(x - \sqrt{\frac{a}{c}}\right)\left(x + \sqrt{\frac{a}{c}}\right).$$

The same result might be deduced by making $b = 0$ in the general expressions for m and n.

When $a = 0$, the expression is reduced to $bx + c$, which is made equal to nothing by one value of x only, that is $-\frac{c}{b}$. If we take the general expressions for m and n, and make $a = 0$ in them, that is, in $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$, and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$, we find as the results $\frac{0}{0}$ and $\frac{-2b}{0}$. These have been already explained. The first may either indicate that any value of x will solve the problem which produced the equation $ax^2 + bx + c = 0$, or that we have applied a rule to a case which was not contemplated in its formation, and have thereby created a factor in the numerator and denominator of x, which, in attempting to apply the rule, becomes equal to nothing. The student is referred to the problem of the two couriers, solved in the preceding part of this treatise. The latter is evidently the case here, because in returning to the original equation, we find it reduced to $bx + c = 0$, which gives a rational value for x, namely, $-\frac{c}{b}$. The second value, or $-\frac{2b}{0}$, which in algebraical language is called infinite, may indicate, that though there is no other value of x, except $-\frac{c}{b}$, which solves the equation, still that the greater the number which is taken for x, the more nearly is a second solution obtained. The use of these expressions is to point out the cases in which there is anything remarkable in the general problem; to the problem itself we must resort for further explanation.

The importance of the investigations connected with the expression $ax^2 + bx + c$, can hardly be over-rated, at least to those students who pursue mathematics to any extent. In the higher branches, great familiarity with these results is indispensable. The student is therefore recommended not to proceed until he has completely mastered the details here given, which have been hitherto too much neglected in English works on algebra.

In solving equations of the second degree, we have obtained a new species of result, which indicates that the problem cannot be solved at all. We refer to those results which contain the square root of a negative quantity. We find that by multiplication the squares of $c - d$ and of $d - c$ are the same, both being $c^2 - 2cd + d^2$. Now either $c - d$ or $d - c$ is positive, and since they both have the same square, it appears that the squares of all quantities, whether positive or negative, are positive. It is therefore absurd to suppose that there is any quantity which x can represent, and which satisfies the equation $x^2 = -a^2$, since that would be supposing that x^2, a positive quantity, is equal to the negative quantity $-a^2$. The solution is then said to be impossible, and it will be easy to show an instance in which such a result is obtained, and also to show that it arises from the absurdity of the problem.

Let a number a be divided into any two parts, one of which is greater than the half, and the other less. Call the first of these $\frac{a}{2} + x$, then the second must be $\frac{a}{2} - x$, since the sum of both

parts must be a. Multiply these parts together, which gives $\left(\frac{a}{2}+x\right)\left(\frac{a}{2}-x\right)$, or $\left(\frac{a}{2}\right)^2 - x^2$. As x diminishes, this product increases, and is greatest of all when $x = 0$, that is, when the two parts, into which a is divided, are $\frac{a}{2}$ and $\frac{a}{2}$, or when the number a is halved. In this case the product of the parts is $\frac{a}{2} \times \frac{a}{2}$, or $\frac{a^2}{4}$, and a number a can never be divided into two parts whose product is greater than $\frac{a^2}{4}$. This being premised, suppose that we attempt to divide the number a into two parts, whose product is b. Let x be one of these parts, then $a - x$ is the other, and their product is $ax - x^2$.

We have, therefore,

$$ax - x^2 = b$$

$$\text{or} \quad x^2 - ax + b = 0.$$

If we solve this equation the two roots are the two parts required, since from what we have proved of the expression $x^2 - ax + b$ the sum of the roots is a and their product b. These roots are $\frac{a}{2}+\sqrt{\frac{a^2}{4}-b}$ and $\frac{a}{2}-\sqrt{\frac{a^2}{4}-b}$*, which are impossible when $\frac{a^2}{4}-b$ is negative, or when b is greater than $\frac{a^2}{4}$, which agrees with what has just been proved, viz. that no number is capable of being divided into two parts whose product is greater than $\frac{a^2}{4}$.

We have shown the symbol $\sqrt{-a}$ to be void of meaning, or rather self-contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility. It depends upon the fact, which must be verified by experience, that the common rules of algebra may be applied to these expressions without leading to any false results. An appeal to experience of this nature appears to be contrary to the first principles laid down at the beginning of this work. We cannot deny that it is so in reality, but it must be recollected that this is but a small and isolated part of an immense subject, to all other branches of which these principles apply in their fullest extent. There have not been wanting some to assert that these symbols may be used as rationally as any others, and that the results derived from them are as conclusive as any reasoning could make them. I leave the student to discuss this question as soon as he has acquired sufficient knowledge to understand the various arguments: at present, let him proceed with the subject as a part of the mechanism of algebra, on the assurance that by careful attention to the rules laid down, he can never be led to any incorrect result. The simple rule is, apply all those rules to such expressions as $\sqrt{-a}$ $a+\sqrt{-b}$, &c. which have been proved to hold good for such quantities as $\sqrt{a}$ $a+\sqrt{b}$, &c. Such expressions as the first of these are called *imaginary*, to distinguish them from the second, which are called *real;* and it must always be recollected that there is no quantity, either positive or negative, which an imaginary expression can represent.

It is usual to write such symbols as $\sqrt{-b}$ in a different form. To the equation $-b = b \times (-1)$ apply the rule derived from the equation $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$, which gives $\sqrt{-b} = \sqrt{b} \times \sqrt{-1}$, of which the first factor is real and the second imaginary. Let $\sqrt{b} = c$, then $\sqrt{-b} = c\sqrt{-1}$. In this way all expressions may be so arranged that $\sqrt{-1}$ shall be the only imaginary quantity which appears in them. Of this reduction the following are examples:

* The general expressions for m and n give $\frac{a \pm \sqrt{a^2-4b}}{2}$ as the roots of $x^2 - ax + b = 0$.

$$\sqrt{-24} = \sqrt{24}\,\sqrt{-1} = 2\sqrt{6}\,\sqrt{-1}$$

$$\sqrt{-a^2} = a\sqrt{-1}$$

$$\sqrt{-a} \times \sqrt{-a} = -a$$

$$\sqrt{2ab - a^2 - b^2} = (a - b)\sqrt{-1}$$

$$\sqrt{-a^2} \times \sqrt{-b^2} = a\sqrt{1-} \times b\sqrt{-1} = -ab.$$

The following tables exhibit other applications of the rules:

$c = a\sqrt{-1}$	$c^5 = a^5\sqrt{-1}$, &c.	$c^{4n-3} = a^{4n-3}\sqrt{-1}$
$c^2 = -a^2$	$c^6 = -a^6$, &c.	$c^{4n-2} = -a^{4n-2}$
$c^3 = -a^3\sqrt{-1}$	$c^7 = -a^7\sqrt{-1}$, &c.	$c^{4n-1} = -a^{4n-1}\sqrt{-1}$
$c^4 = a^4$	$c^8 = a^8$, &c.	$c^{4n} = a^{4n}$.

The powers of such an expression as $a\sqrt{-1}$ are therefore alternately real and imaginary, and are positive and negative in pairs.

$$(a + b\sqrt{-1})^2 = a^2 - b^2 + 2ab\sqrt{-1}$$

$$(a - b\sqrt{-1})^2 = a^2 - b^2 - 2ab\sqrt{-1}$$

$$(a + b\sqrt{-1})\ (a - b\sqrt{-1}) = a^2 + b^2$$

$$\frac{a + b\sqrt{-1}}{a - b\sqrt{-1}} = \frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab\sqrt{-1}}{a^2 + b^2}$$

$$(a + b\sqrt{-1})\ (c + d\sqrt{-1}) = ac - bd + (ad + cb)\sqrt{-1}$$

Let the roots of the equation $ax^2 + bx + c = 0$ be impossible, that is, let $b^2 - 4ac$ be negative and equal to $-k^2$. Its roots, as derived from the rules established when $b^2 - 4ac$ was positive, are

$$\frac{-b + \sqrt{-k^2}}{2a} \text{ and } \frac{-b - \sqrt{-k^2}}{2a}, \text{ or}$$

$$-\frac{b}{2a} + \frac{k}{2a}\sqrt{-1}, \text{ and } -\frac{b}{2a} - \frac{k}{2a}\sqrt{-1}.$$

Take either of these instead of x; for example, let

$$x = -\frac{b}{2a} + \frac{k}{2a}\sqrt{-1}.$$

$$\text{Then } ax^2 = \frac{b^2}{4a} - \frac{bk}{2a}\sqrt{-1} - \frac{k^2}{4a}$$

$$bx = -\frac{b^2}{2a} + \frac{bk}{2a}\sqrt{-1}$$

$$c = c$$

Therefore, $ax^2 + bx + c = \frac{b^2}{4a} - \frac{k^2}{4a} - \frac{b^2}{2a} + c$, in which, if $4ac - b^2$ be substituted instead of k^2, the result is 0. It appears, then, that the imaginary expressions which take the place of the roots when $b^2 - 4ac$ is negative, will, if the ordinary rules be applied, produce the same results as the roots. They are thence called imaginary roots, and we say that every equation of the second degree has two roots, either both real or both imaginary. It is generally true, that wherever an imaginary expression occurs, the same results will follow from the application of these expressions in any process as would have followed had the proposed problem been possible and its solution real.

When an equation arises in which imaginary and real expressions occur together, such as $a + b\sqrt{-1} = c + d\sqrt{-1}$, when all the terms are transferred on one side, the part which is real and that which is imaginary must each of them be equal to nothing. The equation just given when its left side is transposed become $a - c + \overline{b - d}\sqrt{-1} = 0$. Now, if b is not equal to d, let $b - d = e$; then $a - c + e\sqrt{-1} = 0$, and $\sqrt{-1} = \frac{c - a}{e}$; that is, an imaginary expression is equal to a real one, which is absurd. Therefore, $b = d$ and the original equation is thereby reduced to $a = c$. This goes on the supposition

that a, b, c, and d are real. If they are not so there is no necessary absurdity in $\sqrt{-1} = \frac{c-a}{e}$. If, then, we wish to express that two possible quantities a and b are respectively equal to two others c and d, it may be done at once by the equation

$$a + b\sqrt{-1} = c + d\sqrt{-1}$$

The imaginary expression $\sqrt{-a}$ and the negative expression $-b$ have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since $0 - a$ is as inconceivable as $\sqrt{-a}$. What, then, is the difference of signification? The following problems will elucidate this. A father is fifty-six, and his son twenty-nine years old: when will the father be twice as old as the son? Let this happen x years from the present time; then the age of the father will be $56 + x$, and that of the son $29 + x$; and therefore, $56 + x = 2(29 + x) = 58 + 2x$, or $x = -2$. This result is absurd; nevertheless, if in the equation we change the sign of x throughout it becomes $56 - x = 58 - 2x$, or $x = 2$. This equation is the one belonging to the problem: a father is 56 and his son 29 years old; when *was* the father twice as old as the son? the answer to which is, two years ago. In this case the negative sign arises from too great a limitation in the terms of the problem, which should have demanded how many years have elapsed or will elapse before the father is twice as old as his son?

Again, suppose the problem had been given in this last-mentioned way. In order to form an equation, it will be necessary either to suppose the event past or future. If of the two suppositions we choose the wrong one, this error will be pointed out by the negative form of the result. In this case the negative result will arise from a mistake in reducing the problem to an equation. In either case, however, the result may be interpreted, and a rational answer to the question may be given. This, however, is not the case in a problem, the result of which is imaginary. Take the instance above solved, in which it is required to divide a into two parts, whose product is b. The resulting equation is

$$x^2 - ax + b = 0$$

$$\text{or } x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b},$$

the roots of which are imaginary when b is greater than $\frac{a^2}{4}$. If we change the sign of x in the equation it becomes

$$x^2 + ax + b = 0$$

$$\text{or } x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b},$$

and the roots of the second are imaginary, if those of the first are so. There is, then, this distinct difference between the negative and the imaginary result. When the answer to a problem is negative, by changing the sign of x in the equation which produced that result, we may either discover an error in the method of forming that equation or show that the question of the problem is too limited, and may be extended so as to admit of a satisfactory answer. When the answer to a problem is imaginary this is not the case.

Chapter XI.

On Roots in general, and Logarithms.

The meaning of the terms square root, cube root, fourth root, &c. has already been defined. We now proceed to the difficulties attending the connexion of the roots of a with the powers of a. The following table will refresh the memory of the student with respect to the meaning of the terms :—

	Name of x.		Name of x.
Square of a - - -	$x = aa$	Square Root of a - -	$xx = a$
Cube . . . . - - -	$x = aaa$	Cube Root . . . . - -	$xxx = a$
Fourth Power - - -	$x = aaaa$	Fourth Root . . . - -	$xxxx = a$
Fifth Power - - -	$x = aaaaa$	Fifth Root . . . . - -	$xxxxx = a$

The different powers and roots of a have hitherto been expressed in the following way :—

Powers $a^2\ a^3\ a^4\ a^5\ ..\ a^m\ ..\ a^{m+n}$, &c.

Roots $\sqrt[2]{a}$* $\sqrt[3]{a}\ \sqrt[4]{a}\ \sqrt[5]{a}\ \sqrt[m]{a}\ \sqrt[m+n]{a}$, &c.

which series are connected together by the following equation, $(\sqrt[n]{a})^n = a$.

There has hitherto been no connexion between the manner of expressing powers and roots, and we have found no properties which are common both to powers and roots. Nevertheless, by the extension of rules, we shall be led to a method of denoting the raising of powers, the extraction of roots, and combinations of the two, to which algebra has been most peculiarly indebted, and the importance of which will justify the length at which it will be treated here.

Suppose it required to find the cube of $2a^2b^3$; that is, to find $2a^2b^3 \times 2a^2b^3 \times 2a^2b^3$. The common rules of multiplication give, as the result, $8a^6b^9$, which is expressed in the following equation,

$$(2a^2b^3)^3 = 8a^6b^9$$

$$\text{Similarly } (3a^4b^3)^4 = 81a^{16}b^{12}$$

$$\left(\frac{1}{2}\,\frac{b^4}{a}\right)^6 = \frac{1}{64}\,\frac{b^{24}}{a^6}$$

and the general rule by which any single term may be raised to the power whose index is n, is, raise the coefficient to the power n, and multiply the index of every letter by n, that is,

$$(a^p\,b^q\,c^r)^n = a^{np}\,b^{nq}\,c^{nr}.$$

In extracting the root of any simple term, we are guided by the manner in which the corresponding power is found. The rule is, extract the required root of the coefficient, and divide the index of each letter by the index of the root. Where these divisions do not give whole numbers as the quotients, the expression whose root is to be extracted does not admit of the extraction without the introduction of some new symbol. For example, extract the fourth root of $16a^{12}\,b^8\,c^4$, or find $\sqrt[4]{16a^{12}\,b^8\,c^4}$. The expression here given is the same as the following:—

$$2a^3b^2c \times 2a^3b^2c \times 2a^3b^2c \times 2a^3b^2c,$$

or $(2a^3b^2c)^4$, the fourth root of which is $2a^3bc$, conformably to the rule.

Any root of a product, such as AB, may be extracted by extracting the root of each of its factors. Thus, $\sqrt[3]{AB} = \sqrt[3]{A}\,.\,\sqrt[3]{B}$. For, raise $\sqrt[3]{A}\,\sqrt[3]{B}$ to the third power, the result of which is $\sqrt[3]{A}\,\sqrt[3]{B} \times \sqrt[3]{A}\,\sqrt[3]{B} \times \sqrt[3]{A}\,\sqrt[3]{B}$, or $\sqrt[3]{A}\,\sqrt[3]{A}\,\sqrt[3]{A} \times \sqrt[3]{B}\,\sqrt[3]{B}\,\sqrt[3]{B}$, or AB. In the same way it may be proved generally, that $\sqrt[n]{ABC} = \sqrt[n]{A}\,\sqrt[n]{B}\,\sqrt[n]{C}$. The most simple way of representing any root of any expression is the dividing it into two factors, one of which is the highest which it admits of whose root can be extracted by the rule just given. For example, in finding $\sqrt[3]{16a^4b^7c}$ we must observe that 16 is 8×2, a^4 is $a^3 \times a$, b^7 is $b^6 \times b$, and the expression is $8a^3b^6 \times 2abc$, the cube root of which, found by extracting the cube root of each factor, is $2ab^2\sqrt[3]{2abc}$. The second factor has no cube root which can be expressed by means of the symbols hitherto used, but when the numbers which a, b, and c stand for are known, $\sqrt[3]{2abc}$ may be found either exactly, or, when that is not possible, by approximation.

We find that a power of a power is found by affixing, as an index, the product of the indices of the two powers. Thus $(a^2)^4$ or $a^2 \times a^2 \times a^2 \times a^2$ is a^8, or $a^{4\times 2}$. This is the same as $(a^4)^2$, which is $a^4 \times a^4$, or a^8. Therefore, generally $(a^m)^n = (a^n)^m = a^{mn}$. In the same manner, a root of a root is the root whose index is the product of the indices of the two roots. Thus $\sqrt[3]{\sqrt[2]{a}} = \sqrt[6]{a}$. For since $a = \sqrt[6]{a}\,\sqrt[6]{a}\,\sqrt[6]{a} \times \sqrt[6]{a}\,\sqrt[6]{a}\,\sqrt[6]{a}$, the square root of a is $\sqrt[6]{a}\,\sqrt[6]{a}\,\sqrt[6]{a}$, the cube root of which is $\sqrt[6]{a}$. This is the same as $\sqrt[2]{\sqrt[3]{a}}$, and generally

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$$

Again, when a power is raised and a root extracted, it is indifferent which is done first. Thus $\sqrt[3]{a^2}$ is the same thing as $(\sqrt[3]{a})^2$. For since $a^2 = a \times a$, the cube root may be found by taking the cube root of each of these factors, that is $\sqrt[3]{a^2} = \sqrt[3]{a} \times \sqrt[3]{a} = (\sqrt[3]{a})^2$, and generally

* The 2 is usually omitted, and the square root is written thus $\sqrt{a}$.

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

In the expression $\sqrt[n]{a^m}$, both n or m may be multiplied by any number, without altering the expression, that is $\sqrt[np]{a^{mp}} = \sqrt[n]{a^m}$. To prove this, recollect that $\sqrt[np]{a^{mp}} = \sqrt[n]{\sqrt[p]{a^{mp}}}$. But a^{mp} is $(a^m)^p$, and by definition, $\sqrt[p]{(a^m)^p} = a^m$. Therefore $\sqrt[np]{a^{mp}} = \sqrt[n]{a^m}$. This multiplication is equivalent to raising a power of $\sqrt[n]{a^m}$, and afterwards reducing the result, to its former value, by extracting the corresponding root, in the same way as $\frac{mp}{np}$ signifies that $\frac{m}{n}$ has been multiplied by p, and the result has been restored to its former value by dividing it by p.

The following equations should be established by the student to familiarize him with the notation and principles hitherto laid down.

$$\sqrt[n]{(a-b)^{n-2}} \times \sqrt[3n]{(a-b)^6} = a-b$$

$$\sqrt[n+m]{(a+b)^{n-m}} \quad \sqrt[n-m]{(a-b)^{n+m}} = (a^2-b^2) \left(\frac{\sqrt[n-m]{a-b}}{\sqrt[n+m]{a+b}}\right)^{2m}$$

$$\sqrt[n]{\frac{ab}{cd}} = \frac{\sqrt[n]{ab}}{\sqrt[n]{cd}} = \frac{\sqrt[n]{a}\sqrt[n]{b}}{\sqrt[n]{c}\sqrt[n]{d}} = \sqrt[n]{\frac{a}{c}} \quad \sqrt[n]{\frac{b}{d}}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{ab^{n-1}}}{b} = \frac{a}{\sqrt[n]{a^{n-1}b}}$$

The quantity $\sqrt[n]{a^m}$ is a simple expression when m can be divided by n, without remainder, for example $\sqrt[2]{a^{12}} = a^6$ $\sqrt[5]{a^{20}} = a^4$, and, in general, whenever m can be divided by n without remainder. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$. This symbol, viz. a letter which has an exponent appearing in a fractional form, has not hitherto been used. We may give it any meaning which we please, provided it be such that when $\frac{m}{n}$ is fractional in form only, and not in reality, that is, when m is divisible by n, and the quotient is p, $a^{\frac{m}{n}}$ shall stand for a^p, or $aaa\ldots\ldots(p)$*. It will be convenient to let $a^{\frac{m}{n}}$ always stand for $\sqrt[n]{a^m}$, in which case the condition alluded to is fulfilled, since when $\frac{m}{n} = p$ $a^{\frac{m}{n}}$ or $\sqrt[n]{a^m} = a^p$. This extension of a rule, the advantages of which will soon be apparent, is exemplified in the following table, which will familiarize the student with the different cases of this new notation:—

$a^{\frac{1}{2}}$	stands for	$\sqrt[2]{a^1}$ or $\sqrt{a}$	$a^{\frac{m+n}{m-n}}$	stands for	$\sqrt[m-n]{a^{m+n}}$
$a^{\frac{1}{3}}$	,,	$\sqrt[3]{a}$	$\overline{p+q}^{\frac{m-n}{2}}$	,,	$\sqrt{\overline{p+q}^{m-n}}$
$a^{\frac{1}{4}}$	,,	$\sqrt[4]{a}$	$\left(c^{\frac{m}{n}}\right)^{\frac{p}{q}}$	,,	$\sqrt[q]{\left(\sqrt[n]{c^m}\right)^p}$
$a^{\frac{2}{3}}$	,,	$\sqrt[3]{a^2}$ or $\left(\sqrt[3]{a}\right)^2$	$\left(a^{\frac{1}{n}}\right)^{\frac{1}{q}}$	,,	$\sqrt[q]{\sqrt[n]{a}}$
$a^{\frac{7}{2}}$	,,	$\sqrt{a^7}$ or $\left(\sqrt[2]{a}\right)^7$			

* This is a notation in common use, and means that $a\,a\,a\ldots\ldots$ is to be continued until it has been repeated p times. Thus

$$a + a + a + \ldots\ldots(p) = p\,a$$
$$a \times a \times a \ldots\ldots(p) = a^p.$$

The results at which we have arrived in this chapter, translated into this new language, are as follows:—

$$\left(x^{\frac{1}{n}}\right)^n = \left(x^n\right)^{\frac{1}{n}} = x \qquad (1)$$

$$\left(\mathrm{ABC}\right)^{\frac{1}{n}} = \mathrm{A}^{\frac{1}{n}}\,\mathrm{B}^{\frac{1}{n}}\,\mathrm{C}^{\frac{1}{n}} \qquad (2)$$

$$\left(a^{\frac{1}{n}}\right)^{\frac{1}{q}} = a^{\frac{1}{nq}} \qquad (3)$$

$$\left(a^m\right)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}} \qquad (4)$$

$$a^{\frac{m}{n}} = a^{\frac{mp}{np}} \qquad (5)$$

The advantages resulting from the adoption of this notation, are, 1st, that time is saved in writing algehraical expressions; 2dly, all rules which have been shown to hold good for performing operations upon such quantities as a^m, hold good also for performing the same operations upon such quantities as $a^{\frac{m}{n}}$, in which the exponents are fractional. The truth of this last assertion we proceed to establish.

Suppose it required to multiply together $a^{\frac{m}{n}}$ and $a^{\frac{l}{n}}$, or $\sqrt[n]{a^m}$ and $\sqrt[n]{a^l}$. From (2) this is $\sqrt[n]{a^m \times a^l}$, or $\sqrt{a^{m+l}}$, or $a^{\frac{m+l}{n}}$. Suppose it now required to multiply $a^{\frac{m}{n}}$ and $a^{\frac{p}{q}}$. From (5) the first of these is the same as $a^{\frac{mq}{nq}}$, and the second is the same as $a^{\frac{np}{nq}}$. The product of these by the last case is $a^{\frac{mq+np}{nq}}$, or $\sqrt[nq]{a^{mq+np}}$. But $\frac{mq+np}{nq}$ is $\frac{m}{n}+\frac{p}{q}$, and therefore

$$a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}} \qquad (6)$$

This is the same result as was obtained when the indices were whole numbers. The rule is;—to multiply together two powers of the same quantity, add the indices, and make the sum the index of the product. It follows in the same way that

$$\frac{a^{\frac{m}{n}}}{a^{\frac{p}{q}}} = a^{\frac{m}{n}-\frac{p}{q}} = a^{\frac{mq-pn}{nq}} = \sqrt[nq]{a^{mq-pn}}$$

or, to divide one power of a quantity by another, subtract the index of the divisor from that of the dividend, and make the difference the index of the result.

Suppose it required to find $\left(a^{\frac{m}{n}}\right)^p$. It is evident that $a^{\frac{m}{n}} \times a^{\frac{m}{n}} = a^{\frac{m}{n}+\frac{m}{n}} = a^{\frac{2m}{n}}$, or $\left(a^{\frac{m}{n}}\right)^2 = a^{\frac{2m}{n}}$. Similarly $\left(a^{\frac{m}{n}}\right)^3 = a^{\frac{3m}{n}}$, and so on. Therefore $\left(a^{\frac{m}{n}}\right)^p = a^{\frac{mp}{n}}$.

Again to find $\left(a^{\frac{m}{n}}\right)^{\frac{1}{q}}$, or $\sqrt[q]{a^{\frac{m}{n}}}$: Let this be $a^{\frac{x}{y}}$. Then $a^{\frac{x}{y}} = \sqrt[q]{a^{\frac{m}{n}}}$, or $\left(a^{\frac{x}{y}}\right)^q = a^{\frac{m}{n}}$, or $a^{\frac{xq}{y}} = a^{\frac{m}{n}}$. Therefore $\frac{xq}{y} = \frac{m}{n}$, or $\frac{x}{y} = \frac{m}{nq}$, and $\left(a^{\frac{m}{n}}\right)^{\frac{1}{q}} = a^{\frac{m}{nq}}$.

Again to find $\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}}$ or $\sqrt[q]{\left(a^{\frac{m}{n}}\right)^p}$. Apply the last two rules, and it appears that $\left(a^{\frac{m}{n}}\right)^p = a^{\frac{mp}{n}}$, and $\sqrt[q]{a^{\frac{mp}{n}}} = a^{\frac{mp}{nq}}$, therefore $\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = a^{\frac{mp}{nq}} = a^{\frac{m}{n}\times\frac{p}{q}}$.

And the rule is;—to raise one power of a quantity to another power, multiply the indices of the two powers together, and make the product the index of the result. All these rules are exactly those which have been shown to hold good when the indices are whole numbers. But there still remains one remarkable extension, which will complete this subject.

We have proved that whether m and n be whole or fractional numbers, $\frac{a^m}{a^n} = a^{m-n}$. The only cases which have been considered in forming this rule are those in which m is greater than n, being the only ones in which the subtracted indicated is possible. If we apply the rule to any other case, a new symbol is produced, which we proceed to consider. For example, suppose it required to find $\frac{a^3}{a^7}$. If we apply the rule, we find the

result a^{3-7}, or a^{-4}, for which we have hitherto no meaning. As in former cases, we must apply other methods to the solution of this case, and when we have obtained a rational result, a^{-4} may be used in future to stand for this result. Now the fraction $\frac{a^3}{a^7}$ is the same as $\frac{1}{a^4}$, which is obtained by dividing both its numerator and denominator by a^3. Therefore $\frac{1}{a^4}$ is the rational result, for which we have obtained a^{-4} by applying a rule in too extensive a manner. Nevertheless, if a^{-4} be made to stand for $\frac{1}{a^4}$, and a^{-m} for $\frac{1}{a^m}$, the rule will always give correct results, and the general rules for multiplication, division, and raising of powers remain the same as before. For example, $a^{-m} \times a^{-n}$ is $\frac{1}{a^m} \times \frac{1}{a^n}$, or $\frac{1}{a^m a^n}$, which is $\frac{1}{a^{m+n}}$, or $a^{-(m+n)}$, or a^{-m-n}. Similarly $\frac{a^{-m}}{a^{-n}}$, or $\dfrac{\frac{1}{a^m}}{\frac{1}{a^n}}$, is $\frac{a^n}{a^m}$, or a^{n-m} or $a^{-m-(-n)}$. Again $\left(a^m\right)^{-n}$ is $\frac{1}{(a^m)^n}$, or $\frac{1}{a^{mn}}$, or a^{-mn}, and so on. It has before been shown that a^0 stands for 1 whenever it occurs in the solution of a problem. We can now, therefore, assign a meaning to the expression a^m, whether m be whole or fractional, positive, negative, or nothing, and in all these cases the following rules hold good:—

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n} = a^m a^{-n}$$

$$\left(a^m\right)^n = \left(a^n\right)^m = a^{mn}.$$

The student can now understand the meaning of such an expression as $10^{.301}$, where the index or exponent is a decimal fraction. Since .301 is $\frac{301}{1000}$, this stands for $\sqrt[1000]{10^{301}}$, an expression of which it would be impossible to calculate the value by any method which the student has hitherto been taught, but which may be shown by other processes to be very nearly equal to 2.

Before proceeding to the practice of logarithmic calculations, the student should thoroughly understand the meaning of fractional and negative indices, and be familiar with the operations performed by means of them. He should work many examples of multiplication and division in which they occur, for which he can have recourse to any elementary work. The rules are the same as those to which he has been accustomed, substituting the addition, subtraction, &c. of fractional indices, instead of these which are whole numbers.

In order to make use of logarithms, he must provide himself with a table. Either of the following works may be recommended to him:—

1. Taylor's Logarithms.
2. Hutton's Logarithms.
3. Babbage; Logarithms of Numbers.
   Callet; Logarithms of Sines, Cosines, &c.
4. Bagay; Tables Astronomiques et Hydrographiques.

The first and last of these are large works, calculated for the most accurate operations of spherical trigonometry and astronomy. The second and third are better suited to the ordinary student. For those who require a pocket volume, there are Lalande's and Hassler's Tables, the first published in France, the second in the United States.

The definition, theory, and use of logarithms are fully given in the Treatise on Arithmetic and Algebra. The limits of this treatise will not allow us to enter into this subject. There is, however, one consideration connected with the tables, which, as it involves a principle of frequent application, it will be well to explain here. On looking into any table of logarithms it will be seen, that for a series of numbers the logarithms increase in arithmetical progression, as far as the first seven places of decimals are concerned; that is, the difference between the successive logarithms continue the same. For example, the following is found from any tables:

Log. 41713 = 4.6202714
Log. 41714 = 4.6202818
Log. 41715 = 4.6202922

The difference of these successive logarithms and of almost all others in the same page is .0000104. Therefore in this the addition of 1 to the number gives an addition of .0000104 to the logarithm. It is a general rule that,

when one quantity depends for its value upon another, as a logarithm does upon its number, or an algebraical expression, such as $x^2 + x$ upon the letter or letters which it contains, if a very small addition be made to the value of one of these letters, in consequence of which the expression itself is increased or diminished, generally speaking, the increment * of the expression will be very nearly proportional to the increment of the letter whose value is increased, and the more nearly so the smaller is the increment of the letter. We proceed to illustrate this. The product of two fractions, each of which is less than unity, is itself less than either of its factors. Therefore the square, cube, &c. of a fraction less than unity decrease, and the smaller the fraction is the more rapid is that decrease, as the following examples will show:—

$$\begin{aligned} \text{Let } x &= .01 \\ x^2 &= .0001 \\ x^3 &= .000001 \\ &\text{\&c.} \end{aligned} \qquad \begin{aligned} \text{Let } x &= .00001 \\ x^2 &= .0000000001 \\ x^3 &= .000000000000001 \\ &\text{\&c.} \end{aligned}$$

Now quantities are compared, not by the actual difference which exists between them, but by the number of times which one contains the other, and, of two quantities which are both very small, one may be very great as compared with the other. In the second example x^2 and x^3 are both small fractions when compared with unity; nevertheless, x^2 is very great when compared with x^3, being 100,000 times its magnitude. This use of the words small and great sometimes embarrasses the beginner; nevertheless, on consideration, it will appear to be very similar to the sense in which they are used in common life. We do not form our ideas of smallness or greatness from the actual numbers which are contained in a collection, but from the proportion which the numbers bear to those which are usually found in similar collections. Thus of 1000 men we should say, if they lived in one village, that it was extremely large; if they formed a regiment, that it was rather large; if an army, that it was utterly insignificant in point of numbers. Hence, in such an expression as $Ah + Bh^2 + Ch^3$, we may, if h is very small, reject $Bh^2 + Ch^3$, as being very small compared with Ah. An error will thus be committed, but a very small one only, and which becomes smaller as h becomes smaller.

Let us take any algebraical expression, such as $x^2 + x$, and suppose that x is increased by a very small quantity h. The expression then becomes $(x+h)^2 + (x+h)$, or $x^2 + x + (2x+1)\,h + h^2$. But it was $x^2 + x$; therefore, in consequence of x receiving the increment h, $x^2 + x$ has received the increment $\overline{2x+1}\,h + h^2$, for which $(2x+1)\,h$ may be written, since h is very small. This is proportional to h, since, if h were doubled, $\overline{2x+1}\,h$ would be doubled; also, if the first were halved the second would be halved, &c. In general, if y is a quantity which contains x, and if x be changed into $x+h$, y is changed into a quantity of the form $y + Ah + Bh^2 + Ch^3 + \&c.$; that is, y receives an increment of the form $Ah + Bh^2 + Ch^3 + \&c.$ If h be very small, this may, without sensible error, be reduced to its first term, viz. Ah, which is proportional to h. The general proof of this proposition belongs to a higher department of mathematics; nevertheless, the student may observe that it holds good in all the instances which occur in the treatise on Arithmetic and Algebra. For example (Tr. Arith. and Alg. page 89),

$$\overline{x+h}^m = x^m + mx^{m-1}h + m\frac{m-1}{2}x^{m-2}h^2 + \&c.$$

Here $A = mx^{m-1}$ $B = m\dfrac{m-1}{2}x^{m-2}$, &c.; and if h be very small, $\overline{x+h}^m = x^m + mx^{m-1}h$, nearly. Again (page 118), $e^h = 1 + h + \dfrac{h^2}{2} + \dfrac{h^3}{2.3} + \&c.$

Therefore, $e^x \times e^h$ or $e^{x+h} = e^x + e^x\,h + \dfrac{e^x}{2}h^2 + \&c.$

* When any quantity is increased, the quantity by which it is increased is called its *increment*.

And if h be very small, $e^{x+h} = e^x + e^x\ h$, nearly.

Again (page 119), log. $(1 + n') = \text{M}\ (n' - \frac{1}{2} n'^2 + \frac{1}{3} n'^3 - \&c.)$

To each side add log. x, recollecting that log. $x +$ log. $(1 + n') =$ log. $x\,(1 + n')$ $=$ log. $(x + xn')$, and let $xn' = h$ or $n' = \frac{x}{h}$. Making these substitutions, the equation becomes

$$\text{Log. } \overline{x+h} = \log. x + \frac{\text{M}}{x} h - \frac{\text{M}}{2x^2} h^2 + \&c.$$

If h is very small, $\log. \overline{x+h} = \log. x + \frac{\text{M}}{x} h.$

We can now apply this to the logarithmic example with which we commenced this subject. It appears that

Log. 41713 $= 4.6202714$
Log. $(41713 + 1) = 4.6202714 + .0000104$
Log. $(41713 + 2) = 4.6202714 + .0000104 \times 2.$

From which, and the considerations above-mentioned,

Log. $(41713 + h) =$ log. $41713 + .0000104 \times h$,

which is extremely near the truth, even when h is a much larger number, as the tables will show. Suppose, then, that the logarithm of 41713.27 is required. Here $h = .27$. It therefore only remains to calculate $.0000104 \times .27$, and add the result, or as much of it as is contained in the first seven places of decimals, to the logarithm of 41713. This trouble is saved in the tables in the following manner. The difference of the successive logarithms is written down, with the exception of the cyphers at the beginning, in the column marked D or Diff., under which are registered the tenths of that difference, or as much of them as is contained in the first seven decimal places, increasing the seventh figure by 1 when the eighth is equal to or greater than 5, and omitting the cyphers to save room. From this table of tenths the table of hundredth parts may be made by striking off the last figure, making the usual change in the last but one, when the last is equal to or greater than 5, and placing an additional cypher. The logarithm of 41713.27 is, therefore, obtained in the following manner:—

Log. 41713 $= 4.6202714$
$.0000104 \times .2 = .0000021$
$.0000104 \times .07 = .0000007$

Log. 41713.27 $= 4.6202742$

This, when the useless cyphers and parts of the operation are omitted, is the process given in all the books of logarithms. If the logarithm of a number containing more than seven significant figures be sought, for example 219034.717, recourse must be had to a table, in which the logarithms are carried to more than seven places of decimals. The fact is, that in the first seven places of decimals there is no difference between log. 219034.7 and log. 219034.717. For an excellent treatise on the practice of logarithms the reader may consult the preface to Babbage's Table of Logarithms.

Chapter XII.

In this chapter we shall give the student some advice as to the manner in which he should prosecute his studies in algebra. The remaining parts of this subject present a field infinite in its extent and in the variety of the applications which present themselves. By whatever name the remaining parts of the subject may be called, even though the ideas on which they are based may be geometrical, still the mechanical processes are algebraical, and present continual applications of the preceding rules and developments of the subjects already treated. This is the case in Trigonometry, the application of Algebra to Geometry, the Differential Calculus, or Fluxions, &c.

I. The first thing to be attended to in reading any algebraical treatise, is the gaining a perfect understanding of the different processes there exhibited, and of their connexion with one another. This cannot be attained by a mere reading of the book, however great the attention which may be given. It is impossible, in a mathematical work, to fill up every process in the manner in which it must be filled up in the mind of the student before he can be said to have

completely mastered it. Many results must be given, of which the details are suppressed, such are the additions, multiplications, extractions of the square root, &c. with which the investigations abound. These must not be taken on trust by the student, but must be worked by his own pen, which must never be out of his hand while engaged in any algebraical process. The method which we recommend is, to write the whole of the symbolical part of each investigation, filling up the parts to which we have alluded, adding only so much verbal elucidation as is absolutely necessary to explain the connexion of the different steps, which will generally be much less than what is given in the book. This may appear an alarming labour to one who has not tried it, nevertheless we are convinced that it is by far the shortest method of proceeding, since the deliberate consideration which the act of writing forces us to give, will prevent the confusion and difficulties which cannot fail to embarrass the beginner if he attempt, by mere perusal only, to understand new reasoning expressed in new language. If, while proceeding in this manner, any difficulty should occur, it should be written at full length, and it will often happen that the misconception which occasioned the embarrassment will not stand the trial to which it is thus brought. Should there be still any matter of doubt which is not removed by attentive reconsideration, the student should proceed, first making a note of the point which he is unable to perceive. To this he should recur in his subsequent progress, whenever he arrives at anything which appears to have any affinity, however remote, to the difficulty which stopped him, and thus he will frequently find himself in a condition to decypher what formerly appeared incomprehensible. In reasoning purely geometrical, there is less necessity for committing to writing the whole detail of the arguments, since the symbolical language is more quickly understood, and the subject is in a great measure independent of the mechanism of operations; but, in the processes of algebra, there is no point on which so much depends, or on which it becomes an instructor more strongly to insist.

II. On arriving at any new rule or process, the student should work a number of examples sufficient to prove to himself that he understands and can apply the rule or process in question. Here a difficulty will occur, since there are many of these in the books, to which no examples are formally given. Nevertheless, he may choose an example for himself, and his previous knowledge will suggest some method of proving whether his result is true or not. For example, the development of $\overline{a+x}|^{\frac{7}{3}}$ will exercise him in the use of the binomial theorem; when he has obtained the series which is equivalent to $\overline{a+x}|^{\frac{7}{3}}$, let him, in the same way, develop $\overline{a+x}^{\frac{2}{3}}$; the product of these, since $\frac{7}{3}+\frac{2}{3}=3$, ought to be the same as the development of $\overline{a+x}^3$, or as $a^3+3a^2x+3ax^2+x^3$. He may also try whether the development of $\overline{a+x},|^{\frac{1}{2}}$ by the binomial theorem, gives the same result as is obtained by the extraction of the square root of $a+x$. Again, when any development is obtained, it should be seen whether the development possesses all the properties of the expression from which it has been derived. For example, $\frac{1}{1-x}$ is proved to be equivalent to the series $1+x+x^2+x^3+$, &c., *ad infinitum.* This, when multiplied by $1-x$, should give 1; when multiplied by $1-x^2$, should give $1+x$, because

$$\frac{1}{1-x}\times(1-x)=1 \qquad \frac{1}{1-x}\times(1-x^2)=1+x, \text{ \&c.}$$

$$\text{Again, } a = 1+x\,\text{Log}\,a+\frac{x^2\,\overline{\text{Log}\,a}|^2}{2}+\frac{x^3\,\overline{\text{Log}\,a}|^3}{2.3}+\text{ \&c. } \textit{ad inf.}$$

$$a^y = 1+y\,\text{Log}\,a+\frac{y^2\,\overline{\text{Log}\,a}|^2}{2}+, \text{ \&c.}$$

$$a^{x+y} = 1+\overline{x+y}\,\text{Log}\,a+\frac{\overline{x+y}|^2\,\overline{\text{Log}\,a}|^2}{2}+\text{ \&c.}$$

Now, since $a^x \times a^y = a^{x+y}$, the product of the two first series should give the third. Many other instances of the same sort will suggest themselves, and a careful attention to them will confirm the demonstration of the several theorems, which, to a beginner, is often doubtful, on account of the generality of the reasoning.

III. Whenever a demonstration appears perplexed, on account of the number and generality of the symbols, let some particular case be chosen, and let the same demonstration be applied. For example, if the binomial theorem should not appear sufficiently plain, the same reasoning may be applied to the expansion of $\overline{1+x}^{\frac{2}{3}}$, or any other case, which is there applied to $\overline{1+x}^{\frac{m}{n}}$. Again, the general form of the product $(x+a)$, $(x+b)$, $(x+c)$, &c. ... containing n factors, will be made apparent by taking first two, then three, and four factors, before attempting to apply the reasoning which establishes the form of the general product. The same applies particularly to the theory of permutations and combinations, and to the doctrine of probabilities, which is so materially connected with it. In the theory of equations it will be advisable at first, instead of taking the general equation of the form

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Lx + M = 0,$$

to choose that of the third, or at most of the fourth degree, or both, on which to demonstrate all the properties of expressions of this description. But in all these cases, when the particular instances have been treated, the general case should not be neglected, since the power of reasoning upon expressions such as the one just given, in which all the terms cannot be written down, on account of their indeterminate number, must be exercised, before the student can proceed with any prospect of success to the higher branches of mathematics.

IV. When any previous theorem is referred to, the reference should be made, and the student should satisfy himself that he has not forgotten its demonstration. If he finds that he has done so, he should not grudge the time necessary for its recovery. By so doing, he will avoid the necessity of reading over the subject again, and will obtain the additional advantage of being able to give to each part of the subject a time nearly proportional to its importance, whereas, by reading a book over and over again until he is master of it, he will not collect the more prominent parts, and will waste time upon unimportant details, from which even the best books are not free. The necessity for this continual reference is particularly felt in the Elements of Geometry, where allusion is constantly made to preceding propositions, and where many theorems are of no importance, considered as results, and are merely established in order to serve as the basis of future propositions.

V. The student should not lose any opportunity of exercising himself in numerical calculation, and particularly in the use of the logarithmic tables. His power of applying mathematics to questions of practical utility is in direct proportion to the facility which he possesses in computation. Though it is in plane and spherical trigonometry that the most direct numerical applications present themselves, nevertheless the elementary parts of algebra abound with useful practical questions. Such will be found resulting from the binomical theorem, the theory of logarithms, and that of continued fractions. The first requisite in this branch of the subject, is a perfect acquaintance with the arithmetic of decimal fractions; such a degree of acquaintance as can only be gained by a knowledge of the principles as well as of the rules which are deduced from them. From the imperfect manner in which arithmetic is usually taught, the student ought in most cases to recommence this study before proceeding to the practice of logarithms.

VI. The greatest difficulty, in fact almost the only one of any importance which algebra offers to the reason, is the use of the isolated negative sign in such expressions as $-a$, a^{-x}, and the symbols which we have called imaginary. It is a remarkable fact, that the first elements of the mathematics, sciences which demonstrate their results with more certainty than any others, contain difficulties which have been the subjects of discussion for centuries. In geometry, for example, the theory of parallel lines has never yet been freed from the difficulty which presented itself to Euclid, and obliged him to assume, instead of proving the 12th axiom of his first book. Innumerable as have been the attempts

to elude or surmount this obstacle, no one has been more successful than another. The elements of fluxions or the differential calculus, of mechanics, of optics, and of all the other sciences, in the same manner contain difficulties peculiar to themselves. These are not such as would suggest themselves to the beginner, who is usually embarrassed by the actual performance of the operations, and no ways perplexed by any doubts as to the foundations of the rules by which he is to work. It is the characteristic of a young student in the mathematical sciences, that he sees, or fancies that he sees, the truth of every result which can be stated in a few words, or arrived at by few and simple operations, while that which is long is always considered by him as abstruse. Thus while he feels no embarrassment as to the meaning of the equation $+ a \times - a = - a^2$, he considers the multiplication of $a^m + a^n$ by $b^m + b^n$ as one of the difficulties of algebra. This arises, in our opinion, from the manner in which his previous studies are usually conducted. From his earliest infancy, he learns no fact from his own observation, he deduces no truth by the exercise of his own reason. Even the tables of arithmetic, which, with a little thought and calculation, he might construct for himself, are presented to him ready made, and it is considered sufficient to commit them to memory. Thus a habit of examination is not formed, and the student comes to the science of algebra fully prepared to believe in the truth of any rule which is set before him, without other authority than the fact of finding it in the book to which he is recommended. It is no wonder, then, that he considers the difficulty of a process as proportional to that of remembering and applying the rule which is given, without taking into consideration the nature of the reasoning on which the rule was founded. We are not advocates for stopping the progress of the student by entering fully into all the arguments for and against such questions, as the use of negative quantities, &c., which he could not understand, and which are inconclusive on both sides; but he might be made aware that a difficulty does exist, the nature of which might be pointed out to him, and he might then, by the consideration of a sufficient number of examples, treated separately, acquire confidence in the results to which the rules lead. Whatever may be thought of this method, it must be better than an unsupported rule, such as is given in many works on algebra.

It may perhaps be objected that this is induction, a species of reasoning which is foreign to the usually received notions of mathematics. To this it may be answered, that inductive reasoning is of as frequent occurrence in the sciences as any other. It is certain that most great discoveries have been made by means of it; and the mathematician knows that one of his most powerful engines of demonstration is that peculiar species of induction which proves many general truths by demonstrating that, if the theorem be true in one case, it is true for the succeeding one. But the beginner is obliged to content himself with a less rigorous species of proof, though equally conclusive, as far as moral certainty is concerned. Unable to grasp the generalizations with which the more advanced student is familiar, he must satisfy himself of the truth of general theorems by observing a number of particular simple instances which he is able to comprehend. For example, we would ask any one who has gone over this ground, whether he derived more certainty as to the truth of the binomial theorem from the general demonstration (if indeed he was suffered to see it so early in his career), or from observation of its truth in the particular cases of the development of $\overline{a+b}^2$, $\overline{a+b}^3$, &c., substantiated by ordinary multiplication. We believe firmly, that to the mass of young students, general demonstrations afford no conviction whatever; and that the same may be said of almost every species of mathematical reasoning, when it is entirely new. We have before observed, that it is necessary to learn to reason; and in no case is the assertion more completely verified than in the study of algebra. It was probably the experience of the inutility of general demonstrations to the very young student that caused the abandonment of reasoning which prevailed so much in English works on elementary mathematics. Rules which the student could follow in practice supplied the place of arguments which he could not, and no pains appear to have been taken to adopt a middle course, by suiting the nature of the proof to the student's capacity. The objection to this appears

to have been the necessity which arose for departing from the appearance of rigorous demonstration. This was the cry of those, who not having seized the spirit of the processes which they followed, placed the force of the reasoning in the forms. To such the authority of great names is a strong argument; we will therefore cite the words of LAPLACE on this subject.

"Newton extended to fractional and negative powers the analytical expression which he had found for whole and positive ones. You see in this extension one of the great advantages of algebraic language, which expresses truths much more general than those which were at first contemplated, so that by making the extensions of which it admits, there arises a multitude of new truths out of formulæ which were founded upon very limited suppositions. At first, people were afraid to admit the general consequences with which analytical formulæ furnished them; *but a great number of examples having verified them*, we now, without fear, yield ourselves to the guidance of analysis through all the consequences to which it leads us, and the most happy discoveries have sprung from the boldness. We must observe, however, that precautions should be taken to avoid giving to formulæ a greater extension than they really admit, and that it is always well to demonstrate rigorously the results which are obtained."

We have observed, that beginners are not disposed to quarrel with a rule which is easy in practice, and verified by examples, on account of difficulties which occur in its establishment. The early history of the sciences presents occasion for the same remark. In the work of Diophantus, the first Greek writer on algebra, we find a principle equivalent to the equations $+ a \times - b = - a b$, and $- a \times - b = + a b$, admitted as an axiom, without proof or difficulty. In the Hindoo works on algebra, and the Persian commentators upon them, the same thing takes place. It appears, that struck with the practical utility of the rule, and certain by induction of its truth, they did not scruple to avail themselves of it. A more cultivated age, possessed of many formulæ whose developments presented striking examples of an universality in algebraic language not contemplated by its framers, set itself to inquire more closely into the first principles of the science. Long and still unfinished discussions have been the result, but the progress of nations has exhibited throughout a strong resemblance to that of individuals.

VII. The student should make for himself a syllabus of results only, unaccompanied by any demonstration. It is essential to acquire a correct memory for algebraical formulæ, which will save much time and labour in the higher departments of the science. Such a syllabus will be a great assistance in this respect, and care should be taken that it contain only the most useful and most prominent formulæ. Whenever that can be done, the student should have recourse to the system of tabulation, of which he will have seen several examples in this treatise. In this way he should write the various forms which the roots of the equation $ax^2 + bx + c = 0$ assume, according to the signs of a, b, and c, &c. Both the preceptor and the pupil, but especially the former, will derive great advantage from the perusal of *Lacroix, Essais sur l'Enseignement en général et sur celui des Mathématiques en particulier, Condillac, La Langue des Calculs*, and the various articles on the elements of algebra in the French Encyclopedia, which are for the most part written by D'Alembert. The reader will here find the first principles of algebra, developed and elucidated in a masterly manner. A great collection of examples will be found in most elementary works, but particularly in Hirsch, *Sammlung von Beispielen*, &c. translated into English under the title of *Self Examinations in Algebra, &c.*, London, Black, Young, and Young, 1825. The student who desires to carry his algebraical studies farther than usual, and to make them the stepping-stone to a knowledge of the higher mathematics, should be acquainted with the French language. A knowledge of this, sufficient to enable him to read the simple and easy style in which the writers of that nation treat the first principles of every subject, may be acquired in a short time. When that is done, we recommend to the student the algebra of M. Bourdon, a work of eminent merit, though of some difficulty to the English student, and requiring some previous habits of algebraical reasoning.

VIII. The height to which algebraical studies should be carried must depend upon the purpose to which they are to

be applied. For the ordinary purposes of practical mathematics, algebra is principally useful as the guide to trigonometry, logarithms, and the solution of equations. Much and profound study is not therefore requisite; the student should pay great attention to all numerical processes, and particularly to the methods of approximation which he will find in all the books. His principal instrument is the table of logarithms, of which he should secure a knowledge both theoretical and practical. The course which should be adopted preparatory to proceeding to the higher branches of mathematics is different. It is still of great importance that the student should be well acquainted with numerical applications; nevertheless, he may omit with advantage many details relative to the obtaining of approximative numerical results, particularly in the theory of equations of higher degrees than the second. Instead of occupying himself upon these, he should proceed to the application of algebra to geometry, and afterwards to the differential calculus. When a competent knowledge of these has been obtained, he may then revert to the subjects which he has neglected, giving them more or less attention according to his own opinion of the use which he is likely to have for them. This applies particularly to the theory of equations, which abounds with processes of which very few students will afterwards find the necessity.

We shall proceed in the next number to the difficulties which arise in the study of Geometry and Trigonometry.

CHAPTER XIII.

On the Definitions of Geometry.

In this treatise on the difficulties of Geometry and Trigonometry, we propose, as in the former part of the work, to touch on those points only which, from novelty in their principle, are found to present difficulties to the student, and which are frequently not sufficiently dwelt upon in elementary works. Perhaps it may be asserted, that there are no difficulties in geometry which are likely to place a serious obstacle in the way of an intelligent beginner, except the temporary embarrassment which always attends the commencement of a new study; that, for example, there is nothing in the elements of pure geometry comparable, in point of complexity, to the theory of the negative sign, of fractional indices, or of the decomposition of an expression of the second degree into factors. This may be true; and were it only necessary to study the elements of this science for themselves, without reference to their application, by means of algebra, to higher branches of knowledge, we should not have thought it necessary to call the attention of our readers to the points which we shall proceed to place before them. But while there is a higher study in which elementary ideas, simple enough in their first form, are so generalized as to become difficult, it will be an assistance to the beginner who intends to proceed through a wider course of pure mathematics than forms part of common education, if his attention is early directed, in a manner which he can comprehend, to future extensions of what is before him.

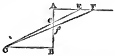
The reason why geometry is not so difficult as algebra, is to be found in the less general nature of the symbols employed. In algebra a general proposition respecting numbers is to be proved. Letters are taken which may represent any of the numbers in question, and the course of the demonstration, far from making any use of a particular case, does not even allow that any reasoning, however general in its nature, is conclusive, unless the symbols are as general as the arguments. We do not say that it would be contrary to good logic to form general conclusions from reasoning on one particular case, when it is evident that the same considerations might be applied to any other, but only that very great caution, more than a beginner can see the value of, would be requisite in deducing the conclusion. There occurs also a mixture of general and particular propositions, and the latter are liable to be mistaken for the former. In geometry on the contrary, at least in the elementary parts, any proposition may be safely demonstrated by reasonings on any one particular example. For though in proving a property of a triangle many truths regarding that triangle may be asserted as having been proved before, none are brought forward which are not general, that is, true for all instances of the same kind. It also affords some facility that the results of elementary geometry are in many cases sufficiently evident of themselves to the eye; for instance, that two sides of a triangle are greater than the third, whereas in algebra many rudimentary propositions derive no evidence from the senses; for example, that $a^3 - b^3$ is always divisible without remainder by $a - b$.

The definitions of the simple terms point, line, and surface have given rise to much discussion. But the difficulties which attend them are not of a nature to embarrass the beginner, provided he will rest content with the notions which he has already derived from observation. No explanation can make these terms more intelligible. To them may be added the words straight line, which cannot be mistaken for one moment, unless it be by means of the attempt to explain them by saying that a straight line is 'that which lies evenly between its extreme points.'

The line and surface are distinct species of magnitude, as much so as the yard and the acre. The first is no part of the second, that is, no number of lines can make a surface. When therefore a surface is divided into two parts by a line, the dividing line is not to be considered as forming a part of either. That the idea of the line or boundary necessarily enters into the notion of the division is very true; but if we conceive the line abstracted, and thus get rid of the idea of division, neither surface is

increased or diminished, which is what we mean when we say that the line is not a part of the surface. The same considerations apply to a point, considered as the boundary of the divisions of a line.

The beginner may perhaps imagine that a line is made up of points, that is, that every line is the sum of a number of points, a surface the sum of a number of lines, and so on. This arises from the fact, that the things which we draw on paper as the representatives of lines and points, have in reality three dimensions, two of which, length and breadth, are perfectly visible. Thus the point, such as we are obliged to represent it, in order to make its position visible, is in reality a part of our line, and our points, if sufficiently multiplied in number and placed side by side, would compose a line of any length whatever. But taking the mathematical definition of a point, which denies it all magnitude, either in length, breadth, or thickness, and of a line, which is asserted to possess length only without breadth or thickness, it is easy to shew that a point is no part of a line, by making it appear that the shortest line can be cut in as many points as the longest, which may be done in the following manner. Let A B be any straight line, from the ends of which, A and B, draw

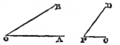


two lines, A F and C B, parallel to one another. Consider A F as produced without limit, and in C B take any point C, from which draw lines C E, C F, &c., to different points in A F. It is evident that for each point E in A F there is a distinct point in A B, viz., the intersection of C E with A B;—for, were it possible that two points, E and F in AF, could be thus connected with the same point of A B, it is evident that two straight lines would enclose a space, viz., the lines C E and CF, which both pass through C, and would, were our supposition correct, also pass through the same point in A B. There can then be taken as many points in the finite or bounded line A B as in the indefinitely extended line A F.

The next definition which we shall consider is that of a *plane surface*. The word plane or flat is as hard to define, without reference to any thing but the idea we have of it, as it is easy to understand. Nevertheless the practical method of ascertaining whether or no a surface is plane, will furnish a definition, not such, indeed, as to render the nature of a plane surface more evident, but which will serve, in a mathematical point of view, as a basis on which to rest the propositions of solid geometry. If the edge of a ruler, known to be perfectly straight, coincides with a surface throughout its whole length, in whatever direction it may be placed upon that surface, we conclude that the surface is plane. Hence the definition of a plane surface is that in which, any two points being taken, the straight line joining these points lies wholly upon the surface.

Two straight lines have a relation to one another independent altogether of their length. This we commonly express (for among the most common ideas are found the germ of every geometrical theory) by saying that they are in the same or different *directions*. By the *direction* of the needle we ascertain the *direction* in which to proceed at sea, and by the *direction* in which the hands of a clock are placed we tell the hour. It remains to reduce this common notion to a more precise form.

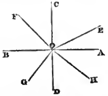
Suppose a straight line O A to be given in magnitude and position, and to remain fixed while another line O B, at



first coincident with O A, is made to move round O A, so as continually to vary its direction with respect to O A. The process of opening a pair of compasses will furnish an illustration of this, but the two lines need not be equal to one another. In this case the opening made by the two will continually increase, and this opening is a species of magnitude, since one opening may be compared with another, so as to ascertain which of the two is the greater. Thus if the fig. CPD be removed from its place, without any other change, so that the point P may fall on O, and the line P C may lie upon and become a part of O A, or OA of PC, according to which is the longer of the two, then if the opening C P D is the same as the opening A O B, P D will lie upon A B at the same time

as P C lies upon O A. But if P D does not then lie upon O B, but falls between O B and OA, the opening C P D is less than the opening A O B, and if P D does not fall between O A and O B, or on O B, the opening C P D is greater than the opening B O A. To this species of magnitude, the opening of two lines, the name of angle is given, that is B O is said to make an angle with O A. The difficulty here arises from this magnitude being one, the measure of which has seldom fallen under observation of those who begin geometry. Every one has measured one line by means of another, and has thus made a number the representative of a length; but few, at this period of their studies, have been accustomed to the consideration, that one opening may be contained a certain number of times in another, or may be a certain fraction of another. Nevertheless we may find measures of this new species of magnitude either by means of time, length, or number.

One magnitude is said to be a measure of another, when, if the first be doubled, trebled, halved, &c., the second is doubled, trebled, or halved, &c.; that is, when any fraction or multiple of the first corresponds to the same fraction or multiple of the second in the same manner as the first does to the second. The two quantities need not be of the same kind: thus, in the barometer the height of the mercury (a length) measures the pressure of the atmosphere (a weight); for if the barometer which yesterday stood at 28 inches, to-day stand at 29 inches, in which case the height of yesterday is increased by its 28th part, we know that the atmospheric pressure of yesterday is increased by its 28th part to-day. Again, in a watch, the *number of hours* elapsed since twelve o'clock is measured by the *angle* which a hand makes with the position it occupied at twelve o'clock. In the spring balances a *weight* is measured by an *angle*, and



many other similar instances might be given.

This being premised, suppose a line which moves round another as just described, to move uniformly, that is, to describe equal openings or angles in equal times. Suppose the line OA to move completely round, so as to reassume its first position in twenty-four hours. Then in twelve hours the moving line will be in the position O B, in six hours it will be in O C, and in eighteen hours in O D. The line O C is that which makes equal angles with O A and O B, and is said to be at right angles, or perpendicular to O A and O B. Again, O A and O B which are in the same right line, but on opposite sides of the point O, evidently make an opening or angle which is equal to the sum of the angles A O C and C O B, or equal to two right angles. A line may also be said to make with itself an opening equal to four right angles, since after revolving through four right angles, the moving line reassumes its original position. We may even carry this notion further: for if the moving line be in the position O E when P hours have elapsed, it will recover that position after every twenty-four hours, that is, for every additional four right angles described; so that the angle A O E is equally well represented by any of the following angles:

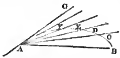
4 right angles + A O E
8 right angles + A O E
12 right angles + A O E
&c. &c. &c.

These formulæ which suppose an opening greater than any *apparent* opening, and which take in and represent the fact that the moving line has attained its position for the second, third, fourth, &c., time, since the commencement of the motion, are not of any use in elementary geometry; but as they play an important part in the application of algebra to the theory of angles, we have thought it right to mention them here.

It is plain also that we may conceive the line OE to make two openings or angles with the original position O A 1. that through which it has moved to recede from OA; 2. that through which it must move to reach O A again. The first (in the position in which we have placed O A) is what is called in geometry the angle AO E; the second is more simply described as composed of the openings or angles EOC, COB, BOD, DOA, and is not used except in the ap-

plication of algebra above mentioned. Of the two angles just alluded to, one must be less than two right angles, and the second greater; the first is the one usually referred to.

It is plain that the angle or opening made by two lines does not depend upon their length but upon their position; if either be shortened or lengthened, the angle still remains the same; and if while the angle increases or decreases one of the straight lines containing it is diminished, the angle so contained may have a definite and given magnitude at the moment when the straight line disappears altogether and becomes nothing. For example, take two points of any curve A B, and join A and B by a straight line. Let the point B move towards A; it is evident that the angle made by the moving line with A B increases continually, while as much of



one of the lines containing it as is intercepted by the curve, diminishes without limit. When this intercepted part disappears entirely, the line in which it would have lain had it had any length, has reached the line A G, which is called the tangent of the curve.

In elementary geometry two equal angles lying on different sides of a line, such as A O E, A O H, would be considered as the same. In the application of algebra, they would be considered as having different signs, for reasons stated at length in pages 37, &c., of the first part of this Treatise. It is also common in the latter branch of the science, to measure angles in one direction only; for example, in the same figure, the angles made by O E, O F, O G, and O H, if measured upwards from O A, would be the openings through which a line must move *in the same direction* from O A, to attain those positions; and the second, third, and fourth angles would be greater than one, two, and three right angles respectively.

We proceed to the method of reasoning in geometry, or rather to the method of reasoning in general, since there is, or ought to be, no essential difference between the manner of deducing results from first principles, in any science.

CHAPTER XIV.

On Geometrical Reasoning.

It is evident that all reasoning, of what form soever, can be reduced at last to a number of simple propositions or assertions; each of which, if it be not self-evident, depends upon those which have preceded it. Every assertion can be divided into three distinct parts. Thus the phrase 'all right angles are equal,' consists of—1, the *subject* spoken of, viz., right angles, which is here spoken of universally, since every right angle is a part of the subject;—2, the *copula*, or manner in which the two are joined together, which is generally the verb *is*, or *is equal to*, and can always be reduced to one or the other: in this case the copula is affirmative;—3, the *predicate*, or thing asserted of the subject, viz., equal angles. The phrase, thus divided, stands as written below, and is called universally affirmative. The second is called a particular affirmative proposition; the third a universal negative; the fourth a particular negative.

All right angles are equal, (magnitudes).

Some triangles are equilateral, (figures).

No circle is convex to its diameter.

Some triangles are not equilateral, (figures).

Many assertions appear in a form which, at first sight, cannot be reduced to one of the preceding: the following are instances of the change which it is necessary to make in them.

Parallel lines never meet, or parallel lines are lines which never meet. The angles at the base of an isosceles triangle are equal, or an isosceles triangle is a triangle having the angles at the base equal.

The different species of assertions, and the arguments which are compounded of them, may be distinctly conceived by referring them all to one species of subject and predicate. Since every assertion, generally speaking, includes a number of individual cases in its subject, let the points of a circle be the subject and those of a triangle the predicate. These figures being drawn, the four species of assertions just alluded to are as follows:—

1. Every point of the circle is a point of the triangle, or the circle is contained in the triangle.

2. Some points of the circle are points of the triangle, or part of the circle is contained in the triangle.

3. No point of the circle is a point of the triangle, or the circle is entirely without the triangle.

4. Some points of the circle are not points of the triangle, or part of the circle is outside the triangle.

On these we observe that the second follows from the first, as also the fourth from the third, since that which is true of all is true of some or any; while the first and third do not follow from the second and fourth, *necessarily*, since that which is true of some only need not be true of all. Again, the second and fourth are not necessarily inconsistent with each other for the same reason. Also two of these assertions must be true and the others untrue. The first and the third are called contraries, while the first and fourth, and the second and third are contradictory. The *converse* of a proposition is made by changing the predicate into the subject, and the subject into the predicate. No mistake is more common than confounding together a proposition and its converse, the tendency to which is rather increased in those who begin geometry, by the number of propositions which they find, the converses of which are true. Thus all the definitions are necessarily conversely true, since the identity of the subject and predicate is not merely asserted, but the subject is declared to be a name *given* to *all* those magnitudes which have the properties laid down in the predicate, and to no others. Thus a square is a four-sided figure having equal sides and one right angle, that is, let every four-sided figure having &c., *be called* a square, and let no other figure be called by that name, whence the truth of the converse is evident. Also many of the facts proved in geometry are conversely true. Thus all equilateral triangles are equiangular, from which it is proved that *all* equiangular triangles are equilateral. Of the first species of assertion, the universal affirmative, the converse is not necessarily true. Thus 'every point in figure A is a point of B,' does not imply that 'every point of B is a point of A,' although this may be the case, and is, if the two figures coincide entirely. The second species, the particular affirmative, is conversely true, since if some points of A are points of B, some points of B are also points of A. The first species of assertion is conversely true, if the converse be made to take the form of the second species: thus from 'all right angles are equal,' it may be inferred that '*some* equal magnitudes are right angles.' The third species, the universal negative, is conversely true, since if 'no point of B is a point of A,' it may be inferred that 'no point of A is a point of B.' The fourth species, the particular negative, is not necessarily conversely true. From 'some points of A are not points of B,' or A is not entirely contained within B, we can infer nothing as to whether B is or is not entirely contained in A. It is plain that the converse of a proposition is not necessarily true, if it says more either of the subject or predicate than was said before. Now 'every equilateral triangle is equiangular,' does not speak of all equiangular triangles, but asserts that *among* all possible equiangular triangles are to be found *all* the equilateral ones. There may then, for any thing to the contrary to be discovered in our assertion, be classes of equiangular triangles not included under this assertion, of which we can therefore say nothing. But in saying 'no right angles are unequal,' that which we exclude, we exclude from all unequal angles, and therefore 'no unequal angles are right angles' is not more general than the first.

The various assertions brought forward in a geometrical demonstration must be derived in one of the following ways.

I. From definition. This is merely substituting, instead of a description, the name which it has been agreed to give to whatever bears that description. No definition ought to be introduced until it is certain that the thing defined is really possible. Thus though parallel lines are defined to be 'lines which are in the same plane, and which being ever so far produced never meet,' the mere agreement to call such lines, should they exist, by the name of parallels, is no sufficient ground to assume that they do exist. The definition is therefore inadmissible until it is really shewn that there are such things as lines which being in the same plane never meet. Again, before applying the name, care must be taken that all the circumstances connected with the definition have been attended to. Thus, though in plane geometry, where all lines are in one plane, it is sufficient that two lines would never meet though ever so far produced, to call them parallel, yet in solid geometry the first circumstance must be attended to, and it must be shewn that lines are in the same plane before the name can be

applied. Some of the axioms come so near to definitions in their nature, that their place may be considered as doubtful. Such are, 'the whole is greater than its part,' and 'magnitudes which entirely coincide are equal to one another.'

II. From hypothesis. In the statement of every proposition, certain connexions are supposed to exist from which it is asserted that certain consequences will follow. Thus 'in an isosceles triangle the angles at the base are equal,' or, 'if a triangle be isosceles the angles at the base will be equal.' Here the hypothesis or supposition is that the triangle has two equal sides, the consequence asserted is that the angles at the base or third side will be equal. The consequence being only asserted to be true when the angle is isosceles, such a triangle is supposed to be taken as the basis of the reasonings, and the condition that its two sides are equal, when introduced in the proof, is said to be introduced by hypothesis.

In order to establish the result it may be necessary to draw other lines, &c., which are not mentioned in the first hypothesis. These, when introduced, form what is called the construction.

There is another species of hypothesis much in use, principally when it is required to deduce the converse of a theorem from the theorem itself. Instead of proving the consequence directly, the contradictory of the consequence is assumed to hold good, and if from this new hypothesis, supposed to exist together with the old one, any evidently absurd result can be derived, such as that the whole is greater than its part, this shews that the two hypotheses are not consistent, and that if the first be true, the second cannot be so. But if the second be not true, its contradictory is true, which is what was required to be proved.

III. From the evidence of the assertions themselves. The propositions thus introduced without proof are only such as are in their nature too simple to admit of it. They are called axioms. But it is necessary to observe, that the claim of an assertion to be called an axiom does not depend only on its being self-evident. Were this the case many propositions which are always proved might be assumed; for example, that two sides of a triangle are greater than the third, or that a straight line is the shortest distance between two points. In addition to being self-evident, it must be incapable of proof by any other means, and it is one of the objects of geometry to reduce the demonstrations to the least possible number of axioms. There are only two axioms which are distinctly geometrical in their nature, viz., 'two straight lines cannot enclose a space,' and 'through each point outside a line, not more than one parallel to that line can be drawn.' All the rest of the propositions commonly given as axioms are either arithmetical in their nature; such as 'the whole is greater than its part,' 'the doubles of equals are equals,' &c.; or mere definitions, such as 'magnitudes which entirely coincide are equal;' or theorems admitting of proof, such as 'all right angles are equal.' There is however one more species of self-evident proposition, the postulate or self-evident problem, such as the possibility of drawing a right line, &c.

IV. From proof already given. What has been proved once may be always taken for granted afterwards. It is evident that this is merely for the sake of brevity, since it would be possible to begin from the axioms and proceed direct to the proof of any one proposition, however far removed from them; and this is an exercise which we recommend to the student. Thus much for the legitimate use of any single assertion or proposition. We proceed to the manner of deducing a third proposition from two others.

It is evident that no assertion can be the direct and necessary consequence of two others, unless those two contain something in common, or which is spoken of in both. In many, nay most cases of ordinary conversation and writing, we leave out one of the assertions, which is, usually speaking, very evident, and make the other assertion followed by the consequence of both. Thus, 'geometry is useful, and therefore ought to be studied,' contains not only what is expressed, but also the following, 'that which is useful ought to be studied;' for were this not admitted, the former assertion would not be necessarily true. This may be written thus,—

Every thing useful is what ought to be studied.

Geometry is useful, therefore geometry is what ought to be studied.

This, in its present state, is called a syllogism, and may be compared with the following, from which it only differs

in the *things* spoken of, and not in the *manner* in which they are spoken of.

Every point of the circle is a point of the triangle.

The point B is a point of the circle. Therefore the point B is a point of the triangle. Here a connexion is established between the point B and the points of the triangle (viz. that the first is one of the second) by comparing them with the points of the circle; that which is asserted of every point of the circle in the first can be asserted of the point B, because from the second B is one of these points. Again, in the former argument, whatever is asserted of every thing useful is true of geometry, because geometry is useful.

The common term of the two propositions is called the *middle term*, while the *predicate* and *subject* of the conclusion are called the *major* and *minor* terms, respectively. The two first assertions are called the *major* and *minor premises*, and the last the *conclusion*. Suppose now the two premises and conclusion of the syllogism just quoted to be varied in every possible way from affirmative to negative, from universal to particular, and vice versâ, where the number of changes will be $4 \times 4 \times 4$, or 64 (called moods); since each proposition may receive four different forms, and each form of one may be compounded with any of the other two. And these may be still further varied, if instead of the middle term being the subject of the first, and the predicate of the second, this order be reversed, or if the middle term be the subject of both, or the predicate of both, which will give four different figures, as they are called, to each of the sixty-four moods above mentioned. But of these very few are correct deductions, and without entering into every case we will state some general rules, being the methods which common reason would take to ascertain the truth or falsehood of any one of them, collected and generalised *.

I. The middle term must be the same in both premises, by what has just been observed; since in the comparison of two things with one and the same third thing, in order to ascertain their connexion or discrepancy, consists the whole of reasoning. Thus, the deduction without further process of the equation $a^2 + b^2 = c^2$ from the proposition, which proves that the sum of the *squares* described on the sides of a right-angled triangle is equal to the square on its hypothenuse, a, b, and c being the number of linear units in the sides and the hypothenuse, is incorrect, since syllogistically stated the argument would stand thus:—

The sum of the *squares* of the lines a and b and the *square* of the line c } are {equal quantities,

$a^2 + b^2$ and c^2 } are {the sum of the *squares* of a and b, and the *square* of c.

Therefore $a^2 + b^2$ and c^2 } are {equal quantities.

Here the term square in the major premiss has its geometrical, and in the minor its algebraical sense, being in the first a geometrical figure, and in the second an arithmetical operation. The term of comparison is not therefore the same in both, and the conclusion does not therefore follow from the premises.

The same error is committed if all that can be contained under the middle term be not spoken of either in the major or minor premiss. For if each premiss only mentions a part of the middle term, these parts may be different, and the term of comparison really different in the two, though passing under the same name in both. Thus,

All the triangle is in the circle,

All the square is in the circle,

proves nothing, since the square may, consistently with these conditions, be either wholly, partly, or not at all, contained in the triangle. In fact, as we have before shewn, each of these assertions speaks of a part of the circle only. The following is of the same kind.

Some of the triangle is in the circle.

Some of the circle is not in the square, &c.

II. If both premises are negative, no conclusion can be drawn. For it can evidently be no proof either of agreement or disagreement that two things both disagree with a third. Thus the following is inconclusive,—

None of the circle is in the triangle.

None of the square is in the circle.

III. If both premises are particular,

* Whately's Logic, page 76, third edition. A work which should be read by all mathematical students.

no conclusion can be drawn, as will appear from every instance that can be taken: thus,—

Some of the circle is in the triangle,
Some of the square is not in the circle, proves nothing.

IV. In forming a conclusion, where a conclusion can be formed, nothing must be asserted more generally in the conclusion than in the premises. Thus, if from the following,

All the triangle is in the circle,
All the circle is in the square,

we would draw a conclusion in which the square should be the subject, since the whole square is not mentioned in the minor premiss, but only part of it, the conclusion must be,

Part of the square is in the triangle.

V. If either of the premises be negative, the conclusion must be negative. For as both premises cannot be negative, there is asserted in one premiss an agreement between the term of the conclusion and the middle term, and in the other premiss a disagreement between the other term of the conclusion, and the same middle term. From these nothing can be inferred but a disagreement or negative conclusion. Thus, from

None of the circle is in the triangle,
All the circle is in the square,

can only be inferred,

Some of the square is *not* in the triangle.

VI. If either premiss be particular, the conclusion must be particular. For example, from

None of the circle is in the triangle,
Some of the circle is in the square,

we deduce,

Some of the square is not in the triangle.

If the student now applies these rules, he will find that of the sixty-four moods eleven only are admissible, in any case; and in applying these eleven moods to the different figures he will also find that some of them are not admissible in every figure, and some not necessary, on account of the conclusion, though true, not being as general as from the premises it might be. This he may do either by reasoning or by actual inspection of the figures, drawn and arranged according to the premises. The admissible moods are nineteen in number, and are as follows, where A at the beginning of a proposition signifies that it is a universal affirmative, E a universal negative, I a particular affirmative, O a particular negative.

Figure I. The middle term is the subject of the major, and the predicate of the minor premiss.

1.	* A	All the ○ is in the △
	A	All the □ is in the ○
∴	A	All the □ is in the △
2.	E	None of the ○ is in the △
	A	All the □ is in the ○
∴	E	None of the □ is in the △
3.	A	All the ○ is in the △
	I	Some of the □ is in the ○
∴	I	Some of the □ is in the △
4.	E	None of the ○ is in the △
	I	Some of the □ is in the ○
∴	O	Some of the □ is not in △

Figure II. The middle term is the predicate of both premises.

1.	E	None of the △ is in the ○
	A	All the □ is in the ○
∴	E	None of the □ is in the △
2.	A	All the △ is in the ○
	E	None of the □ is in the ○
∴	E	None of the □ is in the △
3.	E	None of the △ is in the ○
	I	Some of the □ is in the ○
∴	O	Some of the □ is not in △
4.	A	All the △ is in the ○
	O	Some of the □ is not in ○
∴	O	Some of the □ is not in △

Figure III. The middle term is the subject of both premises.

1.	A	All the ○ is in the △
	A	All the ○ is in the □
∴	I	Some of the □ is in the △
2.	I	Some of the ○ is in the △
	A	All the ○ is in the □
∴	I	Some of the □ is in the △
3.	A	All the ○ is in the △
	I	Some of the ○ is in the □
∴	I	Some of the □ is in the △
4.	E	None of the ○ is in the △
	A	All the ○ is in the □
	O	Some of the □ is not in △
5.	O	Some of the ○ is not in △
	A	All the ○ is in the □
∴	O	Some of the □ is not in △
6.	E	None of the ○ is in the △
	I	Some of the ○ is in the □
∴	O	Some of the □ is not in △

Figure IV. The middle term is the predicate of the major, and the subject of the minor premiss.

* This, and 3, are the most simple of all the combinations, and the most frequently used, especially in geometry.

1	A	All the △ is in the ○
	A	All the ○ is in the □
∴	I	Some of the □ is in the △
2.	A	All the △ is in the ○
	E	None of the ○ is in the □
∴	E	None of the □ is in the △
3.	I	Some of the △ is in the ○
	A	All the ○ is in the □
∴	I	Some of the □ is in the △
4.	E	None of the △ is in the ○
	A	All the ○ is in the □
∴	O	Some of the □ is not in △
5.	E	None of the △ is in the ○
	I	Some of the ○ is in the □
∴	O	Some of the □ is not in △

We may observe that it is sometimes possible to condense two or more syllogisms into one argument, thus:

Every A is B (1),
Every B is C (2),
Every C is D (3),
Every D is E (4),
Therefore Every A is E (5),

is equivalent to three distinct syllogisms of the form Figure I.; these syllogisms at length being 1, 2, *a*; *a*, 3, *b*; *b*, 4, 5.

The student, when he has well considered each of these, and satisfied himself, first by the rules, and afterwards by inspection, that each of them is legitimate; and also that all other moods, not contained in the above, are not allowable, or at least do not give the most general conclusion, should form for himself examples of each case, for instance of Fig. III. 3.

The axioms constitute part of the basis of geometry.

Some of the axioms are grounded on the evidence of the senses.

∴ Some evidence derived from the senses is part of the basis of geometry.

He should also exercise himself in the first principles of reasoning by reducing arguments as found in books to the syllogistic form. Any controversial or argumentative work will furnish him with a sufficient number of instances.

Inductive reasoning is that in which a universal proposition is proved by proving separately every one of its particular cases. As where, for example, a figure, A B C D, is proved to be a rectangle by proving each of its angles separately to be a right angle, or proving all the premises of the following, from which the conclusion follows necessarily.

The angles at A, B, C, and D are all the angles of the figure A B C D.

A is a right angle,
B is a right angle,
C is a right angle,
D is a right angle,

Therefore all the angles of the figure A B C D are right angles.

This may be considered as one syllogism of which the minor premiss is,

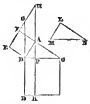
A, B, C, and D are right angles, where each part is to be separately proved.

Reasoning *a fortiori*, is that contained in Fig. I. 1. in a different form, thus: A is greater than B, B is greater than C; *a fortiori* A is greater than C; which may be also stated as follows:

The whole of B is contained in A,
The whole of C is contained in B,
Therefore C is contained in A.

The premises of the second do not necessarily imply as much as those of the first; the complete reduction we leave to the student.

The elements of geometry present a collection of such reasonings as we have just described, though in a more condensed form. It is true that, for the convenience of the learner, it is broken up into distinct propositions, as a journey is divided into stages; but nevertheless, from the very commencement, there is nothing which is not of the nature just described. We present the following as a specimen of a geometrical proposition reduced nearly to a syllogistic form. To avoid multiplying petty syllogisms, we have omitted some few which the student can easily supply. (Treatise on Geometry, page 21.)



Hypothesis. A B C is a right-angled triangle the right angle being at A.

Consequence. The squares on AB and AC are together equal to the square on BC.

Construction. Upon BC and BA describe squares, produce DB to meet EF, produced, if necessary, in G, and draw HK parallel to BD.

Demonstration.

I. Conterminous sides of a square are at right angles to one another. (Definition.)

E B and B A are conterminous sides of a square. (Construction.)

∴ E B and B A are at right angles.

II. A similar syllogism to prove that D B and B C are at right angles, and another to prove that G B and B C are at right angles.

III. Two right lines drawn perpendicular to two other right lines make the same angle as those others (already proved); E B and BG and A B and B C are two right lines &c., (I. II).

∴ The angle E B G is equal to A B C.

IV. All sides of a square are equal. (Definition.)

A B and B E are sides of a square. (Construction.)

∴ A B and B E are equal.

V. All right angles are equal. (Already proved.)

B E G and B A C are right angles. (Hypothesis and construction.)

∴ B E G and B A C are equal angles.

VI. Two triangles having angles of one equal to two angles of the other, and the interjacent sides equal, are equal in all respects. (Proved.)

B E G and B A C are two triangles having B E G and E B G respectively equal to B A C and A B C and the sides EB and B A equal. (III. IV. V.)

∴ The triangles B E G, B A C are equal in all respects.

VII. B G is equal to B C. (VI.)
B C is equal to B D. (Proved as IV.)

∴ B G is equal to B D.

VIII. A four-sided figure whose opposite sides are parallel is a parallelogram. (Definition.) B G H A and B P K D are four-sided figures &c. (Construction.)

∴ B G H A and B P K D are parallelograms.

IX. Parallelograms upon the same base and between the same parallels are equal. (Proved.) E B A F and B G H A, are parallelograms &c. (Construction.)

∴ E B A F and B G H A are equal.

X. Parallelograms on equal bases and between the same parallels, are equal. (Proved.)

B G H A and B D K P are parallelograms &c. (Construction.)

∴ B G H A and B D K P are equal.

XI. EBAF is equal to BGHA. (IX.)
BGHA is equal to B D K P. (X.)

∴ E B A F (that is the square on A B) is equal to B D K P.

XII. A similar argument from the commencement to prove that the square on A C is equal to the rectangle C P K.

XIII. The rectangles B K and C K are together equal to the square on AB. (Self-evident from the construction.)

The squares on B A and A C are together equal to the rectangles B K and C K. (Self-evident from XI and XII.)

∴ The squares on B A and A C are together equal to the square on B A.

Such is an outline of the process, every step of which the student must pass through before he has understood the demonstration. Many of these steps are not contained in the book, because the most ordinary intelligence is sufficient to suggest them, but the least is as necessary to the process as the greatest. Instead of writing the propositions at this length, the student is recommended to adopt the plan which we now lay before him.

Hyp.	1		A B C is a triangle, right angled at A.
Constr.	2	*a*	On B A describe a square B A F E.
	3	*a*	On B C describe a square.
	4		Produce B D to meet E F, produced if necessary in G.
	5	*b*	Through A draw H A K parallel to B D.
Demonst.	6	2, Def.	E B A is a right angle.
	7	3	G B C is a right angle.
	8	6, 7, *c*	∠ E B G is equal to ∠ A B C.
	9	2, 1, *d*	∠ B E G is equal to ∠ B A C.
	10	2	E B is equal to A B.
	11	8, 9, 10, *e*	The triangles B E G and A B C are equal.
	12	11, 3	B G is equal to B D.
	13	5, 2, Def.	A H G B is a parallelogram.
	14	5, 3, Def.	B P D K is a parallelogram.
	15	13, 2, *f*	A H G B and A B E F are equal.
	16	13, 14, *g*	A H G B and B P D K are equal.
	17	15, 16	B P D K and the square on A B are equal.

18 { By similar reasoning } C P K and the square on C A are equal.

19 17, 18 The square on B C is equal to the square on B A and A C.

a, b Here refer to the necessary problems.

c If two lines be drawn at right angles to two others, the angles made by the first and second pair are equal.

d All right angles are equal.

e Two triangles which have two angles of one equal to two angles of the others, and the interjacent side equal, are equal in all respects.

f, g Parallelograms on the same or equal bases, and between the same parallels, are equal.

The explanation of this is as follows: the whole proposition is divided into distinct assertions, which are placed in separate consecutive paragraphs, which paragraphs are numbered in the first column on the left; in the second column on the left we state the reasons for each paragraph, either by referring to the preceding paragraphs from which they follow, or the preceding propositions in which they have been proved. In the latter case a letter is placed in the column, and at the end, the enunciation of the proposition there used is written opposite to the letter. By this method, the proposition is much shortened, its more prominent parts are brought immediately under notice, and the beginner, if he recollect the preceding propositions perfectly well, is not troubled by the repetition of prolix enunciations, while in the contrary case he has them at hand for reference.

In all that has been said, we have taken instances only of direct reasoning, that is, where the required result is immediately obtained without any reference to what might have happened if the result to be proved had not been true. But there are many propositions in which the only possible result is one of two things which cannot be true at the same time, and it is more easy to shew that one is *not* the truth, than that the other *is*. This is called indirect reasoning; not that it is less satisfactory than the first species, but because, as its name imports, the method does not appear so direct and natural. There are two propositions of which it is required to shew that whenever the first is true the second is true; that is, the first being the hypothesis the second is a necessary conclusion from it, whence the hypothesis in question, and any thing contradictory to, or inconsistent with the conclusion cannot exist together. In indirect reasoning, we suppose that, the original hypothesis existing and being true, something inconsistent with or contradictory to the conclusion is true also. If from combining the consequences of these two suppositions, something evidently erroneous or absurd is deduced, it is plain that there is something wrong in the assumptions. Now care is taken that the only doubtful point shall be the one just alluded to, namely, the supposition that one proposition and the contradictory of the other are true together. This then is incorrect, that is, the first proposition cannot exist with any thing contradictory to the second, or the second must exist wherever the first exists, since if any proposition be not true its contradictory must be true, and vice versâ. This is rather embarrassing to the beginner, who finds that he is required to admit, for argument's sake, a proposition which the argument itself goes to destroy. But the difficulty would be materially lessened, if instead of assuming the contradictory of the second proposition positively, it were hypothetically stated, and the consequences of it asserted with the verb 'would be,' instead of 'is.' For example: suppose it to be known that if A is B, then C must be D, and it is required to shew indirectly that when C is not D, A is not B. This put into the form in which such a proposition would appear in most elementary works, is as follows.

It being granted that if A is B, C is D, it is required to shew that when C is not D, A is not B. If possible, let C be not D, and let A be B. Then by what is granted, since A is B, C is D; but by hypothesis C is not D, therefore both C *is* D and *is not* D, which is absurd; that is, it is absurd to suppose that C *is not* D and A *is* B, consequently when C is not D, A is not B. The following, which is exactly the same thing, is plainer in its language. Let C be not D. Then if A *were* B, C *would be* D by the proposition granted. But by hypothesis C is

not D, &c. This sort of indirect reasoning frequently goes by the name of *reductio ad absurdum.*

In all that has gone before we may perceive that the validity of an argument depends upon two distinct considerations,—1, the truth of the relations assumed, or represented to have been proved before; 2, the manner in which these facts are combined so as to produce new relations; in which last the *reasoning* properly consists. If either of these be incorrect in any single point, the result is certainly false; if both be incorrect, or if one or both be incorrect in more points than one, the result, though not at all to be depended on, is not certainly false, since it may happen and has happened, that of two false reasonings or facts, or the two combined, one has reversed the effect of the other and the whole result has been true; but this could only have been ascertained after the correction of the erroneous fact or reasoning. The same thing holds good in every species of reasoning, and it must be observed, that however different geometrical argument may be in form from that which we employ daily, it is not different in reality. We are accustomed to talk of mathematical *reasoning* as above all other, in point of accuracy and soundness. This, if by the term *reasoning* we mean the comparing together of different ideas and producing other ideas from the comparison, is not correct, for in this view mathematical reasonings and all other reasonings correspond exactly. For the real difference between mathematics and other studies in this respect we refer the student to the first chapter of this treatise.

In what then, may it be asked, does the real advantage of mathematical study consist? We repeat again, in the actual certainty which we possess of the truth of the facts on which the whole is based, and the possibility of verifying every result by actual measurement, and not in any superiority which the method of reasoning possesses, since there is but one method of reasoning. To pursue the illustration with which we opened this work (page the first), suppose this point to be raised, was the slaughter of Cæsar justifiable or not? The actors in that deed justified themselves by saying, that a tyrant and usurper, who meditated the destruction of his country's liberty, made it the duty of every citizen to put him to death, and that Cæsar was a tyrant and usurper, &c. Their *reasoning* was perfectly correct, though proceeding on premises then extensively, and now universally denied. The first premiss, though correctly used in this reasoning, is now asserted to be false, on the ground that it is the duty of every citizen to do nothing which would, were the practice universal, militate against the general happiness; that were each individual to act upon his own judgment, instead of leaving offenders to the law, the result would be anarchy and complete destruction of civilization, &c. Now in these reasonings and all others, with the exception of those which occur in mathematics, it must be observed that there are no premises so certain, as never to have been denied, no first principles to which the same degree of evidence is attached as to the following, that 'no two straight lines can enclose a space.' In mathematics, therefore, we reason on certainties, on notions to which the name of innate can be applied, if it can be applied to any whatever. Some, on observing that we dignify such simple consequences by the name of reasoning, may be loth to think that this is the process to which they used to attach such ideas of difficulty. There may, perhaps, be many who imagine that reasoning is for the mathematician, the logician &c., and who, like the Bourgeois Gentilhomme, may be surprised on being told, that, well or ill, they have been reasoning all their lives. And yet such is the fact; the commonest actions of our lives are directed by processes exactly identical with those which enable us to pass from one proposition of geometry to another. A porter, for example, who being directed to carry a parcel from the City to a street which he has never heard of, and who on inquiry, finding it is in the Borough, concludes that he must cross the water to get at it, has performed an act of reasoning, differing nothing in kind from those by a series of which, did he know the previous propositions, he might be convinced that the square of the hypothenuse of a right-angled triangle, is equal to the sum of the squares of the sides.

CHAPTER XV.

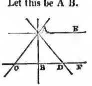
On Axioms.

GEOMETRY, then, is the application of strict logic to those properties of space and figure which are self-evident, and which therefore cannot be disputed.

But the rigour of this science is carried one step further; for no property, however evident it may be, is allowed to pass without demonstration, if that can be given. The question is therefore to demonstrate all geometrical truths with the smallest possible number of assumptions. These assumptions are called axioms, and for an axiom it is requisite,—1, that it should be self-evident; 2, that it should be incapable of being proved from the other axioms. In fulfilling these conditions, the number of axioms which are really geometrical, that is, which have not equal reference to Arithmetic, is reduced to two, viz., two straight lines cannot enclose a space, and through a given point not more than one parallel can be drawn to a given straight line. The first of these has never been considered as open to any objection; it has always passed as perfectly self-evident. It is on this account made the proposition on which are grounded all reasonings relative to the straight line, since the definition of a straight line is too vague to afford any information. But the second, viz., that through a given point not more than one parallel can be drawn to a given straight line, has always been considered as an assumption not self-evident in itself, and has therefore been called the defect and disgrace of geometry. We proceed to place it on what we conceive to be the proper footing.

By taking for granted the arithmetical axioms only, with the first of those just alluded to, the following propositions may be strictly shewn.

I. One perpendicular and only one can be let fall from any point A to a given line C D. Let this be A B.



II. If equal distances B C and B D be taken on both sides of B, A C and A D are equal, as also the angles B A C and B A D.

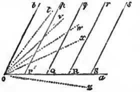
III. Whatever may be the length of B C and B D, the angles B A C and B A D are each less than a right angle.

IV. Through A, a line may be drawn parallel to C D (that is, by definition, never meeting C D, though the two be ever so far produced), by drawing any line A D and making the angle D A E equal to the angle A D B, which it is before shewn how to do.

From proposition (IV) we should at first see no reason against there being as many parallels to C D, to be drawn through A, as there are different ways of taking A D, since the direction for drawing a parallel to C D is, 'take *any line* A D cutting C D and make the angle D A E equal to A D B.' But this our senses immediately assure us is impossible.

It appears also a proposition to which no degree of doubt can attach, that if the straight line A B, produced indefinitely both ways, set out from the position A B and revolve round the point A, moving first towards A E; then the point of intersection D will first be on one side of B and afterwards on the other, and there will be one position where there is no point of intersection either on one side or the other, and *one such position only*. This is in reality the assumption of Euclid; for having proved that A E and B F are parallel when the angles B D A and D A E are equal, or, which is the same thing, when E A D and A D F are together equal to two right angles, he further assumes that they will be parallel in no other case, that is, that they will meet when the angles E A D and A D F are together greater or less than two right angles; which is really only assuming that the parallel which he has found, is the only one which can be drawn. The remaining part of his axiom, namely, that the lines A E and D E, if they meet at all, will meet upon that side of D A on which the angles are less than two right angles, is not an assumption but a consequence of his proposition, which shews that any two angles of a triangle are together less than two right angles, and which is established before any mention is made of parallels. It has been found by the experience of two thousand years, that some assumption of this sort is indispensable. Every species of effort has been made to avoid or elude the difficulty, but hitherto without success, as some assumption has always been involved, at least equal, and in most cases superior in difficulty to the one already made by Euclid. For example, it has been proposed to define parallel lines as those which are equidistant from one another at every point. In this case, before the name parallel can be allowed to belong to any thing, it must be proved that there are lines such that a perpendicular to one is always perpen-

dicular to the other, and that the parts of these perpendiculars intercepted between the two are always equal. A proof of this has never been given without the previous assumption of something equivalent to the axiom of Euclid. Of this last, indeed, a proof has been given, but involving considerations not usually admitted into geometry, though it is more than probable that had the same come down to us, sanctioned by the name of Euclid, it would have been received without difficulty. The Greek geometer confines his notion of equal magnitudes to those which have boundaries. Suppose this notion of equality extended to all such spaces as can be made to coincide entirely in all their extent, whatever that extent may be; for example, the unbounded spaces contained between two equal angles whose sides are produced without end, which by the definition of equal angles might be made to coincide entirely by laying the sides of one angle upon those of the other. In the same sense we may say, that, one angle being double of another, the spaces contained by the sides of the first is double that contained by the sides of the second, and so on. Now suppose two lines



Oa and Ob, making any angle with one another, and produced *ad infinitum* *. On O a take off the equal spaces O P, P Q, Q R, &c., *ad infinitum*, and draw the lines Pp, Q q, Rr, &c., so that the angles O Pp, O Q q &c., shall be equal to one another, each being such as with b O P will make two right angles. Then O b, Pp, Qq, &c., are parallel to one another, and the infinite spaces b O Pp, p P Q q, q Q R r, &c., can be made to coincide, and are equal. Also no finite number whatever of these spaces will fill up the infinite space b O a, since O P, P Q, &c. may be continued *ad infinitum* upon the line O a. Let there be any line O t, such that the angles t O P and p P O are together less than two right angles, that is, less than b O P and p P O; whence t O P is less than b O P and tO falls between bO and aO. Take the angles tOv, vOw, wOx, &c., each equal to bOt, and continue this until the last line Oz falls beneath Oa, so that the angle bOz is greater than bOa. That this is possible needs no proof, since it is manifest that any angle being continually added to itself the sum will in time exceed any other given angle; again, the infinite spaces bOt, tOv, &c., are all equal. Now on comparing the spaces bOt and bOPp, we see that a certain number of the first is more than equal to the space bOa, while no number whatever of the second is so great. We conclude, therefore, that the space bOt is greater than bOPp, which cannot be unless the line Ot cuts Pp at last; for if Ot did never cut Pp, the space bOt would evidently be less than bOPp, as the first would then fall entirely within the second. Therefore two lines which make with a third angles together less than two right angles will meet if sufficiently produced.

This demonstration involves the consideration of a new species of magnitude, namely, the whole space contained by the sides of an angle produced without limit. This space is unbounded, and is greater than any number whatever of finite spaces, of square feet, for example. No comparison, therefore, as to magnitude, can be instituted between it and any finite space whatever, but that affords no reason against comparing this magnitude with others of the same kind.

Any thing may become the subject of mathematical reasoning, which can be increased or diminished by other things of the same kind; this is, in fact, the definition given of the term magnitude; and geometrical reasoning, in all other cases at least, can be applied as soon as a criterion of equality is discovered. Thus the angle, to beginners, is a perfectly new species of magnitude, and one of whose measure they have no conception whatever; they see, however, that it is capable of increase or diminution, and also that two of the kind can be equal, and how to discover whether this is so or not, and nothing more is necessary for them. All that can be said of the introduction of the angle in geometry holds with some, to us it appears an equal force, with regard to

* Every line in this figure must be produced *ad infinitum*, from that extremity at which the small letter is placed.

these unlimited spaces; the two are very closely connected, so much so, that the term angle might even be defined as 'the unlimited space contained by two right lines,' without alteration in the truth of any theorem in which the word angle is found. But this is a point which cannot be made very clear to the beginner.

The real difficulties of geometry begin with the theory of proportion, to which we now proceed. The points of discussion which we have hitherto raised, are not such as to embarrass the elementary student, however much they may perplex the metaphysical inquirer into first principles. The theory to which we are coming abounds in difficulties of both classes.

CHAPTER XVI.

On Proportion.

In the first elements of geometry, two lines, or two surfaces, are mentioned in no other relation to one another, than that of equality or non-equality. Nothing but the simple fact is announced, that one magnitude is equal to, greater than, or less than another, except occasionally when the sum of two equal magnitudes is said to be double of one of them. Thus in proving that two sides of a triangle are together greater than the third, the fact that they are *greater* is the essence of the proposition; no measure is given of the excess, nor does anything follow from the theorem as to whether it is, or may be, small or great. We now come to the doctrine of proportion, in which geometrical magnitude is considered in a new light. The subject has some difficulties, which have been materially augmented by the almost universal use, in this country at least, of the theory laid down in the fifth book of Euclid. Considered as a complete conquest over a great and acknowledged difficulty of principle, this book of Euclid well deserves the immortality of which its existence, at the present moment, is the guarantee; nay, had the speculations of the mathematician been wholly confined to geometrical magnitude, it might be a question whether any other notions would be necessary. But when we come to apply arithmetic to geometry, it is necessary to examine well the primary connexion between the two; and here difficulties arise, not in comprehending that connexion, so much as in joining the two sciences by a chain of demonstration as strong as that by which the propositions of geometry are bound together, and as little open to cavil and disputation.

The student is aware that before pronouncing upon the connexion of two lines with one another, it is necessary to *measure* them, that is, to refer them to some third line, and to observe what number of times the third is contained in the other two. Whether the two first are equal or not is readily ascertained by the use of the compasses, on principles laid down with the utmost strictness in Euclid, and other elementary works. But this step is not sufficient; to say that two lines are not equal, determines nothing. There are an infinite number of ways in which one line may be greater or less than a given line, though there is only one in which the other can be equal to the given one. We proceed to shew how, from the common notion of measuring a line, the more strict geometrical method is derived.

To measure the line AB apply to it another line (the edge of a ruler), which is divided into equal parts (as inches), each of which parts is again subdivided into ten equal parts, as in the figure. This division is made to take place in practice until the last subdivision gives a part so small, that any thing less may be neglected as inconsiderable. Thus a carpenter's rule is divided into tenths or

eighths of inches only, while in the tube of a barometer a process must be employed which will mark a much less difference. In talking of accurate measurement, therefore, any where but in geometry, or algebra, we only mean accurate as far as the senses are concerned, and as far as is necessary for the object in view. The ruler in the figure shews that the line AB contains more than two and less than three inches; and closer inspection shews that the excess above two inches is more than six-tenths of an inch, and less than seven. Here, in practice, the process stops; for, as the subdivision of the ruler was carried only to tenths of

inches, because a tenth of an inch is a quantity which may be neglected in ordinary cases, we may call the line two inches and six-tenths, by doing which the error committed is less than one-tenth of an inch. In this way lines may be compared together with a common degree of correctness; but this is not enough for the geometer. His notions of accuracy are not confined to tenths or hundredths, or hundred-millionth parts of any line, however small it may be at first. The reason is obvious; for although to suit the eye of the generality of readers, figures are drawn, in which the least line is usually more than an inch, yet his theorems are asserted to remain true, even though the dimensions of the figure are so far diminished as to make the whole imperceptible in the strongest microscope. Many theorems are obvious upon looking at a moderately-sized figure; but the reasoning must be such as to convince the mind of their truth when, from excessive increase or diminution of the scale, the figures themselves have past the boundary even of imagination. The next step in the process of measurement is as follows, and will lead us to the great and peculiar difficulty of the subject.

The inch, the foot, and the other lengths, by which we compare lines with one another, are perfectly arbitrary. There is no reason for their being what they are, unless we adopt the commonly received notion that our inch is derived from our Saxon ancestors, who observed that a barley-corn is always of the same length, or nearly so, and placed three of them together as a common standard of measure, which they called an inch. Any line whatever may be chosen as the standard of measure, and it is evident that when two or more lines are under consideration, exact comparisons of their lengths can only be obtained from a line which is contained an exact number of times in them all. For even exact fractional measures are reduced to the same denominator, in order to compare their magnitudes. Thus, two lines which contain $\frac{2}{11}$ and $\frac{3}{7}$ of a foot, are better compared by observing that $\frac{2}{11}$ and $\frac{3}{7}$ being $\frac{14}{77}$ and $\frac{33}{77}$, the given lines contain one 77th part of a foot 14 and 33 times respectively. Any line which is contained an exact number of times in another is called in geometry a measure of it, and a common measure of two or more lines is that which is contained an exact number of times in each.

Again, a line which is measured by another is called a multiple of it, as in arithmetic.

The same definition, *mutatis mutandis*, applies to surfaces, solids, and all other magnitudes; and though, in our succeeding remarks, we use lines as an illustration, it must be recollected that the reasoning applies equally to every magnitude which can be made the subject of calculation.

In order that two quantities may admit of comparison as to magnitude, they must be of the same sort; if one is a line, the other must be a line also. Suppose two lines A and B each of which is measured by the line C; the first containing it five times and the second six. These lines A and B, which contain the same line C five and six times respectively, are said to have to one another the ratio of five to six, or to be in the proportion of five to six. If then we denote the first by A *, and the second by B, and the common measure by C, we have

$$A = 5C, \quad \text{or } 6A = 30C,$$
$$B = 6C, \quad 5B = 30C,$$
$$\text{whence } 6A = 5B, \quad \text{or } 6A - 5B = 0.$$

Generally, when $mA - nB = 0$, the lines, or whatever they are, represented by A and B, are said to be in the proportion of n to m, or to have the ratio of n to m.

Let there be two other magnitudes P and Q, of the same kind with one another, either differing from the first in kind or not; thus A and B may be lines, and P and Q surfaces, &c., and let them contain a common measure R, just as A and B contain C, viz.: Let P contain R five times, and let Q contain R six times, we have by the same reasoning

$$6P - 5Q = 0,$$

and P and Q, being also in the ratio of

* The student must distinctly understand that the common meaning of algebraical terms is departed from in this chapter, wherever the letters are large instead of small. For example, A, instead of meaning the number of *units* of some sort or other contained in the line A, stands for *the line A itself*, and mA (the small letters throughout meaning *whole* numbers) stands for the line made by taking A, m times. Thus such expressions as $mA + B$, $mA - nB$, &c., are the only ones admissible. AB, $\frac{A}{B}$, A^2, &c., are unmeaning, while $\frac{A}{m}$ is the line which is contained m times in A, or the mth part of A. The capital letters throughout stand for concrete quantities, not for their representations in abstract numbers.

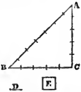
five to six, as well as A and B, are said to be proportional to A and B, which is denoted thus,

$$A : B :: P : Q,$$

by which *at present* all we mean is this, that there are some two whole numbers m and n such that, at the same time

$$mA - nB = 0,$$
$$mP - nQ = 0,$$

Nothing more than this would be necessary for the formation of a complete theory of proportion, if the common measure, which we have supposed to exist in the definition, did always really exist. We have however no right to assume, that two lines A and B, whatever may be their lengths, both contain some other line an exact number of times. We can moreover produce a direct instance in which two lines have no common measure whatever, in the following manner.



Let A B C be an isosceles right-angled triangle, the side B C and the hypothenuse have no common measure whatever. If possible let D be a common measure of B C and A B; let B C contain D, n times, and let A B contain D m times. Let E be the square described on D. Then since A B contains D, m times, the square described on A B contains E, $m \times m$ or m^2 times. Similarly the square described on B C contains E $n \times n$ or n^2 times: (Treatise on Geometry, Prop. 29.) But, because A B is an isosceles right-angled triangle, the square on A B is double of that on B C (Prop. 36,) whence $m \times m = 2n \times n$ or $m^2 = 2n^2$. To prove the impossibility of this equation, (when m and n are whole numbers,) observe that m^2 must be an even number, since it is twice the number n^2. But $m \times m$ cannot be an even number unless m is an even number, since an odd number multiplied by itself produces an odd number *. Let m (which has been shewn to be even) be double of m' or $m = 2m'$. Then $2m' \times 2m' = 2n^2$ or $4m'^2 = 2n^2$ or $n^2 = 2m'^2$. By repeating the same reasoning we show that n is even. Let it be $2n'$. Then. $2n' \times 2n' = 2m'^2$ or $m'^2 = 2n'^2$. By the same reasoning m' and n' are both even, and so on *ad infinitum*. This reasoning shows that the whole numbers which satisfy the equation $n^2 = 2m^2$ (if such there be) are divisible by 2 without remainder, *ad infinitum*. The absurdity of such a supposition is manifest: there are then no such whole numbers, and consequently no common measure to B A and B C.

Before proceeding any further, it will be necessary to establish the following proposition.

If the greater of two lines A and B, be divided into m equal parts, and one of these parts be taken away; if the remainder be then divided into m equal parts, and one of them be taken away, and so on,—the remainder of the line A, shall in time become less than the line B, how small soever the line B may be.

Take a line which is less than B, and call it C. It is evident that, by a continual addition of the same quantity to C, this last will come in time to exceed A; and still more will it do so, if the quantity added to C be increased at each step. To simplify the proof, we suppose that 20 is the number of equal parts into which A and its remainders are successively divided, so that 19 out of the 20 parts remain after subtraction.

Divide C into 19 equal parts and add to C a line equal to one of these parts. Let the length of C, so increased, be C′. Divide C′ into 19 equal parts and let C′, increased by its 19th part, be C″. Now since we add more and more each time to C, in forming C′, C″, &c., we shall in time exceed A. Let this have been done, and let D be the line so obtained, which is greater than A. Observe now that C′ contains 19, and C″, 20 of the same parts, whence C′ is made by dividing C″ into 20 parts and removing one of them. The same of all the rest. Therefore we may return from D to C by dividing D into 20 parts, removing one of them, and repeating the process continually. But C is less than B by hypothesis. If then we can, by this process, reduce D below B, still more can we do so with A, which is less than D, by the same method.

This depends on the obvious truth, that if, at the end of any number of subtractions (D being taken), we have left $\frac{p}{q}$D,

* Every odd number, when divided by 2, gives a remainder 1, and is therefore of the form $2p + 1$, where p is a whole number. Multiply $2p + 1$ by itself, which gives $4p^2 + 4p + 1$, or $2(2p^2 + 2p) + 1$, which is an odd number, since, when divided by 2, it gives the quotient $2p^2 + 2p$, a whole number, and the remainder 1.

at the end of the same number of subtractions (A being taken), we shall have $\frac{p}{q}$ A, since the method pursued in both cases is the same. But since A is less than D, $\frac{p}{q}$ A is less than $\frac{p}{q}$ D, which becomes equal to C, therefore $\frac{p}{q}$ A becomes less than C *.

We now resume the isosceles right angled triangle. The lines B C and A B, which were there shown to have no common measure, are called *incommensurable* quantities, and to their existence the theory of proportion owes its difficulties. We can nevertheless show that A and B being incommensurable, a line can be found as near to B as we please, either greater or less, which is commensurable with A. Let D be any line taken at pleasure, and therefore as small as we please. Divide A into two equal parts, each of those parts into two equal parts, and so on. We shall thus at last find a part of A which is less than D. Let this part be E, and let it be contained m times in A. In the series E, 2E, 3E, &c., we shall arrive at last at two consecutive terms, pE and $\overline{p+1}\,E$ of which the first is less, and the second greater than B. Neither of these differs from B by so much as E; still less by so much as D; and both pE and $\overline{p+1}\,E$ are commensurable with A, that is with mE, since E is a common measure of both. If therefore A and B are incommensurable, a third magnitude can be found, either greater or less than B, differing from B by less than a given quantity, which magnitude shall be commensurable with A.

We have seen that when A and B are incommensurable, there are no whole values of m and n, which will satisfy the equation $mA - nB = 0$; nevertheless we can prove that values of m and n can be found which will make $mA - nB$ less than any given magnitude C, of the same kind, how small soever it may be. Suppose, that for certain values of m and n*, we find $mA - nB = E$, and let the first multiple of E, which is greater than B, be pE, so that $pE = B + E'$ where E' is less than E, for were it greater, $\overline{p-1}\,E$, or $pE - E$, which is $B + (E' - E)$, would be greater than B, which is against the supposition.

The equation $mA - nB = E$ gives

$$pmA - pnB = pE = B + E';$$

whence $pmA - (pn+1)B = E'$

let $pm = m'$ and $pn + 1 = n'$,

whence $m'A - n'B = E'$.

We have therefore found a difference of multiples which is less than E. Let $p'E'$ be the first multiple of E' which is greater than B, where p' must be *at least as great* as p, since E being greater than E', it cannot take *more*† of E than of E' to exceed B.

Let $p'E' = B + E''$,

then, as before,

$$m'p'A - \overline{n'p'+1}\,B = E'',$$

or $m''A - n''B = E''$;

we have therefore still further diminished the difference of the multiples; and the process may be repeated any number of times; it only remains to show that the diminution may proceed to any extent.

This will appear superfluous to the beginner, who will probably imagine that a quantity diminished at every step, must, by continuing the number of steps, at last become as small as we please. Nevertheless if any number, as 10, be taken and its square root extracted, and the square root of that square root, and so on, the result will not be so small as unity, although ten million of square roots should have been extracted. Here is a case of continual diminution, in which the diminution is not *without limit*. Again, from the point D in the line A B draw DE, making an angle with A B less than half a right angle. Draw B E perpendicular to A B, and take

* Algebraically, let a be the given line, and let $\frac{1}{m}$th part of the remainder be removed at every subtraction. The first quantity taken away is $\frac{a}{m}$ and the remainder $a - \frac{a}{m}$ or $a\left(1 - \frac{1}{m}\right)$, whence the second quantity removed is $\frac{a}{m}\left(1 - \frac{1}{m}\right)$, and the remainder $a - \frac{a}{m}\left(1 - \frac{1}{m}\right)$ or $a\left(1 - \frac{1}{m}\right)^2$. Similarly the nth remainder is $a\left(1 - \frac{1}{m}\right)^n$. Now since $1 - \frac{1}{m}$ is less than unity, its powers decrease, and a power of so great an index may be taken as to be less than any given quantity.

* It is necessary here to observe, that in speaking of the expression $mA - nB$ we more frequently refer to its form, than to any actual value of it, derived from supposing m and n to have certain known values. When we say that $mA - nB$ can be made smaller than C, we mean that some values can be given to m and n such that $mA - nB < C$, or that *some* multiple of B subtracted from some multiple of A is less than C. The following expressions are all of the same form, viz.: that of some multiple of B subtracted from some multiple of A.

$$mA - nB$$
$$mpA - (np + 1)B$$
$$2mA - 4mB, \text{ \&c. \&c.}$$

† It may require *as many*. Thus it requires as many of 7 as of 8 to exceed 83, though 7 is less than 8.

B C = B E. Draw C F perpendicular to A B, and take C C′= C F, and so on.

The points C, C′, C″, &c., will always be further from A than D is; and all the lines AC, AC′, AC″, &c., though diminished at every step, will always remain greater than A D. Some such species of diminution, for any thing yet proved to the contrary, may take place in $mA - nB$.

To compare the quantities E, E′, &c., we have the equations

$$\begin{aligned} pE &= B + E' \\ p'E' &= B + E'' \\ p''E'' &= B + E''' \\ \&c. &\quad \&c. \end{aligned}$$

The numbers p, p', p'', &c., do not diminish; the lines E, E′, E″, &c., diminish at every step. If then we can show that p, p', &c., can only remain the same for a finite number of steps, and must then increase, and after the increase can only remain the same for another finite number of steps, and then must increase again, and so on, we shew that the process can be continued, until one of them is as great as we please; let this be $p^{(z)}$, where z is not an exponent, but marks the number which our notation will have reached, and indicates the $(z+1)^{\text{th}}$ step of the process. Let $E^{(z)}$ be the corresponding remainder from the former step. Then, since $p^{(z)}E^{(z)}$ is the first multiple of $E^{(z)}$, which exceeds the given quantity B, if $p^{(z)}$ can be as great as we please, $E^{(z)}$ can be as small as we please. To shew that $p^{(z)}$ can be as great as we please, observe, that p, p', p'', &c., must remain the same, or increase, since, as appears from their method of formation, they cannot diminish. Let them remain the same for some steps, that is, let $p = p' = p''$, &c. The equations become

$$\begin{aligned} pE &= B + E' \\ pE' &= B + E'' \\ pE'' &= B + E''' \\ \&c. &\quad \&c. \end{aligned}$$

Then by subtraction,—

$$\begin{aligned} E' - E'' &= p(E - E') \\ E'' - E''' &= p(E' - E'') = pp\,(E - E') \\ E''' - E'''' &= p(E'' - E''') = ppp(E - E') \\ \&c. &\qquad \&c. \end{aligned}$$

$$\begin{aligned} \text{Now, } E - E'' &= E - E' + E' - E'' = (E - E')(1 + p) \\ E - E''' &= E - E' + E' - E'' + E'' - E''' = (E - E')(1 + p + p^2) \\ \&c. &\qquad \&c. \qquad \&c. \end{aligned}$$

$$\begin{aligned} \text{Generally, } E - E^{(w)} &= E - E' + E' - E'' + \ldots\ldots + E^{(w-1)} - E^{(w)} \\ &= (E - E')(1 + p + p^2 + \ldots\ldots + p^{w-1}), \end{aligned}$$

which is derived from w steps of the process. Now, if this can go on *ad infinitum*, it can go on until $1 + p + p^2 + \ldots + p^{w-1}$ is as great as we please; for, since p is not less than unity, the continual addition of its powers will, in time, give a sum exceeding any given number. This is absurd, from the step at which $1 + p + p^2 + \ldots + p^{w-1}$ becomes greater than the number of times which $E - E'$ is contained in E; for, from the above equation, $E - E'$ is contained in $E - E^{(w)}$, $1 + p + p^2 + \ldots + p^{w-1}$ times; and it is contradictory to suppose that $E - E'$ should be contained in $E - E^{(w)}$ more times than it is contained in E.

To take an example: suppose that B is 55 feet, and E is 54 feet; the first equation is

$$2 \times 54' = 55' + 53',$$

where $E' = 53'$ and $E - E' = 1'$, and is contained in E 54 times. If, then, we continue the process, 2 cannot maintain its present place through so many steps of the process as will, if the same number of terms be taken, give $1 + 2 + 2^2 + 2^3 +$, &c., greater than 54; that is, it cannot be the same for *six* steps. And we find, on actually performing the operations,

$$\begin{aligned} 2 \times 54' &= 55' + 53' \\ 2 \times 53' &= 55' + 51' \\ 2 \times 51' &= 55' + 47' \\ 2 \times 47' &= 55' + 39' \\ 2 \times 39' &= 55' + 23' \\ 3 \times 23' &= 55' + 14' \end{aligned}$$

We do not say that p, p', &c. *will* remain the same until $1 + p + p^2 + \ldots$ would be greater than the number of times which E contains $E - E'$, but only that they cannot remain the same longer. By repetition of the same process, we can show that a further and further increase must take place, and so on until we have attained a quantity greater than any given one. And it has already been shown to be a consequence of this, that $mA - nB$ can be diminished to any ex-

tent we please. Similarly it may be shewn that when A and B are incommensurable, $mA-nB$ may be brought as near as we please to any other quantity C, of the same kind as A and B, so as not to differ from C by so much as a given quantity E. For let m and n be taken, by the last case, so that $mA-nB$ may be less than E, and let $mA-nB$, in this case, be equal to E′. Let C lie between pE' and $\overline{p+1}\,E'$, neither of which can differ from C by so much as E′, and therefore not by so much as E.

Then since $mA-nB=E'$;

therefore $\overline{pmA-pnB}=pE'$,

and $\overline{p+1}\,mA-\overline{p+1}\,nB=\overline{p+1}\,E'$.

Both which last expressions differ from C by a quantity less than E, the first being less and the second greater than C, and both are of the *form* $mA-nB$, m and n being changed for other numbers.

The common ideas of proportion are grounded entirely upon the false notion that all quantities of the same sort are commensurable. That the supposition is practically correct, if there are any limits to the senses, may be shewn, for let any quantity be rejected as imperceptible, then since a quantity can be found as near to B as we please, which is commensurable with A, the difference between B and its approximate commensurable magnitude, may be reduced below the limits of perceptible quantity. Nevertheless, inaccuracy to some extent must infest all *general* conclusions drawn from the supposition that A and B being two magnitudes, whole numbers, m and n, can *always* be found such that $mA-nB=0$. We have shewn that this can be brought as near to the truth as we please, since $mA-nB$ can be made as small as we please. This, however, is not a perfect answer, at least it wants the unanswerable force of all the preceding reasonings in geometry. A definition of proportion should therefore be substituted, which, while it reduces itself, in the case of commensurable quantities to the one already given, is equally applicable to the case of incommensurables. We proceed to examine the definition already given with a view to this object.

Resume the equations—

$$mA-nB=0,\text{ or } A=\frac{n}{m}B$$

$$mP-nQ=0,\qquad P=\frac{n}{m}Q$$

If we take any other expressions of the same sort $\frac{n'}{m'}B$ and $\frac{n'}{m'}Q$, it is plain that, according as the arithmetical fraction $\frac{n}{m}$ is greater than, equal to, or less than $\frac{n'}{m'}$, so will $\frac{n}{m}B$ be greater than, equal to, or less than $\frac{n'}{m'}B$, and the same of $\frac{n}{m}Q$ and $\frac{n'}{m'}Q$. Let the symbol

$$\left.\begin{matrix}x\\z\end{matrix}\right\} > = < \left\{\begin{matrix}y\\w\end{matrix}\right.$$

be the abbreviation of the following sentence—"when x is greater than y, z is greater than w; when x is equal to y, z is equal to w; when x is less than y, z is less than w." The following conclusions will be evident:—

If $\left.\begin{matrix}a\\c\end{matrix}\right\} > = < \left\{\begin{matrix}b\\d\end{matrix}\right.$

and $\left.\begin{matrix}a\\e\end{matrix}\right\} > = < \left\{\begin{matrix}b\\f\end{matrix}\right.$

Then $\left.\begin{matrix}c\\e\end{matrix}\right\} > = < \left\{\begin{matrix}d\\f\end{matrix}\right.$ (1)

And from the first of these alone it follows that

$$\left.\begin{matrix}ma\\nc\end{matrix}\right\} > = < \left\{\begin{matrix}mb\\nd\end{matrix}\right. \quad (2)$$

We have just noticed the following—

$$\left.\begin{matrix}\frac{n}{m}\\ \frac{n}{m}B\end{matrix}\right\} > = < \left\{\begin{matrix}\frac{n'}{m'}\\ \frac{n'}{m'}B\end{matrix}\right.$$

and $$\left.\begin{matrix}\frac{n}{m}\\ \frac{n}{m}Q\end{matrix}\right\} > = < \left\{\begin{matrix}\frac{n'}{m'}\\ \frac{n'}{m'}Q\end{matrix}\right.$$

Therefore (1) $$\left.\begin{matrix}\frac{n}{m}B\\ \frac{n}{m}Q\end{matrix}\right\} > = < \left\{\begin{matrix}\frac{n'}{m'}B\\ \frac{n'}{m'}Q\end{matrix}\right.$$

or $$\left.\begin{matrix}A\\ P\end{matrix}\right\} > = < \left\{\begin{matrix}\frac{n'}{m'}B\\ \frac{n'}{m'}Q\end{matrix}\right.$$

Therefore (2) $\left.\begin{matrix}m'A\\ m'P\end{matrix}\right\} > = < \left\{\begin{matrix}n'B\\ n'Q\end{matrix}\right.$

Or, if four magnitudes are proportional, according to the common notion, it follows that the same multiples of the first and third being taken, and also of the second and fourth, the multiple of the first is greater than, equal to, or less than,

that of the second, according as that of the third is greater than, equal to, or less than, that of the fourth. This property* necessarily follows from the equations $\begin{matrix} mA-nB=0 \\ mP-nQ=0 \end{matrix}$, but it does not therefore follow that the equations are necessary consequences of the property, since the latter may possibly be true of incommensurable quantities, of which, by definition, the former is not. The existence of this property is Euclid's definition of proportion: he says, let four magnitudes, two and two, of the same kind, *be called proportional*, when, if equimultiples be taken of the first and third, &c., repeating the property just enunciated. What is lost and gained by adopting Euclid's definition may be very simply stated; the gain is an entire freedom from all the difficulties of incommensurable quantities, and even from the necessity of inquiring into the fact of their existence, and the removal of the inaccuracy attending the supposition that, of two quantities of the same kind, each is a determinate arithmetical fraction of the other; on the other hand, there is no obvious connexion between Euclid's definition and the ordinary and well-established ideas of proportion; the definition itself is made to involve the idea of infinity, since *all possible multiples* of the four quantities enter into it; and lastly, the very existence of the four quantities, called proportional, is matter for subsequent demonstration, since to a beginner it cannot but appear very unlikely that there are any magnitudes which satisfy the definition. The last objection is not very strong, since the learner could read the first proposition of the sixth book immediately after the definition, and would thereby be convinced of the existence of proportionals; the rest may be removed by shewing another definition, more in consonance with common ideas, and demonstrating that, if four magnitudes fall under either of these definitions, they fall under the other also. The definition which we propose is as follows:—" Four magnitudes, A, B, P, and Q, of which B is of the same kind as A, and Q as P, are said to be proportional, if magnitudes B+C and Q+R can be found *as near as we please* to B and Q, so that A, B+C, P and Q+R, are proportional according to the common notion, that is, if whole numbers m and n can satisfy the equations $mA-n(B+C)=0$

$$mP-n(Q+R)=0.$$

We have now to shew that Euclid's definition follows from the one just given, and also that the last follows from Euclid, that is, if there are four magnitudes which fall under either definition, they fall under the other also. Let us first suppose that Euclid's definition is true of A, B, P, and Q, so that

$$\left.\begin{matrix} mA \\ mP \end{matrix}\right\} > = < \left\{\begin{matrix} nB \\ nQ \end{matrix}\right. .$$

This being true, it will follow that we can take m and n, so as not only to make $mA-nB$ less than a given magnitude E, which may be as small as we please, but also so that $mP-nQ$ shall at the same time be less than a given magnitude F, however small this last may be. For if not, while m and n are so taken as to make $mA-nB$ less than E (which it has been proved can be done, however small E may be) suppose, if possible, that the same values of m and n will never make $mP-nQ$ less than some certain quantity F, and let pF be the first multiple of F which exceeds Q, and also let E be taken so small that pE shall be less than B, still more then shall $p(mA-nB)$, or $pmA-pnB$ be less than B. But since pF is greater than Q, and $mP-nQ$ is, by hypothesis greater than F, still more shall $mpP-npQ$ be greater than Q. We have then, if our last supposition be correct, some value of mp and np, for which

$mpA-npB$ is less than B,

while

$mpP-npQ$ is greater than Q,

or

mpA is less than $(np+1)$B,

mpP is greater than $(np+1)$Q,

which is contrary to our first hypothesis respecting A, B, P, and Q, that hypothesis being Euclid's definition of proportion, from which if

mpA is less than $\overline{np+1}$ B

mpP is *less* than $\overline{np+1}$ Q.

We must therefore conclude that if the four quantities A, B, P, and Q, satisfy Euclid's definition of proportion, then m and n may be so taken that $mA-nB$ and $mP-nQ$ shall be as small as we please. Let

$$mA-nB=E \text{ and } E=nC$$
$$mP-nQ=F \qquad F=nR.$$

Then $mA-n(B+C)=0$

$$mP-n(Q+R)=0,$$

and since E and F can, by properly assuming m and n, be made as small as we please, much more can the same be

* It would be expressed algebraically by saying, that if $mA-nB$ and $mP-nQ$ are nothing for the same values of m and n, they are either both positive or both negative, for every other value of m and n.

done with C and R, consequently we can produce $B+C$ and $Q+R$ as near as we please to B and Q, and proportional to A and P, according to the common arithmetical notion. In the same way it may be proved, that on the same hypothesis $B-C$ and $Q-R$ can be found as near to B and Q as we please, and so that A, $B-C$, P and $Q-R$, are proportional according to the ordinary notion. It only remains to shew that if the last-mentioned property be assumed, Euclid's definition of proportion will follow from it. That is, if quantities can be exhibited as near to P and Q as we please, which are proportional to A and B, according to the ordinary notion, it follows that

$$\left.\begin{matrix} mA \\ mP \end{matrix}\right\} > = < \left\{\begin{matrix} nB \\ nQ. \end{matrix}\right.$$

For let $B+C$ and $Q+R$ be two quantities, such that

$$fA - g(B+C) = 0$$
$$fP - g(Q+R) = 0,$$

in which, by the hypothesis, f and g can be so taken that C and R are as small as we please. We have already shewn that in this case (m and n being any numbers whatever) mA is never greater or less than $n(B+C)$, without mP being at the same time the same with regard to $n(Q+R)$. That is, if

mA is greater than $nB+nC$,

then

mP . . . . $nQ+nR$.

Take some *given** values for m and n, fulfilling the first condition; then, since C and R may be as small as we please, the same is true of nC and nR; if then

mA is greater than nB

mP . . . . nQ.

For if not, let $mA=nB+x$, while $mP=nQ-y$, x and y being some definite magnitudes. Then if

$$nB+x>nB+nC$$
$$nQ-y>nQ+nR,$$

which last equation is evidently impossible; therefore if $mA>nB$, $mP>nQ$. In the same way it may be proved, that if $mA<nB$, $mP<nQ$, &c., so that Euclid's definition is shewn to be a necessary consequence of the one proposed.

The definition of proportion which we have here given, and the methods by which we have established its identity with the one in use, bear a close analogy to the process used by the ancients, and denominated by the moderns the *method of exhaustions*. We have seen that the common definition of proportion fails in certain cases, where the magnitudes are what we have called incommensurable, but at the same time we have shewn, that though in this case we can never take m and n, so that $mA=nB$, or $mA-nB=0$, we can nevertheless find m and n, so that mA shall differ from nB by a quantity less than any which we please to assign. We therefore extend the definition of the word proportion, and make it embrace not only those magnitudes which fulfil a given condition, but also others, of which it is impossible that they should fulfil that condition, provided always, that whatever magnitudes we call by the name of proportionals, they must be such as to admit of other magnitudes being taken as near as we please to the first, which are proportional, according to the common arithmetical notion. It is on the same principle that in algebra we admit the existence of such a quantity as $\sqrt{2}$, and use it in the same manner as a definite fraction, although there is no such fraction in reality as, multiplied by itself, will give 2 as the product. But, however small a quantity we may name, we can assign a fraction, which, multiplied by itself, shall differ less from 2 than that quantity.

Having established the properties of rectilinear figures, as far as their proportions are concerned, it is necessary to ascertain the properties of curvilinear figures in this respect. And here occurs a difficulty of the same kind as that which met us at the outset, for no rectilinear figure, how small soever its sides may be, or how great soever their number, can be called curvilinear. Nevertheless, it may be shewn, that in every curve a rectilinear figure may be inscribed, whose area and perimeter shall differ from the area and perimeter of the curve by magnitudes less than any assigned magnitudes. The circle is the only curve whose properties are considered in elementary geometry, and this is shewn in the Treatise on Geometry, III. 3t. Indeed, for this or any other curve, the proposition is almost self-evident. This being granted, the properties of curvilinear figures are established by help of the following theorem.

If A, B, C, and D are always proportional, and of these, if C and D may be made as near as we please to P and Q,

* It is very necessary to recollect, that the relations just expressed are true for every value of m and n; and therefore true for any particular case. In this investigation f and g may both be very great in order that C and R may be sufficiently small, and we must suppose them to vary with the values we give to C and R, or rather the limits which we assign to them; but m and n are *given*.

than which they are always both greater or both less, then A, B, P, and Q are proportional.

Let $C = P + P'$, and $D = Q + Q'$, where by hypothesis P' and Q' may be made as small as we please, and A, B, $P + P'$, and $Q + Q'$, are proportionals. If A, B, P, and Q are not proportionals, let P and $Q + R$ be proportional to A and B. Then, since A and B are proportional to $P + P'$ and $Q + Q'$, and also to P and $Q + R$,

$$P + P' : Q + Q' :: P : Q + R$$

in which all the magnitudes are of the same kind. Now, let P' and Q' be so taken, that Q' is less than R, which may be done, since by hypothesis Q' can be as small as we please. Hence $Q + Q'$ is less than $Q + R$, and therefore $P + P'$ is less than P, which is absurd. In the same way it may be proved that P is not to $Q - R$ in the proportion of A to B, and consequently P is to Q in the proportion of A to B. This theorem, with those which prove that the surfaces, solidities, areas, and lengths, of curve lines and surfaces, may be represented as nearly as we please by the surfaces, &c., of rectilinear figures and solids, form the method of exhaustions. We may refer for instances to the Treatise on Geometry, III. 32, 33. IV. 30. V. 2, 15, 16, &c. In this method are the first germs of that theory, which, under the name of Fluxions, or the Differential Calculus, contains the principles of all the methods of investigation now employed, whether in pure or mixed mathematics.

Chapter XVII.

Application of Algebra to the measurement of Lines, Angles, Proportion of Figures, and Surfaces.

We have already defined a measure, and have noticed several instances of magnitudes of one kind being measured by those of another. But the most useful measure, and that with which we are most familiar, is number. We express one line by the number of times which another line is repeated in it, or if the first is not exactly contained in the second, by the greatest number of the second contained in the first, together with the fraction of the second, which will complete the first. Thus, suppose the line A contains B m times, with a remainder which can be formed by dividing B into q parts, and taking p of them. Then B is to A in the proportion of 1 to $m + \frac{p}{q}$, or as q to $mq + p$, and if B be a fixed line, which is used for the comparison of all lines whatsoever, then the line A is $m + \frac{p}{q}$, or $\frac{mq+p}{q}$, if it be understood that for every unit in m, B is to be taken, and also that for $\frac{p}{q}$ the same fraction of B is to be taken that $\frac{p}{q}$ is of unity. In this case B is called the *linear unit.*

But here we suppose that a line B being taken, the ratio of any other line A to B can be expressed by that of the whole numbers $mq + p$ to q, which we have shewn in some cases to be impossible. If we take one of these cases, $mA - nB$, though it can never be made equal to nothing, can be made as small as we please, by properly assuming m and n. Let $mA - nB = E$, then $A = \frac{n}{m} B + \frac{E}{m}$, and since $\frac{E}{m}$ can be made as small as we please, A can be represented as nearly as we please by a fraction $\frac{n}{m}$, where B is the linear unit. Hence, in practice, an approximation may be found to the value of A, sufficient for any purpose whatever, in the following manner, which will be easily understood by the student, who has a tolerable facility in performing the operations of algebra.

Let A contain B, p times with a rem^r. P
B contain P, q times with a rem^r. Q
P contain Q, r times with a rem^r. R

and so on. If the two magnitudes are commensurable, this operation will end by one of the remainders becoming nothing. For, let A and B have a common measure E, then P has the same measure, for P is $A - pB$, of which both A and pB contain E an exact number of times. Again, because B and P contain the common measure E, Q has the same measure, and so on. All the remainders are therefore multiples of E, and if E be the linear unit, are represented by whole numbers. Now, if a whole number be continually diminished by a whole number, it must, if the operation can be continued without end, eventually become nothing. If, therefore, the remainder never disappears, it is a sign that the magnitudes A and B are incommensurable. Nevertheless, approximate whole numbers can be found whose ratio is as near as we please to the ratio of A and B.

From the suppositions above-mentioned, it appears that—

$$
\begin{aligned}
A &= pB + P^{*} \quad &(a)\\
B &= qP + Q \quad &(b)\\
P &= rQ + R \quad &(c)\\
Q &= sR + S \quad &(d)\\
R &= tS + T \quad &(e)\\
&\text{\&c.} \quad \text{\&c.} &\text{\&c.}
\end{aligned}
$$

Substitute in (b) the value of P derived from (a), find Q from the result, and substitute the values of P and Q in (c); find a value of R from the result, and substitute the values of Q and R in (d), and so on, which give the following series of equations,—

$$
\begin{aligned}
A &= pB + P\\
qA &= (pq+1)B - Q\\
(qr+1)A &= (pqr+p+r)B + R\\
(qrs+q+s)A &= (pqrs+ps+rs+pq+1)B - S\\
qrst+qt+st+qr+1)A &= (pqrst+{}_{rs}t+rst+pqt+pqr+p+r+t)B + T
\end{aligned}
$$

On inspection, it will be found that the coefficients of A and B in these equations may be formed by a very simple law. In each a letter is introduced which was not in the preceding one, and every coefficient is formed from the two preceding, by multiplying the one immediately preceding by the new letter, and adding to the product the one which comes before that. Thus the third coefficient of B is $pqr+p+r$; the new letter is r, and the two preceding coefficients are $pq+1$ and p, and $pqr+p+r = \overline{pq+1}\,r+p$. The remainders enter also with signs alternately positive and negative. Let x, x', and x'' be the n^{th} $\overline{n+1}^{\text{th}}$ and $\overline{n+2}^{\text{th}}$ numbers of the series p, q, r, &c., and X, X′ and X″ the corresponding remainders. Let the corresponding equations be,

$$
\begin{aligned}
a\ A &= b\ B + X\\
a'\ A &= b'\ B - X'\\
a''\ A &= b''\ B + X''
\end{aligned}
$$

Here n must be supposed odd, since, were it even, the first equation would be $aA = bB - X$, as will be seen by reference to the equations deduced. Hence, from the law of formation of the coefficients, x'' being the new letter in the last equation,

$$
\begin{aligned}
a'' &= a'\,x'' + a\\
b'' &= b'\,x'' + b
\end{aligned}
$$

Eliminate x'' from these two, the result of which is $a''b' - a'b'' = ab' - a'b$, the first side of which is the numerator of $\frac{b'}{a'} - \frac{b''}{a''}$ and the second of $\frac{b'}{a'} - \frac{b}{a}$. It appears then that $\frac{b'}{a'}$ is either greater than both $\frac{b}{a}$ and $\frac{b''}{a''}$ or less than both, since $\frac{b'}{a'} - \frac{b''}{a''}$ and $\frac{b'}{a'} - \frac{b}{a}$ will both have the same sign, the numerators being the same and the denominators positive. It may also be proved that $\frac{b''}{a''}$ lies between $\frac{b}{a}$ and $\frac{b'}{a'}$ by means of the following lemma.

The fraction $\frac{m+n}{p+q}$ must lie between $\frac{m}{p}$ and $\frac{n}{q}$; for let $\frac{m}{p}$ be the greater of the two last, or $\frac{m}{p} > \frac{n}{q}$ then $mq > np$, or $\frac{mq}{mp} > \frac{np}{mp}$, or $\frac{q}{p} > \frac{n}{m}$, and $1 + \frac{q}{p} > 1 + \frac{n}{m}$; therefore $\frac{1+\frac{n}{m}}{1+\frac{q}{p}}$ is less than unity, and any fraction multiplied by this, is diminished. But $\frac{m+n}{p+q}$ is $\frac{m}{p} \times \frac{1+\frac{n}{m}}{1+\frac{q}{p}}$, and is therefore less than $\frac{m}{p}$, the greater of the two. In the same way it may be proved to be greater than $\frac{n}{q}$, the least of the two.

This being premised, since $\frac{b''}{a''} = \frac{b'\,x'' + b}{a'\,x'' + a}$, it lies between $\frac{b'\,x''}{a'\,x''}$ and $\frac{b}{a}$ or between $\frac{b'}{a'}$ and $\frac{b}{a}$.

Call the coefficients of A and B in the series of equations, $a_1\ a_2$, &c. $b_1\ b_2$, &c. and form the series of fractions $\frac{b_1}{a_1}\ \frac{b_2}{a_2}\ \frac{b_3}{a_3}$ &c. The two first of these will be $\frac{p}{1}$ and $\frac{pq+1}{q}$, of which the second is the greater, since it is $p + \frac{1}{q}$ Hence by what has been proved $\frac{b_3}{a_3}$ is less than

* Throughout these investigations the capital letters represent the lines themselves, and not the numbers of units, which represent them, while the small letters are whole numbers, as in the last chapter.

$\frac{b_2}{a_2}$ and grealer than $\frac{b_1}{a_1}$; and every fraction is greater or less than the one which comes before it, according as the number of its equation is even or odd. Again, as the numerator of the difference of two successive fractions $\frac{a''}{b''}$ and $\frac{a'}{b'}$ is the same as that of $\frac{a'}{b'}$ and $\frac{a}{b}$, whatever the numerator of the first difference is, the same must be that of the second, third, &c. and of all the rest. But the numerator of the difference of $\frac{p}{1}$ and $\frac{pq+1}{q}$ is 1; therefore either $ab' - a'b$, or $a'b - ab'$, is 1 according as $\frac{b'}{a'}$ or $\frac{b}{a}$ is the greater of the two, that is according as n is odd or even*. Now since the n^{th} and $\overline{n+1}^{th}$ equations, n being odd, are

$$aA = bB + X$$
$$\text{and } a'A = b'B - X';$$

by eliminating A we have

$$(ab'-a'b)\ B = a'X + aX'$$
$$\text{or}\quad B = a'X + aX'$$

since $ab' - a'b = 1$; and since the remainders decrease and the coefficients increase, $a' > a$ and $X > X'$, whence $2aX' < a'X + aX'$, or $2aX' < B$ and $X' < \frac{B}{2a}$; the remainder therefore which comes in the $\overline{n+1}^{th}$ equation is less than the part of B arising from dividing it into twice as many equal parts as there are units in the n^{th} coefficient of A; and as this number of units may increase to any amount whatever, by carrying the process far enough, $\frac{B}{2a}$ may be made as small as we please, and, *à fortiori*, the remainders may be made as small as we please. The same theorem may be proved in a similar way, if we begin at an even step of the process.

Resuming the equations

$$aA = bB + X$$
$$a'A = b'B - X'$$
$$a''A = b''B + X''$$

From the second, $A = \frac{b'}{a'}B - \frac{X'}{a'}$;

and since $X' < \frac{B}{2a}, \frac{X'}{a'} < \frac{B}{2aa'}$, or if B be taken as the linear unit, $\frac{b'}{a'}$ will express the line A with an error less than $\frac{1}{2aa'}$, which last may be made as small as we please by continuing the process.

It is also evident that $\frac{b}{a}$ is too small, while $\frac{b'}{a'}$ is too great; and since X and X' are less than B, $aA < bB + B$, or $\frac{b+1}{a}$ is too great, while $a'A > b'B - B$, or $\frac{b'-1}{a'}$, is too small. Again, $A - \frac{b}{a}B = \frac{X}{a}$ and $\frac{b'}{a'}B - A = \frac{X'}{a'}$. Now $X' < X$ and $a' > a$; whence $\frac{X'}{a'} < \frac{X}{a}$; that is, $\frac{b'}{a'}B$ exceeds A by a less quantity than $\frac{b}{a}$ B falls short of it, so that $\frac{b'}{a'}$ is a nearer representation of A than $\frac{b}{a}$, though on a different side of it.

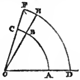
We have thus shewn how to find the representation of a line by means of a linear unit, which is incommensurable with it, to any degree of nearness which we please. This, though little used in practice, is necessary to the theory; and the student will see that the method here followed is nearly the same as that of continued fractions in Algebra.

We now come to the measurement of an angle; and here it must be observed that there are two distinct measures employed, one exclusively in theory, and one in practice. The latter is the well-known division of the right angle into 90 equal parts, each of which is one degree; that of the degree into 60 equal parts, each of which is one minute; and of the minute into 60 parts, each of which is one second. On these it is unnecessary to enlarge, as this division is perfectly arbitrary, and no reason can be assigned, as far as theory is concerned, for conceiving the right angle to be so divided. But it is far otherwise with the measure which we come to consider, to which we shall be naturally led by the theorems relating to the circle. Assume any angle, AOB, as the angular unit, and any other angle, AOC. Let r be the number * of linear units con-

* We might say that $ab' - a'b$ is alternately + 1 and − 1; but we wish to avoid the use of the isolated negative sign.

* It must be recollected that the word number means both *whole* and *fractional* number.

tained in the radius OA, and t and s the lengths, or number of units contained in the arcs AB and AC. Then since



the angles AOB and AOC are proportional to the arcs AB and AC, or to the numbers t and s, we have

Angle AOC is $\frac{s}{t}$ of the angle AOB;

and the angle AOB being the angular unit, the number $\frac{s}{t}$ is that which expresses the angle AOC. This number is the same for the same angle, whatever circle is chosen; in the circle FD the proportion of the arcs DE and DF is the same as that of AB and AC: for since similar arcs of different circles are proportional to their radii,

AB : DE :: OA : OD
Also AC : DF :: OA : OD
∴ AB : DE :: AC : DF;

therefore the proportion of DF to DE is that of s to t, and $\frac{s}{t}$ is the measure of the angle DOF, DOE being the unit, as before. It only remains to choose the angular unit AOB, and here that angle naturally presents itself, whose arc is equal to the radius in length. This, from what is proved in Geometry, will be the same for all circles, since in two circles, arcs which have the same ratio (in this case that of equality) to their radii, subtend the same angle. Let $t = r$, then $\frac{s}{r}$ is the number corresponding to the angle whose arc is s. This is the number which is always employed in theory as the measure of an angle, and it has the advantage of being independent of all linear units; for suppose s and r to be expressed, for example, in feet, then $12s$ and $12r$ are the numbers of inches in the same lines, and by the common theory of fractions $\frac{s}{r} = \frac{12s}{12r}$. Generally, the alteration of the unit does not affect the number which expresses the *ratio* of two magnitudes. When it is said that the angle $= \frac{\text{arc}}{\text{radius}}$, it is only meant that, *on one particular supposition*, namely, that the angle 1 is that angle whose arc is equal to the radius, the number of these *units* in any other angle is found by dividing the number of *linear* units in its arc by the number of *linear* units in the radius. It only remains to give a formula for finding the number of degrees, minutes, and seconds in an angle, whose theoretical measure is given. It is proved in geometry that the ratio of the circumference of a circle to its diameter, or that of half the circumference to its radius, though it cannot be expressed exactly, is between 3.14159265 and 3.14159266. Taking the last of these, which will be more than a sufficient approximation for our purpose, it follows that the radius being r, one half of the circumference is $r \times 3.14159266$; and one-fourth of the circumference, or the arc of a right angle, is $r \times 1.57079633$. Hence the number of units above described, in a right angle, is $\frac{\text{arc}}{\text{radius}}$, or 1.57079633. And the number of seconds in a right angle is $90 \times 60 \times 60$, or 324000. Hence if ϑ be an angle expressed in units of the first kind, and A the number of seconds in the same angle, the proportion of A to 324000 will also be that of ϑ to 1.57079633. To understand this, recollect that the proportion of any angle to the right angle is not altered by changing the units in which both are expressed, so that the numbers which express the two for one unit, are proportional to the like numbers for another.

Hence A : 324000 :: ϑ : 1.57079633;

$$\text{or } A = \frac{324000}{1.57079633} \times \vartheta;$$

or $A = 206265 \times \vartheta$, very nearly.

Suppose, for example, the number of seconds in the theoretical unit itself is required. Here $\vartheta = 1$ and $A = 206265$; similarly if A be 1, $\vartheta = \frac{1}{206265}$, which is the expression for the angle of one second referred to the other unit. In this way, any angle, whose number of seconds is given, may be expressed in terms of the angle, whose arc is equal to the radius, which, for distinction, might be called the *theoretical* unit. This unit is used without exception in analysis; thus, in the formula, for what is called in trigonometry the sine of x, *viz.*,

$$\sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} -, \text{ \&c.}$$

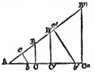
If x be an angle of one second, it is not 1 which must be substituted for x, but $\frac{1}{206265}$.

—

The number 3.14159265, &c. is called π, and is the measure, in theoretical units, of two right angles. Also $\frac{\pi}{2}$ is the measure of one right angle; but it must not be confounded, as is frequently done, with 90°. It is true that they stand for the same angle, but on different suppositions with respect to the unit; the unit of the first being very nearly $\frac{206265}{60 \times 60}$ times that of the second.

There are methods of ascertaining the value of one magnitude by means of another, which, though it varies with the first, is not a measure of it, since the increments of the two are not proportional; for example, when, if the first be doubled, the second, though it changes in a definite manner, is not doubled. Such is the connexion between a number and its common logarithm, which latter increases much more slowly than its number; since, while the logarithm changes from 0 to 1, and from 1 to 2, the number changes from 1 to 10, and from 10 to 100, and so on.

Now, of all triangles which have the same angles, the proportions of the sides are the same. If, therefore, any angle



CAB be given, and from any points B, B′, B″, &c. in one of its sides, and b, b', &c. in the other, perpendiculars be let fall on the remaining side, the triangles BAC, B′AC′, bAc, &c. having a right angle in all, and the angle A common, are equiangular; that is, one angle being given, which is not a right angle, the proportions of every right-angled triangle in which that angle occurs are given also; and, *vice versâ*, if the proportion, or ratio of any two sides of a right-angled triangle are given, the angles of the triangle are given (Geom. II. 33.).

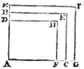
To these ratios names are given; and as the ratios themselves are connected with the angles, so that one of either set being given, *viz.* ratios or angles, all of both are known, their names bear in them the name of the angle to which they are supposed to be referred. Thus, $\frac{BC}{AB}$ or $\frac{\text{side opposite to A}}{\text{hypothenuse}}$, is called the *sine* of A; while $\frac{AC}{AB}$, or $\frac{\text{side opposite to B}}{\text{hypothenuse}}$, or the sine of B, the complement * of A, is called the *cosine* of A. The following table expresses the names which are given to the six ratios, $\frac{BC}{AB}$ $\frac{AC}{AB}$ $\frac{BC}{AC}$ $\frac{AC}{BC}$ $\frac{AB}{AC}$ and $\frac{AB}{BC}$, relatively to both angles, with the abbreviations made use of. The terms opp., adj., and hyp., stand for, opposite side, adjacent side, and hypothenuse, and refer to the angle last mentioned in the table.

	is the	being	or	being	These are written.	
$\frac{BC}{AB}$	sine of A	$\frac{\text{opp.}}{\text{hyp.}}$	cosine of B	$\frac{\text{adj.}}{\text{hyp.}}$	sin A	cos B
$\frac{AC}{AB}$	cosine of A	$\frac{\text{adj.}}{\text{hyp.}}$	sine of B	$\frac{\text{opp.}}{\text{hyp.}}$	cos A	sin B
$\frac{BC}{AC}$	tangent of A	$\frac{\text{opp.}}{\text{adj.}}$	cotangent of B	$\frac{\text{adj.}}{\text{opp.}}$	tan A	cot B
$\frac{AC}{BC}$	cotangent of A	$\frac{\text{adj.}}{\text{opp.}}$	tangent of B	$\frac{\text{opp.}}{\text{adj.}}$	oot A	tan B
$\frac{AB}{AC}$	secant of A	$\frac{\text{hyp.}}{\text{adj.}}$	cosecant of B	$\frac{\text{hyp.}}{\text{opp.}}$	sec A	cosec B
$\frac{AB}{BC}$	cosecant of A	$\frac{\text{hyp.}}{\text{opp.}}$	secant of B	$\frac{\text{hyp.}}{\text{adj.}}$	cosec A	sec B

* When two angles are together equal to a right angle, each is called the complement of the other. Generally, complement is the name given to one part of a whole relatively to the rest. Thus, 10 being made of 7 and 3, 7 is the *complement* of 3 *to* 10.

If all angles be taken, beginning from one minute, and proceeding through 2′, 3′, &c., up to 45°, or 2700′, and tables be formed by a calculation, the nature of which we cannot explain here, of their sines, cosines, and tangents, or of the logarithms of these, the proportions of every right-angled triangle, one of whose angles is an exact number of minutes, are registered. We say sines, cosines, and tangents only, because it is evident, from the table above made, that the cosecant, secant, and cotangent of any angle, are the reciprocals of its sine, cosine, and tangent, respectively. Again, the table need only include 45°, instead of the whole right angle, because, the sine of an angle above 45° being the cosine of its complement, which is less than 45°, is already registered. Now, as all rectilinear figures can be divided into triangles, and every triangle is either right-angled, or the sum or difference of two right-angled triangles, a table of this sort is ultimately a register of the proportions of all figures whatsoever. The rules for applying these lables form the subject of trigonometry, which is one of lhe great branches of the application of algebra to geomelry. In a right-angled triangle, whose angles do not contain an exact number of minutes, the proportions may be found from the lables by the method explained in Chapter XI. of this treatise. It must be observed, that the sine, cosine, &c. are not *measures* of lheir angle; for, though the angle is given when either of them is given, yet, if the angle be increased in any proportion, the sine is not increased in the same proportion. Thus, sin 2A is not double of sin A.

The measurement of surfaces may be reduced to the measurement of rectangles; since every figure may be divided into triangles, and every triangle is half of a rectangle on the same base and altitude. The superficial unit or quantity of space, in terms of which it is chosen lo express all other spaces, is perfectly arbitrary; nevertheless, a common theorem points out the convenience of choosing, as the superficial unit, the square on that line which is chosen as the linear unit. If the sides of a rectangle contain a and b units (Geometry I. 29.), the rectangle itself contains ab of the squares described on the unit. This proposition is true, even when a and b are fractional. Let the number of units in lhe sides be $\frac{m}{n}$ and $\frac{p}{q}$, and lake another unit which is $\frac{1}{nq}$ of the first, or is obtained by dividing the first unit into nq parts, and taking one of them. Then, by the proposition just quoled, the square described on the larger unit contains $nq \times nq$ of that described on the smaller. Again, since $\frac{m}{n}$ and $\frac{p}{q}$ are the same fractions as $\frac{mq}{nq}$ and $\frac{np}{nq}$, they are formed by dividing the first unit into nq parts, and taking one of these parts mq and np times; that is, they contain mq and np of the smaller unit; and, therefore, the rectangle contained by them, contains $mq \times np$ of the square described on the smaller unit. But of these there are $n\,q \times n\,q$ in the square on the longer unit; and, therefore, $\frac{mq \times np}{nq \times nq}$, or $\frac{mp\,nq}{nq\,nq}$, or $\frac{mp}{nq}$, is lhe number of the larger squares conlained in lhe reclangle. But $\frac{mp}{nq}$ is lhe algebraical producl of $\frac{m}{n}$ and $\frac{p}{q}$. This proposition is true in the following sense, where the sides of the rectangle are incommensurable with the unit. Whatever the unit may be, we have shewn that, for any incommensurable magnitude, we cau go on finding b and a lwo whole numbers, so that $\frac{b}{a}$ is too little, and $\frac{b+1}{a}$ too great: until a is as great as we please. Let AB and AC be the sides of a rectangle AK, and let them



be incommensurable with the unit M. Let the lines AF and AG, containing $\frac{b}{a}$ and $\frac{b+1}{a}$ units, be respectively less and grealer than AC; and let AD and AE, containing $\frac{c}{d}$ and $\frac{c+1}{d}$ units, be respectively less and greater than AB; and complete the figure. The rectangles

A H and A I contain, respectively, $\frac{b}{a} \times \frac{c}{d}$, and $\frac{b+1}{a} \times \frac{c+1}{d}$ square units*, and the first is less than the given rectangle, and the second greater; consequently the given rectangle does not differ from either, so much as they differ from one another. But the difference of AH and AI is $\frac{(b+1)(c+1)}{ad} - \frac{bc}{ad}$, or

$$\frac{b+c+1}{ad}, \text{ or } \frac{b}{ad} + \frac{c}{ad} + \frac{1}{ad}, \text{ or}$$

$$\frac{1}{d} \cdot \frac{b}{a} + \frac{1}{a} \cdot \frac{c}{d} + \frac{1}{ad}.$$

Proceed through two †, four, six, &c. steps of the approximation. The linear unit being M, the results will be such, that $\frac{b}{a}$ M will be always less than AC, but continually approaching to it. Hence $\frac{1}{d} \cdot \frac{b}{a}$ M is always less than $\frac{\text{AC}}{d}$; and since AC remains the same, and d is a number which may increase as much as we please, by carrying on the approximation, $\frac{\text{AC}}{d}$ and *à fortiori* $\frac{1}{d}\frac{b}{a}$ M may be made as small a line as we please; that is, $\frac{1}{d} \cdot \frac{b}{a}$ may be made as small as we please, and so may $\frac{1}{a} \cdot \frac{c}{d}$ in the same manner. Also $\frac{1}{ad}$ may be made as small as we please; and, therefore, also, the sum $\frac{1}{d}\frac{b}{a} + \frac{1}{a}\frac{c}{d} + \frac{1}{ad}$. But this number, when the unit is the square unit, represents the difference of the rectangles AH and AI, and is greater than the difference of AK and AI; therefore, the approximate fractions which represent AC and AB may be brought so near, that their product shall, as nearly as we please, represent the number of square units in their rectangle.

In precisely the same manner it may be proved, that if the unit of content or solidity be the cube described on the unit of length, the number of cubical units in any rectangular parallelopiped, is the product of the number of linear units in its three sides, whether these numbers be whole or fractional; and in the sense just established, even if they be incommensurable with the unit.

These algebraical relations between the sides and content of a rectangle or parallelopiped were observed by the Greek geometers; but as they had no distinct science of algebra, and a very imperfect system of arithmetic, while, with them, geometry was in an advanced state; instead of applying algebra to geometry, what they knew of the first was by deduction from the last: hence the names which, to this day, are given to aa, aaa, ab, which are called the *square* of a, the *cube* of a, the rectangle of a and b. The student is thus led to imagine that he has proved that square described on the line, whose number of units is a, to contain aa square units, because he calls the latter the square of a. He must, however, recollect, that squares in algebra and geometry mean distinct things. It would be much better if he would accustom himself to call aa and aaa the second and third powers of a, by which means the confusion would be avoided. It is, nevertheless, too much to expect that a method of speaking, so commonly received, should ever be changed; all that can be done is, to point out the real connexion of the geometrical and algebraical signification. This, if once thoroughly understood, will prevent any future misconception.

* The phrase *square unit* is the abbreviation of square described on the unit.

† This is done, because, by proceeding one step at a time, $\frac{b}{a}$ is alternately too little and too great to represent AC; whereas we wish the successive steps to give results always less than AC.

INDEX.

PREFACE.

In compiling the following pages, my object has been to notice particularly several points in the principles of Algebra and Geometry, which have not obtained their due importance in our elementary works on these sciences. There are two classes of men who might be benefited by a work of this kind, viz., teachers of the elements, who have hitherto confined their pupils to the working of rules, without demonstration, and students, who, having acquired some knowledge under this system, find their further progress checked by the insufficiency of their previous methods and attainments. To such it must be an irksome task to recommence their studies entirely; I have therefore placed before them, by itself, the part which has been omitted in their mathematical education, presuming throughout in my reader, such a knowledge of the rules of algebra, and the Theorems of Euclid, as is usually obtained in schools.

It is needless to say that those who have the advantage of University education, will not find more in this Treatise than a little thought would enable them to collect from the best works now in use, both at Cambridge and Oxford. Nor do I pretend to settle the many disputed points, on which I have necessarily been obliged to treat. The perusal of the opinions of an individual, offered simply as such, may excite many to become enquirers, who would otherwise have been workers of rules and followers of dogmas. They may not ultimately coincide in the views promulgated by the work which first drew their attention, but the benefit which they will derive from it is not the less on that account. I am not, however, responsible for the contents of this treatise, further than for the manner in which they are presented, as most of the opinions here maintained have been found in the writings of eminent mathematicians.

It has been my endeavour to avoid entering into the purely metaphysical part of the difficulties of algebra. The student is, in my opinion, little the better for such discussions, though he may derive such conviction of the truth of results by deduction from particular cases, as no *à priori* reasoning can give to a beginner. In treating, therefore, on the negative sign, on impossible quantities, and on fractions of the form $\frac{0}{0}$ &c., I have followed the method adopted by several of the most esteemed continental writers, of referring the explanation to some particular problem, and shewing how to gain the same from any other. Those who admit such expressions as $-a$, $\sqrt{-a}$ $\frac{0}{0}$ &c., have never produced any clearer method; while those who call them absurdities, and would reject the altogether, must, I think, be forced to admit the fact, that in algebra the

different species of contradictions in problems are attended with distinct absurdities, resulting from them as necessarily as different numerical results from different numerical data. This being granted, the whole of the ninth chapter of this work may be considered as an inquiry into the nature of the different misconceptions, which give rise to the various expressions above alluded to. To this view of the question I have leaned, finding no other so satisfactory to my own mind.

The number of mathematical students, increased as it has been of late years, would be much augmented if those who hold the highest rank in science would condescend to give more effective assistance in clearing the elements of the difficulties which they present. If any one claiming that title should think my attempt obscure or erroneous, he must share the blame with me, since it is through his neglect that I have been enabled to avail myself of an opportunity to perform a task which I would gladly have seen confided to more skilful hands.

AUGUSTUS DE MORGAN.

90, *Guilford-street, Nov.* 22*nd*, 1831.

ERRATA.

Page 11, line 13, *for* rules, *read* rule.
" 14, " 1 from bottom, *for* $\frac{6}{}$, *read* $\frac{6}{10}$.
" 20, " 14, *for* $4\,a\,b^2$, *read* $4\,a^2\,b^2$.
" 28, " 36 from bottom, for $b\,a$, read $b\,a^2$.
" —, " 3 " *for* $+\,6^3$, *read* $-\,6^3$.
" —, " 1 " *for* $-\,6^4$, *read* $+\,6^4$.
" 33, " 3 " *for* equation *read* equations.
" 34, " 3 " for a, read d.
" 35, " 13 " *for* $+\,a$, *read* $+\,a^2$.
" 42, " 1, *for* $=$, *read* $= 1$.
" 46, " 25 and 26, *for* greater, *read* less.
" 47, " end of 2, *for* a full stop, *read* a comma.
" —, 5 from bottom, *for* $\sqrt{}49 - 24$, *read* $\sqrt{49 - 24}$.
" —, 1 from bottom, *for* $\sqrt{}221 \pm 9$, *read* $\sqrt{221 \pm 9}$.
" 49, 21, *for* $+\,9$, *read* $+\,11$.
" 52, 5, *for* $1\,-$, *read* $-\,1$.
" 53, 17, *for* $\sqrt{\frac{a^2}{4}} - 6$, *read* $\sqrt{\frac{a^2}{4} - 6}$.
" 55, 7, *for* $\left(a\right)^p$, *read* $\left(a^m\right)^p$.
" —, 1 from bottom, *for* $\sqrt{a^q}$, *read* $\sqrt[3]{a^q}$.
" 56, 3, *for* $\sqrt{a^{m+b}}$, *read* $\sqrt[n]{a^{m+b}}$
" —, 1 from bottom, *for* $\sqrt[q]{\left(a^{m \atop n}\right)^p}$, *read* $\sqrt[q]{\left(a^{\frac{m}{n}}\right)^p}$.
" —, 22 from bottom, for $\frac{mp}{q}$, read $\frac{mp}{n}$.
" —, 22 " , for $\frac{mp}{na}$, read $\frac{mp}{nq}$.
" —, 6 from bottom, *for* subtracted, *read* subtraction.
" 58, 3 and 4 from bottom, for $\overline{x + h^m}$, read $\overline{x + h}^m$.
" 60, 3 from bottom, for a, read a^n.

(2

ARITHMETIC AND ALGEBRA.

Notation and Definitions.

1. It is by means of numbers that we are able to express the *magnitude*, that is, the size, of any thing, or of any collection of things that are of the same kind. To do this in any case, we first fix on some portion of that kind of thing, the magnitude of which portion is well known; we then state the number of times that this portion is contained in the thing or collection of things in question. When, for instance, we say that a distance is twelve miles, or a weight twelve pounds, we state how often the known portion of distance, one mile, or the known portion of weight, one pound, is contained in the distance or the weight of which we are speaking. This portion so fixed on is called a *unit*. In the examples just referred to, a mile is used as the unit of distance; a pound, as the unit of weight. In expressing the magnitude of anything, or in subjecting it to calculation, we may make use of that unit which we may find most convenient. We may express a distance in miles, yards, feet, or inches, or a weight in pounds, ounces, or grains. But when we have once fixed on a unit, we must keep its nature clearly in view, otherwise we shall often fall into errors.

Sometimes there is something in the nature of the thing whose magnitude is to be expressed, that furnishes us with a unit by which to express it. In stating the size of a crowd of people, or a fleet of ships, every body would employ a single person, or a single ship, as a unit, and would say how many people or how many ships there were. Sometimes, again, there is nothing to make us choose one portion more than another. In the cases of weights and measures, for instance, the units are fixed only by an understanding among the community who use them; such units are said to be *arbitrary*. It is often very convenient that the same unit should be used as extensively as possible. On this account most governments have made laws enjoining a uniformity of weights and measures, and men of science have invented means of finding the standard unit at all places and times. When we consider a number in the abstract, as when we say that 10 and 6 make 16, the number 1 is our unit. It is called *unity*.

2. A portion definite or indefinite, known or unknown, of any thing that can be expressed by means of units, is called a *quantity*; and may be made the subject of the operations of arithmetic or algebra. In arithmetic, we only deal with quantities expressed in numbers, which, consequently, are always known and definite. But in algebra it is very often necessary to consider quantities that are indefinite or unknown. These quantities cannot possibly be expressed by numbers, for if they were they would be no longer indefinite; and if we do not know them, how can we find numbers to express them? To represent such quantities, the letters of the alphabet are used.

Arithmetic and algebra are thus very intimately connected. In treating of them together, we shall find, on the one hand, that the notation of algebra is very useful in explaining the operations of arithmetic; and, on the other, that these operations furnish perhaps the best practical illustrations of the results of algebra.

3. A man can dig a piece of ground in ten days which it takes his son sixteen days to dig; in what time can they dig it if they work together? The time sought, though it be unknown, is yet a fixed definite number of days; a number capable of being halved or doubled, multiplied, divided, or submitted to any other operation that can be performed on known quantities. We cannot denote it by a number, for we do not know what number to take; when we are going to operate on it, then, which

we must do before we can find what it is, we denote it by a letter; x, for instance. x is properly called an *unknown quantity*.

Again; the time in which the father and son can dig the field working together, depends on the time in which the father can dig it alone, and the time in which the son can dig it alone, and on nothing else. Suppose that we wish to find generally, and without reference to any particular instance, in what way the time sought depends on the father's time and on the son's time. If we express the time taken by the father and that taken by the son in numbers, we may find the time taken by both together; but then it will be a number also, and we shall have answered the question for one particular case only; we shall not have found the general relation which we have seen must subsist. But if we call the time taken by the father a, and that taken by the son b, we do find a general relation if we can in any way find the time taken by both together, since a may be any number of days, and b may be any number of days. Here a and b are quantities that are indefinite, that is, quantities that may stand for any numbers whatever. But they are called *known quantities*, because they must be known before we can answer the question in any particular case. It is usual to denote known quantities by the first letters, as a, b, c, &c.; and unknown by the last, as x, y, z.

4. When we have any number of quantities, and wish to find or to express how many units are contained in them all taken together, we do it by *addition*. When quantities are to be added together, we write them with the mark + between them. Thus $5+7+11$ shows that 5, 7, and 11 are to be added together; and when this is done, the result is called the *sum* of these numbers. So $a+b+c$ is the sum of a, b, and c, expressing how many units are contained in these quantities taken together. + is read *plus* (a latin word meaning *more*); it is called the *sign of addition*, or the *positive sign*; and a quantity to which it is prefixed, is called a *positive quantity*.

5. When we have two quantities, and wish to find or to express the number of units in one of them, after the number of units in the other has been taken away from it, we do this by *subtraction*. When one quantity is to be subtracted from another, we write it after the other with the mark − between them. Thus $9-5$ shows that 5 is to be taken from 9, and is called the *difference* of these numbers. $a-b$ is the difference of a and b, and shews that b is to be subtracted from a. − is read *minus* (a latin word meaning *less*); it is called the *sign of subtraction*, or the *negative sign*, and a quantity to which it is prefixed is called a *negative quantity*.

6. When a quantity has neither of these signs annexed to it, + is always supposed to be its sign, and it is a positive quantity.

7. It will often happen, that in adding quantities together we have to add the same quantity to itself once or more. We may, for example, have such sums as $5+5+5$, or $a+a+a+$ &c. where a may be supposed to be written b times, or as often as there are unities in b. The process by which we find such sums, or by which we express this continued addition, is called *multiplication*. To add three fives together is called multiplying 5 by 3; to take a as often as there are unities in b and add them all together, is called multiplying a by b. These processes we express by writing 3×5 or 3.5; $b\times a$, $b.a$, or more frequently $b\,a$. $b\,a$ is called the *product* of b and a; b and a are called the *factors* of $b\,a$. b is the *multiplier*, and a is the *multiplicand*. So $a.b\,c$ is the product of a and $b\,c$.

8. We very often have such a product as $a\,a$, or a multiplied by itself. This is written a^2. So $a\,a\,a$ is written a^3. Thus 2^3 is $2.2.2$, or 8, and 10^4 is $10.10.10.10$, or 10,000. The number written above to the right, shows the number of times that the quantity to which it is annexed is a factor, and is called the *index*, or the *exponent* of that quantity. a^2, a^3, &c. are called *powers* of a. a^2 is the *second power*, or *square* of a; a^3 is its *third power*, or *cube*; and a^5 is its *fifth* power. So a^m is the m^{th} *power* of a; meaning that a is a factor as often as there are unities in m; whatever number m may be.

9. When there is a product such as $3\,a$, or $5\,a^2\,c$, where one of the factors is a number, that number is called a *coefficient*; 3 is the coefficient of a in the expression $3\,a$, and 5 that of a^2 in $5\,a^2\,c$. When the coefficient is unity, it is not usual to write it; thus $1\,b\,c$ and $b\,c$ are the same.

10. If there be two quantities, a and b,

and it be required to find a third quantity, such that when multiplied by b the product shall be a, we do this or express it by *division*. To divide 20 by 4, for instance, is to find a number which, when multiplied by 4, will give 20 for product. This number we know to be 5. Here 20 is called the *dividend*, 4 the *divisor*, and 5 the *quotient*. We express that a is to be divided by b by writing $a \div b$, or more frequently $\frac{a}{b}$; where a is the dividend, b the divisor and $\frac{a}{b}$ the result of the division, or the quotient. Similarly, $\frac{20}{4}$ or $20 \div 4$ means 5.

It will be seen at once, that this account of division corresponds with the uses to which we have been accustomed to apply it in arithmetic. Thus, if we want to find how many shillings there are in 180 pence, we divide 180 by 12. Now, whatever be the number of shillings sought, since there are 12 pence in a shilling, 12 times that number must make 180; so that the number required is the number which, when multiplied by 12, gives 180 for product. Again, if there be 725 men, and we wish to divide them into 25 equal companies, and for that purpose to find how many men are to be in every company; 25 times this number, whatever it may be, must make 725. We seek the number then, which, when multiplied by 25, gives 725 for product, we find it by dividing 725 by 25.

As multiplication is a continued addition, division may be regarded as a continued subtraction. In dividing 30 by 6, for example, we may be said to find out how often 6 can be subtracted from 30; that is, how often 6 is contained in 30.

11. The *square root* of any quantity, a, is a quantity which, when multiplied by itself, produces a. We express the square root of a by writing $^2\sqrt{a}$, or simply $\sqrt{a}$. Thus $\sqrt{4}$ is 2, for $2 . 2$ is 4; $\sqrt{b^2}$ is b, for $b . b$ makes b^2. The *cube root* of a is a quantity which, when multiplied into itself twice, produces a; we express it by writing $^3\sqrt{a}$. Thus $^3\sqrt{27}$ is 3, for $3 . 3 . 3$ is 27. So $^4\sqrt{a}$ is a quantity which, when multiplied into itself 3 times, produces a, and is called the *fourth root* of a. Similarly, $^5\sqrt{a}$ is the *fifth root* of a, and so on; and $^m\sqrt{a}$ is the m^{th} *root* of a, or that quantity which, multiplied by itself as often as there are unities in the number which is one less than m, produces a. The mark $\sqrt{\ }$ is called the *radical sign*, from a Latin word meaning a *root*.

12. When a quantity is made up of other quantities connected with each other by means of the signs $+$ and $-$, the quantities of which it is so composed are called its *terms*. Thus $a c$, $- a b$, and d^2, are the terms of $a c - a b + d^2$.

13. When any of the operations we have been describing is to be performed on a quantity consisting of more than a single term, to prevent ambiguity the quantity is enclosed in brackets, or has a line drawn over it. Thus $a - (c + d + e)$, or $a - \overline{c + d + e}$, means that the whole quantity $c + d + e$ is to be taken from a; whereas if it were written $a - c + d + e$, it would mean that c only is to be taken from a, and d and e added to it. So $(a + b)\ (c + d)$, or $\overline{a + b} \,.\, \overline{c + d}$ means that the whole quantity $c + d$ is to be multiplied by the whole quantity $a + b$. The line drawn over a quantity for this purpose is called a *vinculum*, (the Latin word for a *fetter*) or a *bar*. In the same way $(a^2 + b^2)^3$ means the cube of $a^2 + b^2$.

14. The mark $=$ is called the *sign of equality*; it shows that the quantities between which it stands are equal to one another. Thus $7 + 8 - 9 = 6$, $9 \times 13 = 117$. As further examples of what has been laid down, make $a = 1$, $b = 2$, $c = 3$, and $d = 4$, and we shall find that

$$7 a + 7 c - 3 b - d = 7 + 21 - 6 - 4 = 18,$$

$$(2 a + b c) (4 d - b d) = (2 + 6) . (16 - 8) = 8 . 8 = 64,$$

$$\frac{a b + c d}{2 c - d} = \frac{2 + 12}{6 - 4} = \frac{14}{2} = 7,$$

$$(5 a^2 - 3) d^2 - (d - b)^3 = (5 - 3) 16 - (4 - 2)^3 = 32 - 8 = 24.$$

Of Addition and Subtraction.

15. When quantities, if they differ at all, differ only in their coefficients [art. 9] they are called *like quantities*. When they differ in other respects they are called *unlike quantities*.

$$2\,a\,c \text{ and } 3\,a\,c,\ 5\,a^2 \text{ and } 7\,a^2,$$

$$27\,a^2\,b \text{ and } a^2\,b$$

are severally couples of like quantities;

$$a\ b,\ 5\,a^2,\ 5\,a^2\,b,\ 3\,a\,b$$

are all unlike quantities. When we have to add, or to subtract unlike quantities, we write them in the same line with the sign of addition or of subtraction between them. Thus a added to b we can express only by writing $a + b$; $a\,b^2$ subtracted from $a^2\,b$ we express by writing $a^2\,b - a\,b^2$.

But take the case of like quantities, and suppose that $2\,a$ is to be added to $3\,a$. That is, $a + a$ is to be added to $a + a + a$ [art. 7]. The sum must be $a + a + a + a + a$, or $(3 + 2)\ a$, or $5\,a$. So

$15\,a^2 + a^2 = (15 + 1)\,a^2$ [art 9] or $16\,a^2$.

In the same way we find that

$$5\,a - 3\,a = (5 - 3)\ a, \text{ or } 2\,a;$$

and

$$27\,a^2\,b - 15\,a^2\,b = (27 - 15)\ a^2\,b,$$

or

$$12\,a^2\,b.$$

There is no difficulty then in bringing together by means of addition and subtraction any number of quantities consisting each of one term. Those that are like are to be collected into one term by adding or subtracting their coefficients, according as their signs are positive or negative [arts. 4 and 5]; and this term is to be set down with the sign of addition or subtraction, according as the positive or negative quantities collected into it are greater. If the negative be greater, the quantities to be subtracted are together greater than those to be added, and therefore the single term made up of them all is to be subtracted. Those quantities that are unlike are to be set down, each with its proper sign. For example,

$$a^2 + 25\,a\,b^2 - 10\,a^2 - 8\,x\,b^2 + 17\,a^2$$
$$= (1 - 10 + 17)\,a^2 + 25\,a\,b^2 - 8\,x\,b^2$$
$$= 8\,a^2 + 25\,a\,b^2 - 8\,x\,b^2.$$

16. In the same way we find that

$$3\,a + b - 5\,a + 6\,a - b - 16\,a$$
$$= -12\,a.$$

Here the whole can be collected into a single term which is negative. In algebra we very often come to a negative result. We may, perhaps, find it difficult to attach any meaning to a negative quantity in the abstract, a quantity to be subtracted without any thing to subtract it from. But, whenever we have a result of this kind, we shall always find something in the nature of the question we are considering that enables us to give the negative sign a meaning. Of this we shall have many instances as we proceed; to take a familiar one now, suppose a and b in the last example to be sums of money, and that a man is taking an estimate of his outstanding accounts, if he set down sums due to him with + before them, he should set down sums due by him with −. When he takes the amount, if his debts be the greater, of course the balance will have − before it; with respect to these accounts he has so much less than nothing.

We shall find, that when algebra is applied to questions that occur, there is no such thing as a quantity essentially negative. It is negative only because we choose so to consider it; and we may make it positive, if at the same time we make negative the quantities formerly positive. In the instance just given, the man may consider his debts as positive, and his credits to be subtracted from them as negative. On this supposition his result would have had the positive sign; but this would not at all have changed its nature, a positive debt and a negative credit being algebraically the same thing.

17. We now come to the addition and subtraction of quantities that consist of more than a single term. To add to a the quantity $b + c$, is to add b and c both; the sum therefore is $a + b + c$. To add to a the quantity $b - c$, is to add b diminished by c; the sum therefore is $a + b - c$. We find then, that *to add a quantity consisting of more than a single term, is to take its terms one by one, adding those that are positive and subtracting those that are negative.*

18. To subtract $b + c$ from a, is to subtract b and c both; the result therefore is $a - b - c$. In subtracting $b - c$ from a, if we subtract b alone from a, and write $a - b$, we shall have subtracted too much; for it is not b, but b diminished by c, that is to be subtracted.

We must therefore add c to $a-b$ to have the proper result, which is $a-b+c$. Our two last results, which are very important, are

$$a-(b+c)=a-b-c,$$

and

$$a-(b-c)=a-b+c.$$

It is plain that these apply equally to quantities of more than two terms. *To subtract a quantity then of more than one term, we must change the sign o every one of its terms from + to −, or from − to +, and then proceed as if the quantity so altered were to be added.* To subtract a negative quantity is equivalent to adding a positive.

A single example in numbers will illustrate this:

$$10-(8-5)=10-8+5=7$$

by the rule: again, since $8-5=3$,

$$10-(8-5)=10-3, \text{ or } 7,$$

of course the same result as the rule gave.

Of these operations take the following examples.

1. $3a^2+4bc-e^2+(5a^2-6bc-15)+(21-4a^2-10e^2)$,

the sum is

$$4a^2-2bc-11e^2+6.$$

2. $5a^2+4ab-6xy-(11a^2+6ab-4xy)$,

the difference is

$$-6a^2-2ab-2xy.$$

3. $4a-3b-(5c-a-\overline{5b-e})+(10-\overline{7a+3e})$,

the sum is

$$-2a+2b-5c-4e+10.$$

In this last example, before proceeding to collect the terms together, we observe that the quantities within the brackets are respectively $5c-a-5b+e$, and $10-7a-3e$.

19. In explaining the reasons of the rules for addition and subtraction in arithmetic we must first observe, that all numbers are represented by means of the nine *digits*, as the symbols for the first nine numbers are called, and *zero*, or the symbol for nothing. This is done by agreeing that the value of each digit shall be ten times as great as it would have been if it had held the next place to the right. The first digit to the right stands for so many units, the one next it for so many tens, the next for so many hundreds, and so on. A number then such as 57624 is equivalent to

$$50000+7000+600+20+4.$$

20. The common rule for the addition of numbers is this: *Write the numbers to be added under one another, so that the units' digits shall be all in one column, the tens' all in another, the hundreds' all in another, and so on; cast up the units' column, and write under it the units' digit of its sum, carrying the tens' digit to be added to the next column; cast up the tens' column with this addition, write under it the units' digit of its sum, carrying the tens' digit to be added to the next column; proceed thus with every column till the last, under which write its whole sum.*

The reason of this rule will appear at once by stating an example of it in detail.

$$\left.\begin{array}{r}4765\\8904\\725\\7500\\\hline 21894\end{array}\right\} \text{ or } \left\{\begin{array}{r}4000+700+60+5\\8000+900+4\\700+20+5\\7000+500\\\hline 19000+2800+80+14\end{array}\right.$$

The first result is found by the rule; the other is found by casting up each column separately, and is the same as

$$19000+2000+800+80+10+4.$$

Collecting the numbers of the same denomination into one, which is what we do when we carry as directed by the rule, this becomes

$$21000+800+90+4, \text{ or } 21894.$$

21. The rule for subtracting one number from another is this: *Write the number to be subtracted under the other, as in addition; subtract every digit in the lower number from the one over it in the upper, and write the remainder under it. When the digit in the upper number is the smaller, add ten to it, observing at the same time to increase the digit next to its left in the lower number by unity.*

To prove this, suppose the two numbers to be 8437 and 5974. If we write them thus

$$8000+400+30+7,$$
$$000+900+70+4$$

and proceed to subtract them term from term, we find that we cannot take 70 from 30, or 900 from 400, without having negative quantities appearing in our result, which we wish to avoid. We therefore write them thus

$$7000 + 1300 + 130 + 7,$$
$$5000 + \ \ 900 + \ \ 70 + 4,$$

and then, subtracting them term from term, we find for their difference

$$2000 + 400 + 60 + 3, \text{ or } 2463.$$

It is manifestly the same thing to increase a term in the lower number by unity, as to diminish the corresponding one in the upper by unity, and this consideration brings us to the rule.

22. There is a way in use, of subtracting numbers by means of what is called their *arithmetical complement*. Suppose, for instance, that we have to subtract 817 from 2573; that is, to find the difference $2573 - 817$. This difference will not be affected by adding any quantity to it, if at the same time we subtract the same quantity from it. Let this quantity be 1000; $2573 - 817$ then is the same as $2573 + (1000 - 817) - 1000$, or as $2573 + 183 - 1000$, or as $2573 + \bar{1}183$, if we observe in adding the two numbers to subtract the $\bar{1}$ in respect of the negative sign placed above it; this subtraction from the place that $\bar{1}$ occupies being equivalent to the subtraction of 1000. $\bar{1}183$ is called the arithmetical complement of 817. As we can perform such a subtraction as 817 from 1000 mentally, and write down the result almost as fast as we could write 817 itself, the arithmetical complement may furnish us with a more expeditious way of taking the balance of such a set of numbers as $2573 - 183 + 17856 - 1273 + 534$, than taking the separate sums of the positive and negative terms, and then finding the difference of these sums. By arithmetical complements we have

$$\begin{array}{r} 2573 \\ \bar{1}817 \\ 17856 \\ \bar{1}8727 \\ 534 \\ \hline 19507. \end{array}$$

This way of writing numbers with a negative unit in the left hand digit's place is also in use in logarithms.

Of Multiplication.

23. To multiply b by a we write $a\,b$, and to multiply a by b we write $b\,a$. $a\,b$ and $b\,a$ are equal to each other, just as 4×5 and 5×4 are equal to each other, both being 20. Of the truth of this we may satisfy ourselves thus, $a\,b$ means that the units in b are to be taken as often as there are units in a. [art. 7].

$$\begin{array}{ll} 1+1+1+1+1 & 1+1+1+1 \\ 1+1+1+1+1 & 1+1+1+1 \\ 1+1+1+1+1 & 1+1+1+1 \\ 1+1+1+1+1 & 1+1+1+1 \\ & 1+1+1+1 \end{array}$$

Now write 1 as many times in the same line as there are units in b, and make as many such lines as there are units in a. The value of each line is b, and there are a such lines, therefore the value of the whole is b taken a times, or $a\,b$. Again, write 1 as many times as there are units in a, and make as many such lines as there are units in b. In this case the value of the whole will be $b\,a$. But the one of these sets of units is plainly equal in number to the other, whence $a\,b = b\,a$.

Just in the same way we may conclude, that $a\,b\,c$ is equal to $b\,a\,c$. For write c in the same line b times, and make a such lines. The value of each line is $b\,c$, and the value of the whole is $a\,b\,c$.

$$\begin{array}{ll} c+c+c+c & c+c+c \\ c+c+c+c & c+c+c \\ c+c+c+c & c+c+c \\ & c+c+c \end{array}$$

Again, write c in each line a times, and make b such lines. The value of the whole, which is plainly the same as before, is now $b\,a\,c$. So, since $b\,c$ is equal to $c\,b$, $a\,b\,c$ is equal to $a\,c\,b$.

From all this it follows, that if a quantity be the product of any number of factors, the order in which these factors succeed one another may be altered in any way without changing the value of the quantity.

24. $4 = 2^2 = 2 \times 2$, and $8 = 2^3$, or $2 \times 2 \times 2$ [art. 8]; therefore $4 \times 8 = 2 \times 2 \times 2 \times 2 \times 2$, or 2^5. We find 2^5 and 4×8 to be each 32. So $a^3 = a\,a\,a$, and $a^2 = a\,a$, whence $a^3 . a^2 = a\,a\,a\,.\,a\,a$, or a^5. In the same way we may find that the product of any two powers of the same quantity is that quantity raised to the power expressed by the sum of the exponents in the factors

[arts. 7, 8]. We state this algebraically, by writing

$$a^m . a^n = a^{m+n},$$

where m and n may be any numbers whatever. For the same reason

$$a^m . a^n . a^p = a^{m+n+p};$$

and again,

$$a^m . a^m . a^m = a^{m+m+m},$$

or

$$(a^m)^3 = a^{3m}.$$

In the same way we may find any other power of a power of a quantity. The general expression will be

$$(a^m)^n = a^{mn}.$$

For instance, $(3^2)^3 = 3^6$; and, since $3^2 = 9$, it follows that $9^3 = 3^6$. We find that both are 729.

25. By applying what has been laid down, the reader will have no difficulty in multiplying together any quantities consisting of one term. *The coefficient of the product will be the product of the coefficients of the factors, and factors that are the same, or only differ in their exponents, will be collected into one by adding their exponents, observing that when a factor has no exponent unity is to be supplied.* For examples,

$$3\,a . 5\,b = 3 \times 5\,a\,b = 15\,a\,b,$$

$$7\,a\,b\,c . 5\,b^2\,d . a^n f = 7 \times 5 . a\,a^n\,b\,b^2\,f\,c\,d$$

$$= 35\,a^{n+1}\,b^3\,c\,d\,f,$$

$$a^{3n}\,b . (a^m)^2\,(b^3)^3 = a^{3n}\,a^{2m}\,b\,b^9$$

$$= a^{2m+3n}\,b^{10}.$$

26. We now come to the multiplication of quantities that consist of more than one term. $a\,(b + c)$ means that $b + c$ is to be taken a times. If we take

$$(b + c) + (b + c) + \&c.$$

a times, and then add them all together, we shall have a times b, and a times c; the whole sum will be $a\,b + a\,c$. Therefore

$$a(b + c) = a\,b + a\,c.$$

So, by taking

$$(b - c) + (b - c) + \&c.$$

a times, and adding them all together, we find that

$$a\,(b - c) = a\,b - a\,c.$$

Again, regarding $a + b$ as a single quantity,

$$(a + b)(c + d) = (a + b)\,c + (a + b)\,d,$$

or by art. [23]

$$(a + b)(c + d) = c(a + b) + d(a + b),$$

or finally

$$(a + b)(c + d) = a\,c + b\,c + a\,d + b\,d.$$

In a similar manner we find that

$$(a + b)(c - d) = a\,c + b\,c - a\,d - b\,d,$$

and that

$$(a - b)(c + d) = a\,c - b\,c + a\,d - b\,d.$$

Lastly, let us find the product $(a - b)(c - d)$. Regarding $a - b$ as a single quantity, we have, as before,

$$(a - b)(c - d) = (a - b)\,c - (a - b)\,d.$$

That is, from $(a - b)\,c$ we are to subtract $(a - b)\,d$; or from $a\,c - b\,c$ we are to subtract $a\,d - b\,d$. But by art. [18] this difference is $a\,c - b\,c - a\,d + b\,d$. Therefore

$$(a - b)(c - d) = a\,c - b\,c - a\,d + b\,d.$$

It is easy to see that these principles apply equally when the number of terms in one or both of the factors is more than two. From the results already given we may therefore deduce the following general rule: *Multiply every term in the one factor by every term in the other, and set down the quantities so obtained, every one with its proper sign for the terms of the product. To determine what this sign is, observe that the product of two positive, or of two negative quantities is positive; and that the product of a positive and a negative, or a negative and a positive quantity is negative; or that like signs give a positive, unlike a negative product.*

27. The reasons of the part of this rule that determines the signs of the terms of the product will be more apparent, if we consider, that to multiply by a positive quantity is to add the quantity multiplied so many times, and to multiply by a negative, to subtract it so many times. Hence the product of a positive quantity by a positive, is a positive quantity added a certain number of times, and is positive. The product of a positive quantity by a negative, or of a negative quantity by a positive, is a positive quantity subtracted, or a negative quantity added, a certain number of times, and is therefore negative. Lastly, the product of a negative quantity by a negative, is a negative quantity subtracted a certain number of times, and is therefore positive, by art. [18].

An example may aid the reader in apprehending that $(-d) \times (-b)$ is equal to $+\,d\,b$. A person buys a yards of cloth at c shillings a yard, he keeps

b of the yards and sells the rest at d shillings a yard less than he gave for it; how much did he receive for what he sold? He bought a yards and he keeps b; therefore he sells $a - b$ yards, for which he gets $c - d$ shillings a yard. For all that he sells, then, he gets

$$(a - b)(c - d) = a c - a d - b c + b d$$

shillings, by the rule.

Suppose a to be 12, b to be 3, c to be 10, and d to be 2; then

$$a c - a d - b c + b d$$

$$= 120 - 24 - 30 + 6 = 72 \text{ shillings.}$$

Again, since he bought 12 yards and kept 3, he sold 9; and since he paid 10*s.*, and sold for 2*s.* less, he sold at 8*s.* a yard. Therefore he received 8×9, or 72*s.*, which is of course the same result as before.

28. The following are examples of multiplication:

$$\begin{array}{rl} \text{Multiply} & a + 3c - d \\ \text{By} & 2a - d \\ \hline & 2a^2 + 6ac - 2ad - ad - 3dc + d^2 \end{array}$$

and collecting the terms, the product is

$$2a^2 + 6ac - 3ad - 3cd + d^2.$$

$$\begin{array}{rl} \text{Multiply} & a + x \\ \text{By} & a - x \\ \hline & a^2 + ax - ax - x^2 \end{array}$$

or

$$a^2 - x^2.$$

$$\begin{array}{rl} \text{Multiply} & x^4 + x^3 + x^2 + x + 1 \\ \text{By} & x - 1 \\ \hline & x^5 + x^4 + x^3 + x^2 + x - x^4 - x^3 - x^2 - x - 1 \end{array}$$

or

$$x^5 - 1.$$

29. When there are many factors the product of them all is called their *continued product*. Thus

$$(x + a)(x + b)(x + c)(x + d), \text{ \&c.}$$

is the continued product of these factors. We find it thus:

$$\begin{array}{l} x + a \\ x + b \\ \hline x^2 + (a + b)x + ab \\ x + c \\ \hline x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc \\ x + d \\ \hline x^4 + \left.\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\} x^3 + \left.\begin{matrix} ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{matrix}\right\} x^2 + \left.\begin{matrix} abc \\ abd \\ acd \\ bcd \end{matrix}\right\} x + abcd \end{array}$$

and so on. This continued product forms itself very regularly according to a law which it is not difficult to perceive. We shall be able to make more than one important use of it as we go on.

30. Since the product of every pair of negative factors is positive, whenever the number of negative factors in a product is an even number, that product is positive. On the other hand, whenever the number of negative factors in a product is an odd number, that product is negative; for if we take the product, leaving out one negative factor, there will be an even number of negative factors, and their product is positive; and multiplying this by the negative factor omitted, it becomes negative. It follows from this, that

$$(-a)^{2n} = a^{2n};$$

for $2n$ is always an even number. Similarly,

$$(-a)^{2n+1} = -a^{2n+1};$$

for $2n+1$ is always an odd number. Thus,

$(-3)^4 = (-3).(-3).(-3).(-3) = 81$, and

$(-3)^3 = (-3).(-3).(-3) = -27$.

31. With respect to the multiplication of numbers: the product of any two of the first nine numbers is contained in the following table, so well known by the name of the *Multiplication Table.*

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

This table is said to have been first used by Pythagoras, the famous Grecian philosopher, who lived about 500 years before Christ. It is formed in the first instance by addition; thus, $4+4$, or 8, is 2×4; $8+4$, or 12, is 3×4; and so on.

32. To multiply any number by 10, we add a zero to its right. Thus, since

$$732 = 700 + 30 + 2,$$

we must have by art. [26]

$10 \times 732 = 10 \times 700 + 10 \times 30 + 10 \times 2$.

But by our notation [art. 19] $10 \times 2 = 20$, $10 \times 30 = 300$, and so on. Therefore

$$10 \times 732 = 7000 + 300 + 20 = 7320.$$

Similarly, to multiply any number by 100, or 1000, is to add two, or three digits, respectively, to its right.

To find such a product as 9×80, observe that it is the same as $9 \times 8 \times 10$. [art. 23]. But by the table $9 \times 8 = 72$. Therefore 9×80 is the same as 72×10, or 720. Similarly, $2 \times 8000 = 16000$.

33. By means of the multiplication table, and these properties, we find the product of any two numbers whatever. When the multiplier has only one digit the rule is this: *Write the multiplier under the units' digit of the multiplicand; find in the table the product of the units' digit of the multiplicand by the multiplier; write the units' digit of this product immediately under the multiplier; find the product of the next digit of the multiplicand by the multiplier, add to it the tens' digit of the former product, write the units' digit of the number thus found to the left of the digit last set down, and proceed as before.*

The reason of this rule will appear by one example. To multiply 9271 by 7. That is,

$$\begin{array}{lrr} \text{Multiply} & 9000 + 200 + 70 + 1 & 9271 \\ \text{By} & 7 & 7 \\ \hline & 63000 + 1400 + 490 + 7 & 64897 \end{array}$$

Here we find the several products from the table as in art. [32], their sum by art. [26] is the product sought; and when we collect them into one number, we do what is equivalent to the *carrying* directed by the rule.

When both numbers have more than one digit the rule is as follows: *Write the multiplier under the multiplicand, units under units, tens under tens, &c. Write down the product of the multiplicand by the units' digit of the multiplier, observing that its units' digit shall be under the units' digit of the multiplier. In like manner, write down the product of the multiplicand by the tens' digit, observing that its units' digit shall be under the tens' digit of the multiplier; proceed thus, and, when all the partial products are set down, add them up as they stand, for the whole product.*

As before, an example will explain this rule. Let it be to multiply 4786 by 2783. That is, to take 4786, 2783 times, and add them all together; or to take it 2000 times, 700 times, 80 times, and 3 times, and add the sums together, or to multiply it by 2000, by 700, by 80, and by 3, and add the products together. By art. [32] and the first rule we have

$$\begin{array}{rcl|r} 4786 & & & 4786 \\ 2783 & & & 2783 \\ \hline 14358 & = & 3 \times 4786 & 14358 \\ 38288/0 & = & 80 \times 4786 & 38288 \\ 33502/00 & = & 700 \times 4786 & 33502 \\ 9572/000 & = & 2000 \times 4786 & 9572 \\ \hline 13319438 & = & 2783 \times 4786 & 13319438 \end{array}$$

If the zeros be cut off in the detailed operation it will stand as the rule directs.

Of Division.

34. We have already stated [10], that to divide a by b is to find a third quantity such that the product of it by b shall be a. This third quantity it was agreed to represent by $\frac{a}{b}$. It follows, then, that

$$b \times \frac{a}{b} = a.$$

35. When the divisor is a factor of the dividend, the quotient is simply the dividend with that factor struck out of it. If the dividend be 20, that is 4 . 5, and the divisor 4, the quotient is 5. So if the dividend be $a\,b$, and the divisor b, the quotient is simply a; for a multiplied by b, the divisor, produces $a\,b$, the dividend. So $a^5 \div a^3 = a^2$, and

$$a^m \div a^n = a^{m-n},$$

for $a^{m-n} \times a^n = a^m$. [art. 24]. For example, since $243 = 3^5$, and $27 = 3^3$; then $243 \div 27$ must be $3^{5-3} = 3^2$, or 9.

36. Though the divisor itself be not a factor of the dividend it will often happen that it has some factor that is also a factor in the dividend. This factor may be struck out of both of them without affecting the quotient. If $a\,b$ be the dividend, and $a\,c$ the divisor, $\frac{b}{c}$ will be the quotient. For $a\,c \,.\, \frac{b}{c} = a\,b$ by [34]; that is, $\frac{b}{c}$ multiplied by the divisor produces the dividend. Similarly, the quotient of $6\,a^2\,x\,y \div 3\,a\,b\,x$ is $\frac{2\,a\,y}{b}$, striking out of each the factor $3\,a\,x$.

37. When the dividend has more terms than one, the divisor still remaining a quantity of one term, the quotient is found by dividing each term of the dividend by the divisor, in the manner just laid down. Thus, if the dividend be $a + b + c$, and the divisor d, the quotient will be

$$\frac{a}{d} + \frac{b}{d} + \frac{c}{d};$$

because if we multiply this quantity by d the product will be the dividend $a + b + c$. [art. 26].

38. With respect to the sign of the quotient, we have the same rule as in multiplication, namely, that *when the divisor and dividend have like signs, the quotient is positive, when unlike, it is negative.* This is deduced from the rule for the signs in multiplication [26], thus:

$$+ a \div (+ b) = + \frac{a}{b}, \text{ because } + b \times \left(+ \frac{a}{b}\right) = + a;$$

$$+ a \div (- b) = - \frac{a}{b}, \text{ because } - b \times \left(- \frac{a}{b}\right) = + a;$$

$$- a \div (+ b) = - \frac{a}{b}, \text{ because } + b \times \left(- \frac{a}{b}\right) = - a;$$

$$- a \div (- b) = + \frac{a}{b}, \text{ because } - b \times \left(+ \frac{a}{b}\right) = - a.$$

39. After what has been laid down we shall have no difficulty in finding that

$$12\,a^3\,b^6\,c^5 \div 16\,a\,c^7\,b^4 = \frac{3\,a^2\,b^2}{4\,c^2};$$

$$(25\,a^3\,x - x\,y^2) \div 15\,x\,y = \frac{5\,a^3}{3\,y} - \frac{y}{15};$$

$$27\,x^4 \div (9\,x^3 - 3\,a^2\,x)$$

$$= 27\,x^4 \div 3\,x\,(3\,x^2 - a^2) = \frac{9\,x^3}{3\,x^2 - a^2}.$$

40. When the dividend and divisor are both quantities containing more than one term, the operation of division becomes somewhat more complicated. We shall find it better to explain first the reason of the common rule for dividing one number by another. The rule in algebra depends on the same principles. The arithmetical rule is this: *Write the divisor to the left of the dividend; from the left of the dividend mark off the smallest number of digits that make a number not less than the divisor; find by trials the greatest number of times that the divisor is contained in this period, and write the*

result as the left hand digit of the quotient; multiply the divisor by this digit, and subtract the product from the period marked off; bring down to the right of the remainder the next digit in the dividend, and proceed with the number thus formed as with the first period, writing the result as the second digit in the quotient; proceed in the same way to find the other digits of the quotient.

Take as an example to divide 3978 by 17. We seek the number which when multiplied by 17 gives 3978 for product. Now, we observe that the number must be between 200 and 300, for 17×300, or 5100, is greater, and 17×200, or 3400, is less than 3978. The quotient then is greater than 200, let us call it $200 + a$. Therefore $17 (200 + a)$, or $17 \times 200 + 17 a = 3978$. If from each of these equal quantities we take away 17×200, or 3400, the remainders will be equal, that is, $17 a = 578$, and $a = \frac{578}{17}$ [art. 10]. Again, to find a we observe that it must be between 30 and 40; for 17×40, or 680, is greater, and 17×30, or 510, is less than 578. As before, let a be $30 + b$. Then $17 (30+b)$, or $510 + 17 b = 578$; and subtracting, as before, $17 b = 68$. Finally, we observe that b must be 4, since $4 \times 17 = 68$ exactly. The quotient then is $200 + 30 + 4$, or 234.

We may present this operation thus:

	Divisor. Dividend.	Quotient.
	17) 3978 (	200 + 30 + 4, or 234.
Subtract 17 × 200 =	34|00	
	57|8	
Subtract 17 × 30 =	51|0	
	68	
Subtract 17 × 4 =	68	

If the digits cut off by lines be not written, which they need not be, the operation will stand as the rule directs.

We find the successive digits in the quotient by guessing at them as well as we can. When we multiply the divisor by the digit guessed, if the product be greater than the partial dividend, the digit is too great; when we subtract, if the remainder be greater than the divisor, the digit is too small. When the divisor is 12, or under, the operation is much shortened by going through the multiplication and subtraction mentally. This way is what is called *short division.*

41. It will, of course, very often happen, that the dividend does not contain the divisor any exact number of times; that is, that there is no exact number which when multiplied by the divisor produces the dividend. For instance, if we had to divide 3985 by 17, we should find, that when 17×234 is subtracted, there is 7 remaining. So that

$$3985 = 17 \times 234 + 7;$$

here 7 is called the *remainder.* Now by art. [35]

$$(17 \times 234 + 7) \div 17 = 234 + \frac{7}{17}.$$

In general, if a be any number whatever, b any number less than a, q the exact part of the quotient, and r the remainder, then

$$a = q b + r,$$

and

$$\frac{a}{b} = q + \frac{r}{b}.$$

So that when in dividing any number by another there is a remainder over, it is to be written as divided by the divisor, and set down in the quotient. We shall return to these expressions when we come to treat of fractions.

42. A few examples wrought out at length must make the reasons of this rule of division appear very plainly. We now proceed to the division of algebraical quantities of more than one term. For instance, to divide $11 a b^2 + 6 b^3 + a^3 + 6 a^2 b$ by $a + b$. The course of the operation will show, that, since a stands first in the divisor, it will be most convenient to arrange the terms in the dividend according to the powers of a, putting its higher powers before its lower ones. It will then stand thus,

$$a^3 + 6 a^2 b + 11 a b^2 + 6 b^3.$$

Now a^3 is a term in the dividend, and a is one in the divisor; the term a^3 could only come from the multiplication of a^2 in the quotient by a; therefore a^2 is a term in the quotient. For the same reason as in the arithmetical example we subtract $(a + b) a^2$, or $a^3 + a^2 b$

from the dividend, and there remains

$$5\,a^2 b + 11\,a\,b^2 + 6\,b^3.$$

Again, the term $5\,a^2\,b$ in this remainder could only come from the multiplication of $5\,a\,b$ in the quotient by a in the divisor; therefore $5\,a\,b$ is also a term in the quotient. As before, subtract $(a+b)\,5\,a\,b$, or $5\,a^2\,b + 5\,a\,b^2$, and there remains

$$6\,a\,b^2 + 6\,b^3.$$

$6\,a\,b^2$ could only come from the multiplication of $6\,b^2$ by a, therefore $6\,b^2$ is a term in the quotient; and subtracting $(a+b)\,6\,b^2$, or $6\,a\,b^2 + 6\,b^3$, there remains nothing. The quotient then consists of the three terms we have found, and is

$$a^2 + 5\,a\,b + 6\,b^2.$$

The operation stands thus:

$$\begin{array}{rl}
a+b) & a^3 + 6\,a^2 b + 11\,a\,b^2 + 6\,b^3 (a^2 + 5\,a\,b + 6\,b^2 \\
\text{Subtract } (a+b)\,a^2 = & \underline{a^3 + a^2 b} \\
 & \phantom{a^3 + {}} 5\,a^2 b + 11\,a\,b^2 \\
\text{Subtract } (a+b)\,5\,a\,b = & \phantom{a^3 + {}} \underline{5\,a^2 b + 5\,a\,b^2} \\
 & \phantom{a^3 + 5\,a^2 b + {}} 6\,a\,b^2 + 6\,b^3 \\
\text{Subtract } (a+b)\,6\,b^2 = & \phantom{a^3 + 5\,a^2 b + {}} \underline{6\,a\,b^2 + 6\,b^3}
\end{array}$$

The general rule is this: *Arrange the terms of the dividend and of the divisor according to the powers of some one letter; divide the first term in the dividend by the first in the divisor, and write the result as the first in the quotient; multiply the divisor by this term, and subtract the product from the dividend; proceed to deal with the remainder, if any, in the same way.*

43. By applying the same reasoning as in the last example to all the steps of those which follow, the reader will become familiar with the principles on which the rule rests

$$\begin{array}{rl}
2\,a^3 - 5\,a\,b^2 + 2\,b^3) & 4\,a^6 - 25\,a^2 b^4 + 20\,a\,b^5 - 4\,b^6 (2\,a^3 + 5\,a\,b^2 - 2\,b^3 \\
\text{Subtr. divis.} \times 2\,a^3 = & \underline{4\,a^6 - 10\,a^4 b^2 + 4\,a^3 b^3} \\
 & 10\,a^4 b^2 - 4\,a^3 b^3 - 25\,a^2 b^4 + 20\,a\,b^5 - 4\,b^6 \\
\text{Subtr. divis.} \times 5\,a\,b^2 = & \underline{10\,a^4 b^2 - 25\,a^2 b^4 + 10\,a\,b^5} \\
 & - 4\,a^3 b^3 + 10\,a\,b^5 - 4\,b^6 \\
\text{Subtr. divis.} \times (-2\,b^3) = & \underline{- 4\,a^3 b^3 + 10\,a\,b^5 - 4\,b^6}
\end{array}$$

44.
$$\begin{array}{rl}
a-1) & a^3 - 1 (a^2 + a + 1 \\
 & \underline{a^3 - a^2} \\
 & \phantom{a^3 - {}} a^2 - 1 \\
 & \phantom{a^3 - {}} \underline{a^2 - a} \\
 & \phantom{a^3 - a^2 - {}} a - 1 \\
 & \phantom{a^3 - a^2 - {}} \underline{a - 1}
\end{array}$$

Examining this last example, it is plain that in like manner $a^4 - 1$, or, generally, that $a^n - 1$ may be divided by $a - 1$, without leaving any remainder, and that we should have

$$\frac{a^n - 1}{a - 1} = a^{n-1} + a^{n-2} + \&c. + a + 1.$$

But we should find that $a^2 + 1$, $a^4 + 1$, or generally $a^n + 1$ cannot be divided by $a - 1$ without leaving a remainder.

45. In like manner, if we divide $a^3 + 1$, or $a^5 + 1$, or any similar quantity where the index of a is an odd number, by $a + 1$, we shall find that there is no remainder over; but if we try to divide $a^4 + 1$, or any similar expression where the index of a is an even number, by $a + 1$, there will always be a remainder over. Now if n stand for any number, $2\,n + 1$ will stand for any odd number; for every odd number is twice some other number with 1 added to it. $a^{2n+1} + 1$ then can always be divided exactly by $a + 1$, and we find

$$\frac{a^{2n+1} + 1}{a + 1}$$
$$= a^{2n} - a^{2n-1} + \&c. + a^2 - a + 1.$$

Once more, $a^2 - 1$, $a^4 - 1$, and all similar expressions in which the index of a is an even number, can be divided by $a + 1$ without leaving any remainder; but $a^3 - 1$, $a^5 - 1$, &c., where the index of a is an odd number, always leave a remainder. Now $2\,n$ is the general representative of an even number, and therefore $a^{2n} - 1$ can always be divided without remainder by $a + 1$. We shall find

$$\frac{a^{2n} - 1}{a + 1}$$
$$= a^{2n-1} - a^{2n-2} + \&c. - a^2 + a - 1.$$

46. As a last example, let us propose to divide 1 by $1+x$.

$$\begin{array}{l} 1+x)\,1 \quad (1-x+x^2-x^3+\&c. \\ \underline{1+x} \\ \quad -x \\ \quad \underline{-x-x^2} \\ \qquad x^2 \\ \qquad \underline{x^2+x^3} \\ \qquad\quad -x^3 \end{array}$$

The division, it is plain, will never come to an end, and we have

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-\&c.$$

47. Of course all these results, as they are true for all quantities, are also true for numbers. Thus, in the expression for $\frac{a^{2n+1}+1}{a+1}$ [art. 45] make $n=2$, then

$$\frac{a^5+1}{a+1}=a^4-a^3+a^2-a+1.$$

Now, let $a=4$, then

$$\frac{4^5+1}{4+1}=4^4-4^3+4^2-4+1,$$

or

$$\frac{1024+1}{4+1}=256-64+16-4+1,$$

that is,

$$\frac{1025}{5}=205.$$

We said that a^n+1 cannot be divided by $a-1$ without leaving a remainder. Yet make $a=3$ and $n=4$, and we find that 3^4+1, or 82, is divisible without remainder by $3-1$ or 2. The reason of this seeming discrepancy is simple. If we divide a^4+1 by $a-1$, we find for quotient

$$a^3+a^2+a+1+\frac{2}{a-1}$$

so that our assertion holds true in general; but when 3 is put for a, it happens that $\frac{2}{3-1}$ becomes $\frac{2}{2}$ or 1, so that the remainder, as a distinct part of the quotient, is lost.

In the result in art. [46], let us put -2 for x; then that expression becomes

$$\frac{1}{1-2}=1-(-2)+(-2)^2-(-2)^3+\&c.$$

or by art. [30]

$$-1=1+2+4+8+16+\&c.$$

which is seemingly false. But if we go back to the operation by which we arrived at the expression in art. [46], we shall find that there was always a remainder over, which we included in the &c., and that the true way of writing the result would have been

$$\frac{1}{1+x}=1-\frac{x}{1+x},$$

or

$$\frac{1}{1+x}=1-x+\frac{x^2}{1+x},$$

or

$$\frac{1}{1+x}=1-x+x^2-x^3+\frac{x^4}{1+x},$$

and so on. Now if we take any of these expressions, for example the last, we shall find it true; because, putting -2 for x, it becomes

$$-1=1+2+4+8-16.$$

Once more, if for x in art. [46] 1 be put, the result becomes

$$\frac{1}{2}=1-1+1-1+1-1+1-\&c.$$

This is a result which has puzzled some eminent mathematicians, who thought that the &c. must comprise a set of terms exactly the same as those that go before it, and could not understand how a series of numbers, which, when added together, is alternately 0 or 1, according as an even or an odd number of them is taken, can ever be equal to $\frac{1}{2}$. If the remainder be taken into the account, the difficulty disappears, for we have

$$\frac{1}{1+x}=1-\frac{x}{1+x},$$

or making $x=1$,

$$\frac{1}{2}=1-\frac{1}{2},$$

$$\frac{1}{1+x}=1-x+x^2-x^3+\frac{x^4}{1+x},$$

or making $x=1$,

$$\frac{1}{2}=1-1+1-1+\frac{1}{2},$$

and so on.

These remarks are introduced to show how careful we must be when we use algebraical results, not to forget the steps by which we arrived at them.

Of Whole Numbers.

48. Such numbers as 2, 5, 9, 12, 90, &c. are called *whole numbers*, or *integers*, (a Latin word meaning *whole*,) in distinction to such numbers as $5\frac{3}{4}$, $\frac{2}{3}$, &c.

49. When a whole number is the product of any other whole numbers, any one of them is said to *measure* it; meaning that the number can be divided by that factor without leaving any remainder. Thus, 20 is the product of 2 and 10, or of 4 and 5; therefore 2, 4, 5, and 10 measure 20. The reason of this term is plain. We can measure 20 gallons by means of a vessel that we know to contain 2 gallons, or by means of vessels containing 4, 5, or 10 gallons; but we cannot measure 21 gallons by any of these, nor 20 gallons by a vessel containing 6.

50. A number that is measured by any other is called a *multiple* of that other; thus, 20 is a multiple of 2, of 4, of 5, or of 10. The numbers 2, 4, 5, and 10, again, are called *submultiples* (that is, *under multiples*) of 20.

51. Let $a = q\,b$, then $c\,a$ will be equal to $c\,q\,b$; for if equal quantities be multiplied by the same quantity, the two products must be equal. Therefore b measures $c\,a$ as well as a; so that if one quantity measures another, it measures any multiple of that other. For example, 5 measures 15, therefore it measures 6×15, or 90. On the other hand, if one number be measured by another, it is measured by all the factors of that other.

52. Every number is the product of itself by unity; thus, 19 is the product 1×19. When a number has no other whole factors but itself and unity, it is called a *prime number*. Thus, 1, 2, 3, 5, 7, 29, 31, 101, &c. are all prime numbers.

53. If we wish to discover whether any number is a prime, the only way we can do it is by trying if we can find some number that will measure it; if we are sure that none of the numbers less than itself measures it, we are of course sure that it is a prime. Now, if it be measured by any number that is not a prime we have seen that it must also be measured by all the factors of that number [51], and some of these factors must be primes. If the number in question, then, be not measured by any prime number less than itself, it is not measured by any other number, and it is therefore a prime.

Again, let a be the number in question, and let b be its square root [11]. If a be measured by any number greater than b, it must also be measured by some number less than b; for since b multiplied by b produces a, the number by which any number greater than b must be multiplied so as to produce a, must be less than b. From all this it follows, that if any number be not measured by any of the primes that are not greater than its square root, it is not measured by any number whatever, and is consequently itself a prime. To determine that 47 is a prime, for instance, it is sufficient to be certain that it is not measured by 2, 3, or 5; the only three primes not greater than its square root, which is between 6 and 7. So 167 is a prime, for it is not measured by 2, 3, 5, 7, or 11, the only primes not greater than its square root, which is between 12 and 13.

54. Two numbers are said to be *prime to each other* when there is no number but unity that measures both of them; thus, 35 and 12 are prime to each other, though neither of them is itself a prime. On the other hand, if there be two numbers, both of which are measured by a third number other than unity, they are said to have a *common measure*. 15 and 25 have a common measure 5; 360 and 270 have common measures, 90, 45, 30, 18, 15, 10, 9, 6, 5, 3, and 2.

55. Let $a = b\,c$ and $a' = b'\,c$; here a and a' * have a common measure, which is c. Now

$$a + a' = c\,(b + b'),$$

and

$$a - a' = c\,(b - b');$$

so that c measures $a + a'$ and $a - a'$. If two numbers, then, have a common measure, it also measures their sum and difference. Thus, 63 and 35 have a common measure 7; 7 also measures their sum 98, and their difference 28.

56. If any two numbers a and b have a common measure c, and if on dividing a by b there be a remainder r, c will measure r also. For if q be the whole part of the quotient, then by [41]

$$a = q\,b + r.$$

If from each of these equal quantities $q\,b$ be taken away, the remainders will be equal, and therefore

$$a - q\,b = r.$$

Now c measures a; it also measures b, it therefore measures $q\,b$ [51]; and therefore it measures the difference of those quantities [55]; and that difference is r. Thus, 3 measures 57 and 12

* The notation in Algebra is often made much simpler by denoting different quantities, not by different letters, but by the same letter differently marked, as a, a', a'', &c. or a_0, a_1, a_2, &c.

dividing 57 by 12, there is 9 over, which is also measured by 3.

57. On these principles we may establish the rule for finding the greatest common measure of two numbers. Let the numbers be a and b, of which b is less than a.

Divide a by b and let r be the remainder, then $a = q\,b + r$.
Divide b by r and let r' be the remainder, then $b = q'\,r + r'$.
Divide r by r' and let r'' be the remainder, then $r = q''\,r' + r''$.
Divide r' by r'' and let there be no remainder, then $r' = q'''\,r''$.

We have gone on dividing a by b, b by the remainder r, r by the succeeding remainder r', &c. till r'', the third remainder, is found to divide r', the preceding one, exactly. r'' is the greatest common measure of a and b.

In the first place, it is a common measure of a and b. For it measures r'; it therefore measures $q''\,r'$ [51]; and therefore $q''r' + r''$ [55], that is r. In the same way, since it measures r' and r, it measures $q'\,r + r'$, or b. Finally, since it measures r and b, it measures $q\,b + r$, or a.

In the second place, it is the greatest common measure of a and b. For every common measure of a and b measures r [56]; and every common measure of b and r measures r', and every common measure of r and r' measures r''. Now r'' is itself the greatest number that measures r' and r'', and therefore r'' is the greatest number that measures a and b.

It is plain, that as the quantities r, r', r'', &c. always go on diminishing, if the operation does not stop sooner, we shall at last come to a remainder that shall be unity. When this is the case, the numbers are prime to each other, for they have no common measure but unity.

58. The rule is this: *Divide the greater of the numbers by the smaller; divide the smaller by the remainder of the last division, if any; divide the first remainder by the remainder of the second division, if any; proceed in this till some remainder divide the preceding one, that remainder is the greatest common measure if it be unity, the numbers are prime to each other.*

Of this rule take as an example the numbers 234 and 3348.

```
                  234)3348(14
                      234
                      ----
                      1008
                       936
                      ----
1st remainder           72)234(3
                           216
                           ---
2nd remainder               18)72(4
                               72
                               --
```

Here 18 is the greatest common measure sought. In the same way, it will be found that 47 is the greatest common measure of 2961 and 799, and that 824 and 319 are prime to each other.

59. The rule for finding the greatest common measure of two algebraical expressions is the same. For example, to find the greatest common measure of

$$a^4 - x^4,$$

and

$$a^3 - a^2\,x - a\,x^2 + x^3.$$

Dividing the expression in which the highest power of a occurs, by the other, we have

$$\begin{array}{rl} a^3 - a^2\,x - a\,x^2 + x^3) & a^4 - x^4 \quad (a + x \\ & a^4 - a^3\,x - a^2\,x^2 + a\,x^3 \\ \hline & \quad a^3\,x + a^2\,x^2 - a\,x^3 - x^4 \\ & \quad a^3\,x - a^2\,x^2 - a\,x^3 - x^4 \\ \hline \text{1st remainder} & \quad 2\,a^2\,x^2 \qquad\qquad - 2\,x^4 \end{array}$$

Before dividing by this remainder, we observe that it is measured by $2\,x^2$, which does not measure the first divisor, the quantity to be divided. $2\,x^2$ is therefore not a factor in the common measure of the remainder and first divisor, and therefore cannot be a factor in the common measure we are seeking [57], so that it may be struck out of our new divisor without affecting that common measure.

$$\begin{array}{rl} a^2 - x^2) & a^3 - a^2\,x - a\,x^2 + x^3\,(a - x \\ & a^3 \qquad\quad - a\,x^2 \\ \hline & \quad - a^2\,x \qquad\quad + x^3 \\ & \quad - a^2\,x \qquad\quad + x^3 \end{array}$$

We find that $a^2 - x^2$ divides the former divisor without remainder, it therefore is the greatest common measure sought. This operation for finding the common measure of algebraical expressions is scarcely ever used in practice.

60. If a and b be any two numbers, and c any prime number that neither measures a nor b, then c does not measure their product $a\,b$. Thus 3 measures neither 5 nor 8, so it cannot measure 40, their product. This may be proved as follows.

First, let b be less than c, and suppose, for the sake of argument, that $a\,b$ is

measured by c. We have by [41]

$$c = q b + r,$$

and if we multiply each of these equal quantities by a, the products must be equal; therefore

$$a c = q a b + a r.$$

Now $a c$ and $q a b$ are both measured by c, and therefore as in [56] $a r$ is measured by c. Again, let

$$c = q' r + r',$$

then as before

$$a c = a q' r + a r',$$

and as before $a r'$ is measured by c; and, similarly, if

$$c = q'' r' + r'',$$

we should prove that $a r''$ is measured by c. Now the numbers r, r', r'', &c. always go on diminishing, and at last some one of them must be unity, since c is a prime, and therefore cannot be measured by any number but unity. But on the supposition that $a b$ is measured by c, we have shown that $a r$, $a r'$, &c. must also be measured by c. It would follow, then, that $1 . a$ can be measured by c, which is contrary to our first supposition. So that it is impossible that c can measure $a b$, unless it measure a at the same time.

Next, if b be greater than c we have

$$b = q c + r,$$

where r is less than c, and multiplying by a this becomes

$$a b = q a c + a r.$$

Now, if $a b$ be measured by c, $a r$ must be so too, which we have just proved to be impossible.

61. It follows from this, that if we take any number, and find a set of prime numbers which when multiplied together produce it, that is the only set of prime factors the number can have. For if n be a number the product of a, b, and c all primes, no other prime d can measure it; for d cannot, as we have just seen, measure $a b$, and therefore it cannot measure $a b c$, that is n. No number, for instance, which is the product of 2, 5, and 7, or of any powers of these numbers, can be measured by 3.

So the product of any set of primes is prime to the product of any other set, all of which are different from the first. For no prime that measures the one product can measure the other, and therefore no number not a prime can measure both.

62. Every number then can be reduced into only one set of prime factors. The way in which we so reduce it, is by dividing it continually by all the prime numbers that will measure it, till it be brought down to a number which we find to be a prime. Take, for example, the number 8316; we divide by 2, which we can do twice; then by 3, which we can do three times; it cannot be divided by 5; and when it is divided by 7, the quotient is 11, a prime. The operation stands thus:

$$\begin{array}{r} 2)\underline{8316} \\ 2)\underline{4158} \\ 3)\underline{2079} \\ 3)\underline{693} \\ 3)\underline{231} \\ 7)\underline{77} \\ 11 \end{array}$$

So that $8316 = 2^2 . 3^3 . 7 . 11$. So $360 = 2^3 . 3^2 . 5$, and $210 = 2 . 3 . 5 . 7$.

63. It follows from [61] that all the numbers that measure any number, must necessarily be the products of some of the prime factors of that number. No number can measure 210, but 1, 2, 3, 5, 7, or the products of some of these numbers. The common measures of two numbers must be the prime factors they have in common, and the products of these prime factors. Thus, $420 = 2^2 . 3 . 5 . 7$, and $360 = 2^3 . 3^2 . 5$. The factors they have in common are 2^2, 3, and 5, so that their common measures can have no factors but these. Their greatest common measure is plainly $2^2 . 3 . 5$, or 60. Their other common measures are 2, 3, 4, 5, 6, 10, 12, 15, 20, 30. If one number have a prime factor which is not in another, it may be struck out of the first without affecting their common measure.

64. If a, b, and c be three numbers, and m be the greatest common measure of a and b, the greatest common measure of a, b, and c will be found by finding the greatest common measure of m and c. For, by the last article, m and its factors are the only numbers that measure a and b; and the greatest of these numbers that measures c, that is, the greatest common measure of m and c, will therefore be the greatest that measures a, b, and c. If the numbers be 1512, 588, and 330, the greatest common measure of the two first is 84; and

the greatest common measure of 84 and 330 is 6. Therefore 6 is the greatest common measure of the three. So, if there be a fourth number, the greatest common measure of the four is found by finding the greatest number that measures at once the fourth, and the greatest common measure of the three.

65. A *common multiple* of two numbers is a number which both of them measure. Thus, 90 is a common multiple of 6 and 15. It is useful to know that the least common multiple of two numbers is found by dividing their product by their greatest common measure. The greatest common measure of 210 and 360 is 30, and therefore their least common multiple is $\frac{210 \times 360}{30}$, or 2520. The reason of this appears when we consider that $210 = 2.3.5.7$, and $360 = 2^3.3^2.5$; so that no number less than $2^3.3^2.5.7$ (that is, 2520) can be measured by both of them. But $2^3.3^2.5.7$ is their product with $2.3.5$ struck out of it, that is, their product divided by their greatest common measure. The least common multiple of three or more quantities may, in like manner, be found, by resolving them into their prime factors, and considering these as in the instance above. Thus, the least common multiple of 144, 210, and 360 is 5040; for $144 = 2^4.3^2$ $210 = 2.3.5.7$, and $360 = 2^3.3^2.5$, so that no number less than $2^4.3^2.5.7$ (that is, 5040) can be measured by all of them.

66. We now proceed to another way of considering numbers that leads to important consequences. Let us take any number, as 98237, we may write it in this way, [art. 19]

$$7 + 30 + 200 + 8{,}000 + 90{,}000,$$

that is,

$$7 + 3\;\;10 + 2.10^2 + 8.10^3 + 9.10^4.$$

Now 3.10, 2.10^2, 8.10^3. &c. are all measured by 2 and by 5, and therefore if the term farthest to the left, that is, if the units' digit be divided by 2 or by 5, the remainder must be the same as when the whole number is divided by 2 or by 5. It is only when the units' digit of the number is 0, or a multiple of 2, that this remainder, on dividing by 2, is 0, and therefore it is only in these cases that the number is measured by 2. So it is only when its units' digit is 0 or 5, that a number is measured by 5.

67. Again,

$$3.10 = 3.(10 - 1 + 1) = 3.(10 - 1) + 3,$$
$$2.10^2 = 2.(10^2 - 1 + 1) = 2.(10^2 - 1) + 2,$$
$$8.10^3 = 8.(10^3 - 1 + 1) = 8.(10^3 - 1) + 8,$$

and so on. Therefore the number may be written

$$7 + 3(10 - 1) + 3 + 2.(10^2 - 1) + 2 + 8.(10^3 - 1) + 8 + 9.(10^4 - 1) + 9,$$

or writing the same terms in a different order

$$7 + 3 + 2 + 8 + 9 + 3.(10 - 1) + 2\;(10^2 - 1) + 8\,(10^3 - 1) + 9.(10^4 - 1).$$

Now by art. [44] all the terms $3.(10-1)$, $2\,(10^2 - 1)$, $8.(10^3 - 1)$, &c. are measured by $10 - 1$, that is, by 9. So that the remainder when $7 + 3 + 2 + 8 + 9$ is measured by 9, is the same as the remainder when the original number is measured by 9. The same principles apply to all numbers, and therefore the remainder when any number is divided by 9 is the same as when the sum of its digits is divided by 9.

In like manner, since every number measured by 9 is also measured by 3, the quantities $3.(10 - 1)$, $2.(10^2 - 1)$, &c. are all measured by 3. It follows, as before, that when the sum of the digits of any number is divided by 3, the remainder is the same as when the number itself is divided by 3. As examples of these properties, 17 is the sum of the digits of 278; dividing 17 by 9 the remainder is 8, and dividing it by 3 the remainder is 2, therefore when 278 is divided by 9 or 3, the remainders are respectively 8 or 2. The sum of the digits of 1287 is 18, which is measured by 9, and therefore 1287 is also measured by 9.

68. The common way of verifying the multiplication of two numbers, or proving it to be correct, by casting out the nines, is founded on the property demonstrated in the last article. It is this: *Take the sums of the digits of the multiplier, of the multiplicand, and of their supposed product separately; write down severally the remainders when these three sums are divided by 9; take the product of the first and second of these remainders, the remainder when this product is divided by 9 ought to be the same as the third.* For let N and N′ be the two numbers; when N is divided by 9 let the remainder be r, and

when N' is divided by 9 let the remainder be r'. Then by [41]

$$N = 9q + r,$$
$$N' = 9q' + r'.$$

Multiplying these equal quantities together, the products must be equal, therefore

$$N N' = 81 q q' + 9 q' r + 9 q r' + r r',$$

and as $81 q q'$, $9 q' r$, and $9 q r'$, are all measured by 9, the remainder when $r r'$ is divided by 9, is the same as when $N N'$ is divided by 9.

Now, by art. [67] when we divide the sum of the digits in N by 9 the remainder is r, and when we divide the sum of the digits in N' by 9, the remainder is r'; and, from what precedes, when we divide the sum of the digits in $N N'$ by 9, the remainder ought to be the same as when $r r'$ is divided by 9. If it be not so, we are sure that the multiplication $N N'$ has not been rightly performed. Take the example of multiplication in [33] as an instance.

Sum of dig. of 4786 = 25. Rem. = 7.
Sum of dig. of 2783 = 20. Rem. = 2.
Sum of dig. of 13319438 = 32. Rem. = 5.

The last remainder is the same as when 2×7 is divided by 9.

This proof is not quite perfect, for though it always holds when the multiplication is right, it may sometimes hold when it is wrong. If the product be too great or too small by any multiple of 9, the remainder when it is divided by 9 is the same as if it were right; and therefore the proof will not in such a case show it to be wrong.

69. We found

$$98237 = 7 + 3 + 2 + 8 + 9$$
$$+ 3(10 - 1) + 2(10^2 - 1) + \&c.$$

From each of these equal quantities take away $7 + 3 + 2 + 8 + 9$, or 29, and the remainders will be equal. Therefore

$$98237 - 29$$
$$= 3(10 - 1) + 2(10^2 - 1) + \&c.$$

We have seen that the second expression is measured by 9, and therefore $98237 - 29$, or 98208 is measured by 9. In general, if from any number the sum of its digits be subtracted the remainder is measured by 9.

70. Again, observe that

$$3 \cdot 10 = 3(10 + 1 - 1) = 3 \cdot (10 + 1) - 3,$$
$$2 \cdot 10^2 = 2(10^2 - 1 + 1) = 2 \cdot (10^2 - 1) + 2,$$
$$8 \cdot 10^3 = 8(10^3 + 1 - 1) = 8 \cdot (10^3 + 1) - 8,$$
$$9 \cdot 10^4 = 9(10^4 - 1 + 1) = 9 \cdot (10^4 - 1) + 9;$$

so that the number 98237 may be written

$$7 - 3 + 2 - 8 + 9$$
$$+ 3(10+1) + 2(10^2 - 1) + 8(10^3 + 1)$$
$$+ 9(10^4 - 1).$$

Now by [45] all the quantities $10 + 1$, $10^2 - 1$, $10^3 + 1$, and $10^4 - 1$, are measured by $10 + 1$, that is, by 11. Therefore if $7 - 3 + 2 - 8 + 9$ be divided by 11, the remainder is the same as if the original number were divided by 11. When this remainder is 0, the number is measured by 11. So that if the digits of any number be taken alternately, and the sum of one of the sets be subtracted from that of the other, when the difference is 0, or 11, or any multiple of 11, the number is measured by 11.

Observe that $98237 - (7 - 3 + 2 - 8 + 9)$ must be divisible by 11. In the same way, generally, if N be any number, A the sum of its alternate digits beginning with the units' place, and B the sum of the other digits, the expression

$$N - (A - B)$$

is measured by 11.

71. This way of expressing numbers would lead to many other properties of the same sort. We shall give two more. Take the number 89764; it may be written

$$64 + 9700 + 80,000,$$

or

$$64 + 97 \cdot 100 + 8 \cdot 100^2.$$

Now 100 and all its powers are measured by 4, and therefore since 4 measures 64, it measures 89764 also. In general, whenever 4 measures the two last digits of a number, it measures the number itself. Similarly, if the three last digits of a number be measured by 8, or the four last by 16, the number is measured by 8 or by 16. In the same way, if the two last digits be measured by 25, the number is measured by 25. If the three last be measured by 125, the number is measured by 125, and so on.

72. Take the number 8,937,524,361; it may be written

$$361 + 524 . 1000 + 937 . 1000^2 + 8 . 1000^3,$$

or, as in [70],

$$361 - 524 + 937 - 8 + 524 (1000 + 1) + 937 (1000^2 - 1) + 8 (1000^3 + 1).$$

As before, $1000 + 1$, $1000^2 - 1$, $1000^3 + 1$, are all measured by 1001, and therefore by all the factors of 1001, and therefore by 7, for $7 \times 143 = 1001$. So that the number proposed divided by 7 leaves the same remainder as $361 - 524 + 937 - 8$ does. If, therefore, we take any number and divide it into periods of three digits each beginning from the right, and then take the difference of the sums of the alternate periods, when that difference is measured by 7, the number itself is. For example, take the number 2,724,016,614,837,540,988,

$$988 + 837 + 16 + 2 = 1843,$$

$$540 + 614 + 724 \quad = 1878.$$

The difference of 1878 and 1843 is 35, which shows that the number is measured by 7.

In the same way, since 1001 is measured by 13, if the same difference be measured by 13, the number is measured by 13.

73. We have seen [19], that our way of writing numbers consists in an agreement, that the value of every digit shall be ten times as great as if it held the next place towards the right. We owe a very great deal of our present knowledge to this simple invention, which is so admirably adapted to the ends it has to serve. It came to us from the Arabs about A. D. 1000, and was not known to the ancients, though many of them thought and wrote very profoundly about numbers. Instead of agreeing that the digits should increase in value ten times, it might have been settled that they should increase eight, or twelve, or any other number of times. Ten is called the *base* of our *scale of notation*, as eight would be the base of a scale where the digits increased eight-fold, or twelve, of one where they increased twelve-fold. When we come to consider decimal fractions we shall see reason to think that twelve would have been a more convenient base than ten. Men were, perhaps, led to name the numbers according to a scale proceeding by tens, on account of the facility that the ten fingers would then give them in counting; and, afterwards, when they thought of ciphering they would naturally use the same scale.

74. In our scale, or the *decimal* scale as it is called, we can express any number by means of nine digits, and zero or nothing. In the scale whose base is eight, we can express all numbers by means of seven digits and zero, and so for others.

A number may be transferred from the decimal scale into any other by the following rule: *Divide the number by the base of the new scale, and write the remainder as the units' digit sought; divide the quotient by the base again, and write the remainder as the digit next the units'; proceed in this way till a quotient is obtained less than the base, this quotient is the digit of the highest order in the number in its new form. Whenever there is no remainder, 0 is the corresponding digit.* Let it be required, for instance, to present the number 2931 in the scale whose base is 8. Dividing by 8, we find

$$2931 = 366 \times 8 + 3.$$

Again, dividing 366 by 8, we find

$$2931 = (45 \times 8 + 6) 8 + 3$$
$$= 45 \times 8^2 + 6 \times 8 + 3,$$

and dividing 45 by 8,

$$2931 = 5 . 8^3 + 5 . 8^2 + 6 . 8 + 3.$$

If this be written 5563, and it be understood that the second digit is multiplied by 8, the third by 8^2, and the fourth by 8^3, each digit is 8 times as great as if it stood in the next place to the right, and therefore the number is expressed in a scale whose base is 8.

So if (10) and (11) be the two additional digits necessary in the scale whose base is 12, the number 13583 transferred into that scale becomes 7 (10) 3 (11). Similarly, 139 transferred into the scale whose base is 2, becomes 10001011, which is equivalent to $2^7 + 2^3 + 2 + 1$. Since 54 is equal to $2 . 3^3$, it will be expressed in the scale whose base is 3 by 2000.

75. To reduce a number from any other scale into the decimal, the rule is this: *Multiply the digit farthest to the left by the base of the scale in which the number is expressed, and add the next digit to the product; multiply the sum again by the base, and add the third digit; proceed in this way till the units' digit is added, the result is the number in the decimal scale.* For example, if

3465 be a number expressed in the scale whose base is 9, it is equivalent to

$$3.9^3 + 4.9^2 + 6.9 + 5,$$

and this is the same as

$$(3.9 + 4)\ 9^2 + 6.9 + 5,$$

or as

$$\{(3.9 + 4)\ 9 + 6\ \}\ 9 + 5$$

and this last expression merely indicates the operations directed by the rule. The working is this.

$$\begin{array}{l} 3465 \\ \underline{9} \\ 31 = 9 \times 3 + 4 \\ \underline{9} \\ 285 = 9 \times 31 + 6 \\ \underline{9} \\ 2570 = 9 \times 285 + 5 \end{array}$$

In like manner, 19(11)74 in the base whose scale is 12, is 37960, and 3000, where the base is 4, is 192 in the decimal scale.

76. Some of the properties we have been considering are general, and are the same in every scale of notation; such are those depending on the nature of prime factors. Some, again, are owing to the particular scale in which the number is expressed; such are those furnishing tests of the capacity of a number to be divided by certain others. In the decimal scale a number is measured by 9, when the sum of its digits is measured by 9; and in the scale whose base is 11, a number would be measured by 10, when the sum of its digits was measured by 10. There is a way of exhibiting the results of this latter kind very generally. Let us take the expression

$$N = A_0 + A_1 R + A_2 R^2 + A_3 R^3 + A_4 R^4 + \&c.^*$$

as in art. [70] it may be put into the forms

$$N = A_0 + A_1 + A_2 + A_3 + A_4 + \&c. + A_1 (R - 1) + A_2 (R^2 - 1) + A_3 (R^3 - 1) + A_4 (R^4 - 1) + \&c.$$

$$N = A_0 + A_2 + A_4 + \&c. - (A_1 + A_3 + \&c.) + A_1 (R + 1) + A_2 (R^2 - 1) + A_3 (R^3 + 1) + A_4 (R^4 - 1) + \&c.$$

Every number that measures R, measures $A_1 R$, $A_2 R^2$, $A_3 R^3$. &c. [art 51], and therefore in the first of these expressions every factor of R that measures A_0, measures N [art. 55].

Every number that measures $R - 1$, measures $A_1 (R - 1)$, $A_2 (R^2 - 1)$, &c. [art. 44], and therefore in the second expression every factor of $R - 1$ that measures $A_0 + A_1 + A_2 +$ &c. measures N.

Lastly, every number that measures $R + 1$ measures $A_1 (R + 1)$, $A_2 (R^2 - 1)$, $A_3 (R^3 + 1)$, &c. [art. 45], and therefore in the third expression every factor of $R + 1$ that measures $A_0 + A_2 + A_4 +$ &c. $- (A_1 + A_3 +$ &c.) measures N.

Now, let us suppose r to be the base of the scale of notation in which the number N is expressed, and instead of R in the expressions, let us write r^n. They become

$$N = A_0 + A_1 r^n + A_2 r^{2n} + A_3 r^{3n} + A_4 r^{4n} + \&c.$$

$$N = A_0 + A_1 + A_2 + A_3 + A_4 + \&c. + A_1 (r^n - 1) + A_2 (r^{2n} - 1) + A_3 (r^{3n} - 1) + A_4 (r^{4n} - 1) + \&c.$$

$$N = A_0 + A_2 + A_4 + \&c. - (A_1 + A_3 + \&c.) + A_1 (r^n + 1) + A_2 (r^{2n} - 1) + A_3 (r^{3n} + 1) + A_4 (r^{4n} - 1) + \&c.$$

And, as before, N is measured, first, by all the factors of r^n that measure A_0; secondly, by all the factors of $r^n - 1$ that measure $A_0 + A_1 + A_2 +$ &c.; and, thirdly, by all the factors of $r^n + 1$ that measure

$$A_0 + A_2 + A_4 + \&c - (A_1 + A_3 + \&c.).$$

For instance, let

$$r = 10, \text{ and } n = 2;$$

then

$$r^n = 100, r^n - 1 = 99, \text{ and } r^n + 1 = 101.$$

Also

$$N = A_0 + A_1 100 + A_2 10000 + A_3 1000000 + \&c.$$

Or we suppose the number N, which is now expressed in the decimal scale, to be divided into periods of two digits each, of which A_0 is farthest to the right, A_1 next to it, and so on. Then by the three last mentioned properties N is measured; first, by such of the numbers 2, 4, 5, 10, 20, 25, 50, (the factors of 100,) and 100, as measure A_0; secondly, by such of the numbers 3, 9, 11, 33, (the factors of 99,) and 99, as measure $A_0 + A_1 + A_2$ &c.; and thirdly, by the number 101 (a prime) if it measure

* See note to art. 55.

$A_0 + A_2 + A_4 + \&c. - (A_1 + A_3 + \&c.)$

So if r be 8, and n still 2;

$$r^n = 64,\ r^n - 1 = 63,\ r^n + 1 = 65;$$

and the number is supposed to be expressed in the scale whose base is 8. Dividing it into periods of two digits each, from the right, we have it measured; first, by such of the numbers 2, 4, 8, 16, 32, and 64 as measure A_0; secondly, by such of the numbers 3, 7, 9, 21, and 63 as measure $A_0 + A_1 + A_2 + \&c.$; and thirdly, by such of the numbers 5, 13, and 65 as measure $A_0 + A_2 + \&c. - (A_1 + A_3 + \&c.)$

It is plain, that by varying r and n we may find as many properties of the same kind as we please. These properties are of little practical use, and the expressions are inserted chiefly as affording examples of the comprehensiveness of algebraical language, and showing how little beyond an understanding of the symbols is necessary, to enable us to arrive at results seemingly abstruse and difficult.

Of Fractions

77. It will often happen, when we are expressing the magnitude of any thing by means of a number, that the unit we employ is not contained any exact number of times in the thing in question, but that there is a part over less than the unit. We can still, however, express the magnitude of this part by means of the same unit. We suppose the unit to be broken down into a stated number of equal parts, and then we say how many of these are contained in the portion that is over. This way of expressing the magnitude of things is called a *fraction*, from a Latin word meaning *to break*. We say that a distance is eight miles and two thirds of a mile, meaning, that in addition to eight miles we must divide a mile into three equal parts, and take two of these.

78. Let us consider any fraction, such as nine tenths. Here unity is to be divided into ten equal parts, and of these nine are to be taken. Ten times any one part must make unity; therefore ten times the nine parts must make nine times unity: so that ten times nine tenths make nine, and therefore nine tenths is the number which, when multiplied by ten, gives nine for product. Now it has been agreed [10] to write that number $\frac{9}{10}$, and therefore the fraction is to be written in the same way. So any other fraction as two thirds is to be written $\frac{2}{3}$: The lower number showing into how many parts unity is to be divided, is called the *denominator*; the upper one, showing how many of these parts are to be taken, is called the *numerator*.

On the same principles, whenever there is such a number as $\frac{15}{17}$, resulting from division, it is a fraction, and may be read as such. Its meaning is 15 parts of unity divided into 17 equal parts. Such quantities as $\frac{a}{b}$, $\frac{a^2-c^2}{a+c}$, are called algebraical fractions.

79. We have already seen [36], that when the same quantity is a factor in the divisor and dividend, it may be struck out of both without affecting the value of the quotient; and so when it is a factor in the numerator and denominator of a fraction, it may also be struck out of both. Thus, $\frac{ac}{bc} = \frac{a}{b}$, and $\frac{25}{65} = \frac{5}{13}$. Whenever, then, there is a common measure of the numerator and denominator of a fraction, it can always be reduced into one of equal value, but of which the numerator and denominator are smaller numbers, or less complicated expressions. Thus, $\frac{5005}{43316}$ can be reduced to $\frac{55}{476}$, where the common measure is 91, and

$$\frac{6a^3 - 6a^2y + 2ay^2 - 2y^3}{12a^2 - 15ay + 3y^2}$$

to

$$\frac{6a^2 + 2y^2}{12a - 3y},$$

the common measure being $a - y$. When the numerator and denominator have no common measure, the fraction is in its *lowest terms*, and is called an *irreducible fraction*.

80. For the same reason, if the numerator and denominator of a fraction be multiplied by the same quantity, its value is not changed. For instance, $\frac{9}{10}$

$= \frac{45}{50}$ and $\frac{a}{b} = \frac{a\,c}{b\,c}$. By means of this property, when several fractions have different denominators, they can, without altering their values, be changed into fractions that shall all have the same denominator. To do this, *Multiply the numerator and denominator of every one of the fractions by the product of all the denominators, except its own.* Thus, if the fractions be $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$, the first becomes $\frac{a\,d\,f}{b\,d\,f}$, the second $\frac{b\,c\,f}{b\,d\,f}$, and the third $\frac{b\,d\,e}{b\,d\,f}$; or if they be $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$, they become $\frac{40}{60}$, $\frac{45}{60}$, and $\frac{48}{60}$.

When the denominators of any of the fractions have a common measure, the common denominator resulting from this rule is greater than is necessary. If the least common multiple of all the denominators be found, as directed in art. [65], the fractions may be reduced into others whose common denominator is this multiple. This is done by multiplying the numerator and denominator of every fraction by the common multiple divided by the denominator of the same fraction. Take, for example, the fractions $\frac{5}{12}$, $\frac{3}{4}$, $\frac{4}{9}$. The least common multiple of 12, 4, and 9, is 36. Multiply the numerator and denominator of $\frac{5}{12}$ by $\frac{36}{12}$, or 3, of $\frac{3}{4}$ by $\frac{36}{4}$, or 9, and of $\frac{4}{9}$ by $\frac{36}{9}$, or 4, and they become $\frac{15}{36}$, $\frac{27}{36}$, and $\frac{16}{36}$.

81. Since we may multiply the numerator and denominator of a fraction by any quantity without changing its value, that quantity may be -1. If we multiply the numerator and denominator of $\frac{a-b}{c-d}$ by -1, it becomes $\frac{b-a}{d-c}$. So that without altering its value, the signs of all the terms in the numerator of a fraction may be changed, if, at the same time, the signs of all the terms in its denominator be changed.

82. By [37] $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$, and $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$. *To add or subtract fractions, then, that have the same denominators, we add or subtract their numerators. To add or subtract fractions that have different denominators, we must first reduce them to fractions that have the same denominator.*

$$\frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \frac{7}{12},$$

and

$$\frac{1}{3} - \frac{1}{4} = \frac{1}{12};$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a\,d + b\,c}{b\,d},$$

and

$$\frac{a}{b} - \frac{c}{d} = \frac{a\,d - b\,c}{b\,d};$$

$$\frac{a-3\,b}{c} + \frac{5\,a-b}{2\,c} = \frac{7\,a-7\,b}{2\,c};$$

$$\frac{a+b}{a-b} - \frac{a-b}{a+b}$$

$$= \frac{a^2 + 2\,a\,b + b^2}{a^2 - b^2} - \frac{a^2 - 2\,a\,b + b^2}{a^2 - b^2}$$

$$= \frac{4\,a\,b}{a^2 - b^2}.$$

83. When the numerator of a fraction is less than the denominator, the fraction is less than unity, and is called a *proper fraction*. When the denominator is less than the numerator, the fraction is greater than unity, and is called an *improper fraction*. $\frac{3}{4}$ is a proper, and $\frac{5}{4}$ an improper fraction. A number made up of a whole number and a fraction is called a *mixed number*; as $13 + \frac{1}{3}$, or $7 + \frac{1}{2}$, or, as these are usually written, $13\frac{1}{3}$ and $7\frac{1}{2}$.

84. Every whole number may be considered as an improper fraction whose denominator is unity, and it may be reduced to an improper fraction of any other denomination, by multiplying it by the number that is to be the denominator. Thus, 7 is the same as $\frac{7}{1}$, or

$\frac{70}{10}$, or $\frac{84}{12}$. In this way any mixed number may be written as an improper fraction, by changing the whole part of it into a fraction, with the same denominator as the fractional part. Thus,

$$4\frac{5}{6} = \frac{24}{6} + \frac{5}{6} = \frac{29}{6}. \text{ Similarly}$$

$$a + \frac{b}{c} = \frac{ac+b}{c}.$$

An improper fraction may be reduced to a whole or a mixed number, by dividing out the numerator by the denominator. Thus, $\frac{57}{8} = 7\frac{1}{8}$.

85. To multiply a fraction by a whole number, *multiply its numerator by the whole number.* For $c \times \frac{a}{b}$ means [art. 7]

$$\frac{a}{b} + \frac{a}{b} + \text{\&c. taken } c \text{ times};$$

that is, by [82],

$$\frac{a+a+a+\text{\&c. [taken } c \text{ times]}}{b},$$

or $\frac{c\,a}{b}$. So $10 \times \frac{3}{4} = \frac{30}{4}$.

86. *To multiply by a fraction is to multiply by its numerator, and divide by its denominator.* Suppose we have to multiply c by $\frac{a}{b}$. If we take $a \,.\, c$, we have multiplied c by a quantity b times too great, and therefore the product is b times too great. For the true product, therefore, we must take $\frac{a\,c}{b}$; or the quantity which, when multiplied by b, produces $a\,c$. Similarly,

$$\frac{3}{4} \times 15 = \frac{45}{4}.$$

87. *To divide a fraction by any quantity, is to multiply its denominator by that quantity.* Dividing $\frac{a}{b}$ by c, the quotient is $\frac{a}{b\,c}$; for, multiplying this quotient by c, the divisor, the product [85] is $\frac{a\,c}{b\,c}$, or $\frac{a}{b}$, [79] the dividend.

88. To multiply one fraction by another; *Take the product of their numerators for the numerator of their product, and the product of their denominators for its denominator.* To multiply $\frac{a}{b}$ by $\frac{c}{d}$ is, by [86], to multiply $\frac{a}{b}$ by c, and divide it by d. The result, by [85] and [86], will be $\frac{a\,c}{b\,d}$. So

$$\frac{3}{4} \times \frac{5}{6} = \frac{15}{24}, \text{ or } \frac{5}{8}.$$

89. Since $\frac{a}{b} \times \frac{c}{d}$, and $\frac{c}{d} \times \frac{a}{b}$, both equal $\frac{a\,c}{b\,d}$, it follows that

$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$. We may therefore extend to fractions the proposition in art. [23], which was proved there for whole numbers only.

90. *Any power of a fraction has the numerator raised to that power for a numerator, and the denominator raised to the same power for a denominator.* For instance, $\left(\frac{a}{b}\right)^3 = \frac{a}{b}.\frac{a}{b}.\frac{a}{b}$, or $\frac{a^3}{b^3}$. Similarly, $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, and $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$

91. Let us next propose to divide any quantity, as a, by a fraction, as $\frac{b}{c}$. We seek an expression for $a \div \frac{b}{c}$. Multiply the divisor and the dividend, both by c, and this becomes $a\,c \div c\,.\,\frac{b}{c}$; or, $a\,c \div b$; or, finally, $\frac{a\,c}{b}$. To divide any quantity, then, by a fraction, *multiply it by the denominator of the fraction, and divide the product by its numerator.* Thus,

$$50 \div \frac{9}{10} = \frac{10}{9}.50, \frac{500}{9}, \text{ or } 55\frac{5}{9}.$$

92. When the dividend is a fraction *multiply the numerator of the dividend by the denominator of the divisor for the numerator of the quotient, and the denominator of the dividend by the numerator of the divisor for its denominator.* Thus, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, by [91], $= \frac{a\,d}{b\,c}$. So $\frac{4}{5} \div \frac{5}{6} = \frac{24}{25}$.

93. Observe, that

$$\frac{\left(\frac{a}{b}\right)}{c} \text{ means } \frac{a}{b} \div c = \frac{a}{b\,c},$$

by [87]; that

$$\frac{a}{\left(\frac{b}{c}\right)} \text{ means } a \div \frac{b}{c} = \frac{a\,c}{b},$$

by [91]; that

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} \text{ means } \frac{a}{b} \div \frac{c}{d} = \frac{a\,d}{b\,c},$$

by [92]; that

$$\frac{a\,.\,\frac{b}{c}}{d} = \frac{\left(\frac{a\,b}{c}\right)}{d} = \frac{a\,b}{c\,d};$$

and that

$$\frac{a}{b\,.\,\frac{c}{d}} = \frac{a}{\left(\frac{b\,c}{d}\right)} = \frac{a\,d}{b\,c}.$$

94. To multiply a mixed number by a fraction, or by a mixed number, or to divide a fraction or a mixed number by a mixed number, *Reduce to improper fractions, and then divide or multiply by the rules laid down.* For instance,

$$10\frac{3}{4} \times 7\frac{1}{9} = \frac{43}{4} \times \frac{64}{9} = \frac{2752}{36} = 76\frac{4}{9};$$

$$6\frac{1}{5} \div 2\frac{1}{7} = \frac{31}{5} \div \frac{15}{7}, \text{ or}$$

$$\frac{31}{5} \times \frac{7}{15} = \frac{217}{75}, \text{ or } 2\frac{67}{75}.$$

95. When we multiply any quantity by a proper fraction, we have to divide it by a quantity greater than that by which we multiply, and therefore the product is less than the multiplicand. When we multiply by an improper fraction the product is greater than the multiplicand. Again, when we divide by a fraction, if it be a proper fraction, the quotient is greater; if it be an improper fraction, the quotient is less than the dividend. When the multiplicand is the same, the product increases when the multiplier is increased, and diminishes when it is diminished. When the dividend is the same, the quotient diminishes when the divisor is increased, and increases when it is diminished.

96. Unity divided by any quantity is called the *reciprocal* of that quantity. Thus $\frac{1}{m}$ is the reciprocal of m. $1 \div \frac{4}{5} = \frac{5}{4}$ is the reciprocal of $\frac{4}{5}$. By [91], to divide by any quantity is the same thing as to multiply by its reciprocal. Any quantity multiplied by its reciprocal, must be 1.

97. The sum of two irreducible fractions whose denominators are prime to each other, can never be a whole number. Thus, $\frac{2}{5} + \frac{5}{6}$ cannot possibly be a whole number. For, let the fractions be $\frac{a}{b}$ and $\frac{a'}{b'}$, and let their sum be p. Then

$$\frac{a}{b} + \frac{a'}{b'} = p.$$

Multiply each of these equal quantities by b, and the products will be equal. Therefore

$$a + \frac{a'\,b}{b'} = b\,p.$$

Now a' is prime to b', because $\frac{a'}{b'}$ is an irreducible fraction, and b is prime to b', by supposition; therefore, by [61] and [62], $a'\,b$ is prime to b', and $\frac{a'\,b}{b'}$ is an irreducible fraction. It follows that $b\,p$ cannot be a whole number; for a whole number cannot be equal to the sum of another whole number, and an irreducible fraction: and since $p\,b$ is not a whole number, p cannot be a whole number. It is a consequence of this, that if there be any set of irreducible fractions, and the denominator of one of them be prime to all the rest, their sum cannot be a whole number.

Of Compound Numbers.

98. Instead of expressing the magnitude of a thing by means of one unit and its fractional parts, it is usual to have for each kind of thing a scale of units of different magnitudes. We state how many times the unit of the greatest magnitude not greater than the thing in question is contained in it, then how many times the next greatest is contained in the part over, and so on to the least. In this way we avoid the inconvenience of having only a small unit, which would often make it necessary to employ very large numbers

as well as that of having only a great one, which would incumber our operations with fractions. Units of the same kind but of different magnitudes, as pounds, shillings, pence, and farthings, which are units of value, or miles, furlongs, poles, yards, feet, and inches, which are units of length, are called units of different *denominations*. A magnitude expressed in units of different denominations is called a *compound number*.

99. Numbers are changed from one denomination into another by the rules of *reduction*. To reduce from a higher denomination into a lower, *Multiply the number of the highest denomination by the number of times that the unit of the second denomination is contained in that of the highest, and to the product add the number of the second, if any; multiply this result by the number of times that the unit of the third denomination is contained in that of the second, and add the number of the third, if any; proceed in this way till the number is reduced to the denomination required.*

To reduce $5l.\ 8s.\ 6d.$ to pence, for instance: since there are $20s.$ in a pound, $5l.\ 8s.$ are equal to $(20 \times 5 + 8)s.$, or $108s.$; and since there are $12d.$ in a shilling, $108s.\ 6d.$ are equal to $(12 \times 108 + 6)d.$, or $1302d.$

100. To reduce a fraction of a unit of a higher denomination into a lower, *Multiply the fraction by the number of times that the unit of the lower denomination is contained in that of the higher; the product is a fraction of the lower denomination, and if an improper fraction, may be reduced into a mixed number of that denomination.*

For example, to reduce $\frac{6}{7}$ of a yard into feet and inches. There are three feet in a yard, therefore $\frac{6}{7}$ of a yard are $\frac{6}{7}$ of 3 feet, or $\dfrac{18}{7}$ of one foot, or $2\frac{4}{7}$ feet. Again, $\frac{4}{7}$ of a foot are $\frac{4}{7}$ of 12 inches, or $\dfrac{48}{7}$ of one inch, or $6\frac{6}{7}$ inches. So that $\frac{6}{7}$ of a yard = 2 feet $6\frac{6}{7}$ inches.

101. To reduce a number from a lower denomination to a higher, *Divide it by the number of times that the unit in which it is expressed is contained in that of the next higher denomination, noting the remainder; divide the whole part of this quotient again by the number of times that the unit of its present denomination is contained in that of the next higher, noting the remainder, as before; proceed in this way till the number is raised to the denomination required: the several remainders are the numbers of the several lower denominations.*

For instance, to reduce 87431 seconds to hours. Since $87431 = 60 \times 1457 + 11$, and there are 60 seconds in a minute, 87431 seconds = 1457 minutes, 11 seconds. So, since $1457 = 60 \times 24 + 17$, and there are 60 minutes in an hour, 87431 seconds are 24 hours 17 minutes 11 seconds.

102. To reduce whole or fractional numbers of lower denominations into fractional parts of a higher; *Reduce the numbers into their lowest denomination, and divide the result, whether it be whole or fractional, by the number of times that its unit is contained in the unit of the denomination into which the whole is to be reduced.*

Thus, to reduce $7s.\ 4\frac{3}{5}d.$ to a fraction of a pound; it is the same as $88\frac{3}{5}d.$ [art. 99]. But a penny is $\frac{1}{240}$ of a pound, and therefore $88\frac{3}{5}d. = \dfrac{88\frac{3}{5}}{240}$, or $\dfrac{443}{1200}$ of a pound.

103. To add compound numbers together, *Write them under one another, the numbers of the same denomination being all in the same column; take the sum of the column of the lowest denomination, and reduce it as far as practicable into the next higher; write under the column of the lowest denomination the part of its sum that is of that denomination, and carry the other part to be added to the next column; the sum of this column to be treated in the same way, and so on to the column of the highest denomination.*

To subtract one compound number from another, *Write the number to be subtracted under the other, as in addition. When a number of any denomination in the upper number is less than the one corresponding to it in the lower, add to it as many as will make one unit of the next higher denomination, and increase the number of the next higher denomination in the lower number by unity; then proceed to subtract the lower number part by part.*

104. To multiply a compound number by a simple number, *Multiply every particular part of it by the simple number, and reduce the products as far as is practicable to higher denominations.* This operation, however, is much better performed by the rules of *practice*, explained in the next article.

To divide a compound by a simple number, *Divide the number of the highest denomination by the divisor, and write the whole part of the quotient as the number of that denomination in the general quotient; reduce the remainder, if any, to the next lower denomination, and adding the number of that denomination, if any, proceed as before.*

These rules are founded on the same principles as those for the addition, subtraction, multiplication and division of simple numbers in arts. [20], [21], [33] and [40].

105. Suppose we have to multiply 2*s*. 6*d*. by 36, we may multiply its parts separately, as is directed in the last article. But the operation will be much abridged, if we notice that 2*s*. 6*d*. are $\frac{1}{8}$ of a pound; that therefore we have to find 36 times $\frac{1}{8}$ of a pound, or $\frac{1}{8}$ of 36*l*., which is $4\frac{1}{2}$*l*., or 4*l*. 10*s*.

Again, if we have to multiply 3*s*. 9*d*. by $17\frac{1}{2}$, we observe that 3*s*. 9*d*. = 2*s*. 6*d*. + 1*s*. 3*d*., or $\frac{1}{8}$*l*. + $\frac{1}{16}$*l*. Therefore $17\frac{1}{2} \times$ 3*s*. 9*d*. = $17\frac{1}{2} \times (\frac{1}{8} + \frac{1}{16})$*l*., or $(\frac{1}{8} + \frac{1}{16}) \times 17\frac{1}{2}$*l*., or $(\frac{1}{8} + \frac{1}{16}) \times$ 17*l*. 10*s*. Therefore, if we divide 17*l*. 10*s*. by 8, and then by 16, or, which is the same thing, divide 17*l*. 10*s*. by 8, and the quotient by 2, and then add the two quotients, their sum will be the answer. We have

	l.	*s*.	*d*.
8)	17	10	0
2)	2	3	9
	1	1	$10\frac{1}{2}$
	3	5	$7\frac{1}{2}$

On the same principles, to multiply 1*l*. 14*s*. 3*d*. by 173, is to take the product $(1 + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{10} + \frac{1}{80}) \times 173l.$, as appears below:

	l.	*s*.	*d*.		*l*.	*s*.	*d*.
	1	0	0	5	173	0	0
$\frac{1}{5}$ of 1*l*. =	0	4	0		34	12	0
	0	4	0		34	12	0
	0	4	0	2	34	12	0
$\frac{1}{2}$ of 4*s*. =	0	2	0	8	17	6	0
$\frac{1}{8}$ of 2*s*. =	0	0	3		2	3	3
	1	14	3		296	5	3

Similarly, to multiply 2 feet 10 in. by $11\frac{3}{4}$, is to take the product $(\frac{1}{2}+\frac{1}{4}+\frac{1}{12}+\frac{1}{12}+\frac{1}{36}) \times$ 11 yds. 2 ft. 3 in. Since $\frac{3}{4}$ yd. = 2 feet 3 in.

	yd.	ft.	in.		ft.	in.		yds.	ft.	in.
							2	11	2	3
$\frac{1}{2}$ of	1	0	0	=	1	6	2	5	2	$7\frac{1}{2}$
$\frac{1}{2}$ of	0	1	6	=	0	9	3	2	2	$9\frac{3}{4}$
$\frac{1}{3}$ of	0	0	9	=	0	3		0	2	$11\frac{1}{4}$
					0	3	3	0	2	$11\frac{1}{4}$
$\frac{1}{3}$ of	0	0	3	=	0	1		0	0	$11\frac{3}{4}$
					2	10		11	0	$3\frac{1}{2}$

The submultiples [50] of a unit of a higher denomination, when they can be expressed in whole numbers of lower denominations, are called the *aliquot parts* of that unit. Thus, 10*s*., 6*s*. 8*d*., 5*s*., 4*s*., and 3*s*. 4*d*., which are respectively, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{1}{6}$ of a pound, are aliquot parts of a pound. So 1ft. 6 in., 1ft. 9 in., 6 in., &c., are the aliquot parts of a yard. Every compound number of lower denominations can be reduced in many different ways into aliquot parts of a unit of a higher denomination. A little familiarity with the rule of *practice*, by which we have resolved the foregoing questions, teaches us the most convenient set of aliquot parts into which to reduce any number. The rule is this: *Reduce the compound number to be multiplied into a series of aliquot parts of a unit of some denomination, such, that every aliquot part is either the same as the preceding, or some submultiple of it; consider the multiplier as a number of that denomination, with respect to which the aliquot parts are taken, and reduce its fractional part, if any, into lower denominations; divide the multiplier, in this state, by that number by which the first aliquot part is a submultiple of the standard unit; divide the quotient by that number by which the second aliquot part is a submultiple of the first; proceed in this way till all the aliquot parts are exhausted, observing to repeat a quotient twice or more, when two or more aliquot parts are the same; add all these quotients, and their sum is the product sought.*

106. The process of multiplication is the addition of a quantity to itself a certain number of times, so that whatever the multiplicand be, the multiplier must always be an abstract number, and the product a quantity of the same nature as the multiplicand. It is therefore an absurdity to speak of multiplying such quantities together as pounds and yards, or pounds and pounds, or yards and yards. When there are so many yards of cloth, for instance, at so much a yard, and the price of the whole is required, to find it, we do not multiply money by yards, we take a certain sum of money as often as there are yards of cloth, the sum, of course, is a sum of money. Again, if a pound will buy a certain number of yards, and we have a certain number of pounds, and wish to know how many yards we can buy, we take the number of yards as often as there are pounds, and the sum is a

number of yards. There is one seeming exception to this, namely, that we familiarly multiply feet by feet, or inches by inches, and have for product square feet or square inches. But this only needs a little explanation. Suppose, for instance, that there is a square board, 8 inches each way, to find the size of its surface, we are said to multiply 8 inches by 8 inches, and to have for product 64 square inches. But the proper way to consider the matter is this; suppose the board to be divided into squares, so as to resemble a draught or chess board; then, because there are eight inches along the end, there are eight squares in every row; and because there are eight inches along the side, there are eight rows of squares. All that we do, then, in multiplying 8 by 8, is taking 8 square inches 8 times, and adding them together; and, strictly speaking, this is not multiplying inches by inches, any more than if there were a shilling on each square, it would be multiplying shillings by shillings, to find their number by multiplying the number in every row by the number of rows.

We cannot, therefore, properly speaking, multiply one compound number by another. When we have such a question as to find the value of 8 lbs. 9 oz., at 7*s.* 6*d.* per lb., we must consider, that, as there are 16 oz. in a pound, 9 oz. are $\frac{9}{16}$ of a lb., and therefore cost $\frac{9}{16}$ of the price of a pound; and so, to find the answer, multiply 7*s.* 6*d.* by $8\frac{9}{16}$. In such cases, a little attention will always teach us which of the compound numbers is to be reduced to a simple number before we multiply. The rule of *duodecimal multiplication*, which is another way of multiplying compound numbers of certain kinds, depends on principles that cannot properly be explained here.

107. The product of a compound number by a simple number, is, as we have seen [106], a compound number of the same nature as the multiplicand, and we have also seen [10], that the division of the product by the multiplicand gives us the multiplier for quotient. We may therefore properly divide one compound number by another of the same nature, for in so doing we seek the simple number which must multiply the divisor so as to produce the dividend. The most convenient way of performing this division, is to reduce the divisor to its least denomination, and proceed as is directed for a simple number [104].

Of Simple Equations.

108. The sum of two numbers is 89, and their difference is 31; let it be required to find out what the numbers are. Call the less number x. Then, since their difference is 31, the greater is $31 + x$. Also, their sum is 89, therefore

x [the less] $+ (31 + x)$ [the greater] $= 89$,

that is, adding the greater to the less

$$2x + 31 = 89.$$

If from each of these equal quantities we take away 31, the remainder will be equal. Therefore

$$2x = 89 - 31, \text{ or } 2x = 58.$$

And dividing each of the equal quantities by 2,

$$x = 29, \text{ and therefore } 31 + x = 60.$$

We find the one number to be 29, and the other 60; and these two numbers satisfy the conditions that were given.

In the same way, if the question had been to find two numbers whose sum is a, and difference b, calling the less x, we should have found

$$2x + b = a, \text{ and } x = \frac{a-b}{2}.$$

The numbers are $\frac{a-b}{2}$ and $\frac{a+b}{2}$. In these expressions any numbers may be substituted for a and b, and so an answer found in any particular case.

Two expressions connected by the sign of equality, as in these examples, make an *equation*. The two expressions so connected are called the two *members* of the equation. An equation in which the quantity whose value is sought, occurs in the first power only, is called a *simple equation*; a distinction of which the importance will appear afterwards.

109. In the example in the last article, we removed the number 31 from the left hand member of the equation to the right, changing its sign. In the same way, in any equation, if a term be struck out of one member, and written in the other with its sign changed, the new expressions so formed will be equal to each other.

If in this way we remove every term from the left hand member into the right, and every term from the right hand member into the left, the two members will have the same terms as before, only the sign of every term will

be changed. Thus, if

$$a - n x = m x - b,$$

we should have

$$b - m x = n x - a.$$

We may therefore change the sign of every term in an equation, and the new members will still be equal to each other.

If we remove all the terms from one member into the other, we shall have zero on one side of the equation. Thus

$$a - n x = m x - b$$

becomes

$$a + b - (n + m) x = 0.$$

The meaning of this is, that $a + b$, and $(n + m) x$, are two quantities such that their difference is nothing, that is, they are equal quantities. We shall afterwards find, that this way of stating an equation is sometimes very convenient.

110. Again, let it be required to find that number, the third part of which added to its seventh part makes 20. Let the number be called x, as before. Its third part is $\frac{x}{3}$, and its seventh is $\frac{x}{7}$. Therefore

$$\frac{x}{3} + \frac{x}{7} = 20.$$

Multiply both members by 21; then, since the products must be equal,

$$7 x + 3 x = 420;$$

and adding, as before,

$$10 x = 420.$$

Now divide each member by 10, and we find $x = 42$. The third part of 42 is 14, its seventh part is 6, and these added make 20.

When there are fractional terms in an equation, we can always get rid of them, as in this example, by multiplying both members by the product of all the denominators, or by their least common multiple, if it be less than their product. By [80] all the fractions may be reduced to a common denominator, which will be this common multiple, and multiplying all the terms by it, the fractions will disappear.

111. These examples teach us how a simple equation is to be solved, that is, now we are to extract from it the value of the unknown quantity. *Clear both members of fractions, if there be any,* [110]; *collect into one member all the terms containing the unknown quantity, and into the other all those that do not contain it; collect into one the coefficients of the terms containing the unknown quantity, divide both members by the whole coefficient of the unknown quantity, which will then stand alone in one member.*

We may observe here, that a simple equation can have only one solution. By the rule, every simple equation can be reduced to the form,

$$\mathrm{A} x + \mathrm{B} = 0.$$

If a be a value of x that solves it, we have

$$\mathrm{A} a + \mathrm{B} = 0.$$

Suppose, if possible, that a' is another value of x that solves it, then

$$\mathrm{A} a' + \mathrm{B} = 0.$$

Subtract $\mathrm{A} a' + \mathrm{B}$ from $\mathrm{A} a + \mathrm{B}$, the difference must be nothing, so that

$$\mathrm{A} . (a - a') = 0.$$

But a product can be nothing, only when one of its factors is nothing. So that $a - a' = 0$, that is to say, a' cannot be a solution, but on the condition that it is equal to a.

112. We will now apply this very simple rule to a few questions, with some remarks on the results which we shall obtain. Take the question proposed in art. [3]. Let the time which the father and son take to dig the field together be called x. The father in one day digs $\frac{1}{10}$ of the field, in two days he will have digged $\frac{2}{10}$ of it, in x days he will have digged $\frac{x}{10}$ of it. So the son in x days will have digged $\frac{x}{16}$ of the field. But in x days they will have digged the whole field. Therefore

$$\frac{x}{10} \text{ of the field} + \frac{x}{16} \text{ of the field}$$
$$= \text{the whole field.}$$

The magnitude of the field is a quantity which is a factor in every term of this equation, dividing each member by this quantity

$$\frac{x}{10} + \frac{x}{16} = 1.$$

Whence, by the rule, multiplying by 80, the least common multiple of 10, and 16,

$$8x + 5x = 80,$$

and

$$x = \frac{80}{13} = 6\frac{2}{13} \text{ days}$$

Similarly, if the father's time were a, and the son's b, we should find

$$\frac{x}{a} + \frac{x}{b} = 1,$$

and

$$x = \frac{ab}{a+b}.$$

With regard to this expression for x, observe that a enters into it in the same way as b does; a therefore might be written for b, and b for a, without changing its value. Such an expression is said to be *symmetrical* with respect to a and b. This must necessarily be the case from the question; for if the father took b days, and the son a, to dig the field, the answer would be the same. An examination whether an algebraical result be symmetrical with respect to quantities that enter into the conditions of the question in the same way, often enables us to detect errors.

113. A father is 40 years old, his son is 8; in how many years hence will the father's age be just three times the son's? Let the number of years to that time be called x. In x years the father will be $40 + x$ years old, and the son $8 + x$. But the father's age will be then three times the son's; therefore

$$40 + x = 3(8 + x) = 24 + 3x.$$

Carry x to the one side of the equation, and 24 to the other, and it will become

$$40 - 24 = 3x - x, \text{ or } 16 = 2x,$$

whence $x = 8$. The father's age at the end of 8 years will be 48, and the son's 16.

If the father's age had been called a, and the son's b, to answer the same question we should have had, as before,

$$a + x = 3b + 3x,$$
$$2x = a - 3b,$$
$$x = \frac{a - 3b}{2};$$

an expression which gives a value of x for every different value of a and b. Suppose that the father's age is 40, and the son's 18. Here the time when the father's age was three times that of the son's is already past, so that the question, if put as it stands above, would be absurd. Let us see what our expression for x becomes in this case: making $a = 40$ and $b = 18$,

$$x = \frac{40 - 3 \times 18}{2} = \frac{40 - 54}{2} = -7,$$

a negative quantity. In our original conditions, the father's age was to be $40 + x$, and the son's $18 + x$. These are in the present instance $40 - 7$, or 33, and $18 - 7$, or 11. And 33 is 3 times 11. The negative sign teaches us that the number of years that is to make the father's age 3 times the son's is to be subtracted, not to be added, and accordingly we find, that 7 years ago the father's age was just three times the son's.

Observe, that when $a = 3b$, $x = 0$; that is to say, that when the father is three times as old as the son at the present time, x is neither a quantity to be added nor subtracted.

This question may be made still more general by proposing to find when the father's age will be n times the son's. We should have as before

$$a + x = nb + nx,$$

whence we find

$$x = \frac{a - nb}{n - 1}.$$

114. A and B find a purse with shillings in it, A takes out two shillings and one sixth of what remains: then B takes out three shillings and one sixth of what remains; and then they find that they have taken equal shares. How many shillings were in the purse, and how many did each take? Let x be the number of shillings in the purse at first.

A takes out 2*s.*; there remain $x - 2$.

He takes $\frac{1}{6}$ of this remainder, or $\frac{x-2}{6}$; there remain $\frac{5}{6}$ of the first remainder, or

$$\frac{5x - 10}{6}.$$

B takes out 3*s.*; there remain

$$\frac{5x - 10}{6} - 3, \text{ or } \frac{5x - 28}{6};$$

He takes $\frac{1}{6}$ of this remainder, or

$$\frac{5x - 28}{36}.$$

Now A has taken out $2 + \frac{x-2}{6}$, and B has taken out $3 + \frac{5x - 28}{36}$, and therefore since their shares are equal

$$2 + \frac{x-2}{6} = 3 + \frac{5x-28}{36}.$$

Multiply both members by 36, and they become

$$72 + 6 - 12 = 108 + 5x - 28.$$

And collecting the terms

$$x = 20.$$

There were 20*s*. in the purse. A's share was $2 + \frac{x-2}{6}$, or $2 + \frac{20-2}{6}$, or 5*s*.; and B's was $3 + \frac{5x-28}{36}$ which will be found to be 5*s*. also.

115. There is a certain number consisting of two digits, and their sum is 6. If 18 be added to the number the sum will consist of the same digits, in an inverted order. What is the number? Let its tens' digit be called x, then since the sum of its digits is 6, its units' digit will be $6-x$. The number then is $10x + 6 - x$. Now if 18 be added to the number, its units' digit becomes x, and its tens' $6-x$, and therefore it becomes $10(6-x) + x$. Therefore

$$10x + 6 - x \text{ [the original number] } + 18 = 10(6-x) + x \text{ [the new number]}.$$

And collecting the terms

$$18x = 36, \text{ or } x = 2.$$

The digits are 2 and 4. The first number is 24; and $24 + 18 = 42$.

116. A hare is 80 of her own leaps before a greyhound; she takes three leaps for every two that he takes, but he covers as much ground in one leap as she does in two. How many leaps will the hare have taken before she is caught? Call the number of leaps x. Since the dog takes two leaps only for every three that the hare takes, he will have taken $\frac{2}{3}x$ leaps while she takes x. But since one of his leaps is equal to two of hers, in these $\frac{2}{3}x$ leaps of his he will have covered as much ground as $2 \cdot \frac{2}{3}x$, or $\frac{4}{3}x$ hare's leaps. Now when he has done this, he catches the hare, by our supposition, and he was 80 of her leaps behind her at first, therefore he has run a distance equal to $80 + x$ of her leaps. We have thus found two expressions for the distance run by the dog before he catches the hare, which of course must be equal. So that we have

$$\frac{4}{3}x = 80 + x,$$

and multiplying by 3,

$$4x = 240 + 3x,$$

whence

$$x = 240.$$

Suppose that, all other circumstances being the same, it had been said generally, that m of the greyhound's leaps were equal to n of the hare's. Then his $\frac{2}{3}x$ leaps would have covered as much ground as $\frac{n}{m} \cdot \frac{2}{3}x$ of the hare's and we should have had

$$\frac{2n}{3m}x = 80 + x,$$

$$2nx = 240m + 3mx,$$

$$(2n - 3m)x = 240m,$$

and

$$x = \frac{240m}{2n - 3m}.$$

Now suppose, that m were 3, and n were 4, x would become $\frac{3 \times 240}{8-9}$, or -720. Let us inquire into the meaning of this negative sign. As our question now stands, the hare is 80 leaps before the dog, she takes three leaps for his two, and three of his are equal to four of hers. In this case it is clear, that the hare runs faster than the dog, and that she is continually getting away from him. In turning our question into an equation, the way in which we expressed that in x leaps the hare would be caught, was by expressing that at the end of x leaps the hare and dog would be together, which when the dog runs faster than the hare means the same thing. But just as when the dog runs faster than the hare, we look forward for the time when they will be together; so when the hare runs faster than the dog, we must look back for the time when they were together. Accordingly we shall find, that if the chase have lasted for 720 leaps up to the present time, it began by the dog and hare being together, and in 720 leaps the hare has gained 80 on the dog. This is what is meant by the negative

sign. It is impossible that our algebraical symbols should contemplate any beginning or ending to the chase. What is called the *law of continuity* requires that we should consider it as prolonged indefinitely both ways.

Once more, suppose that in our expression m were 2, and n, 3. In that case, we have the dog making two leaps for the hare's three, while two of his leaps are equal to three of hers. They run with the same speed, then; and if the course be lengthened ever so far either way, they have been, and always will be, at the same distance from each other; so that they never have been, and never will be, together. Let us see how this result is shown by our expression for x. It becomes

$$x = \frac{240}{2\times3-3\times2}, \text{ or } \frac{240}{6-6}, \text{ or } \frac{240}{0}.$$

When the denominator of a fraction becomes less, the fraction itself becomes greater. When the denominator becomes very small, the fraction becomes very great; and no quantity can be named so great but that we can make the fraction greater than it, by taking a quantity small enough for its denominator. It follows, that a fraction whose denominator is nothing, is greater than any quantity that can be named. The value of x just found, is therefore greater than any quantity that can be named; and this shows that the number of leaps taken before the dog and hare come together is greater than any number that can be named, that is, that they never come together.

117. A quantity greater than any quantity that can be named, as explained in the last article, is said to be *infinitely* great. A number infinitely great is called *infinity*. The algebraical symbol for infinity is ∞.

118. We see that the main difficulty of answering such questions as these, lies in finding equations to express them, as soon as that is done the solution is easy. The art of turning such questions as occur into algebraical language, is one for which no general rule can be given; it consists in separating from the question all the circumstances that are not essential to its solution, and can be acquired only by practice and careful thought. Questions that can be solved by a simple equation, and one unknown quantity, can always be answered by one of the arithmetical rules of *single* or *double position*. Of these an account will be given, after we have treated of proportions.*

119. Suppose that there is such an equation as

$$5x + 7y = 43,$$

containing two unknown quantities, x and y. For every different value given to y, x has a different value; so that we can find as many pairs of values of x and y as we please, which, when substituted for them in the left hand member, make it equal to 43, and which are therefore said to *satisfy* the equation. Suppose, again, that there is another equation of the same sort,

$$12x - 8y = 4;$$

we can, in the same way, find as many pairs of values of x and y to satisfy it as we please. Now, of all the pairs of values that satisfy the one of these equations, there is one, and but one, that will satisfy the other. It may be found in this way. Multiply both members of the first equation by 8, the coefficient of y in the second, and both numbers of the second by 7, the coefficient of y in the first. They then become

$$40x + 56y = 344,$$
$$84x - 56y = 28.$$

If equal quantities be added to equal, the sums must be equal; the sum of the left hand members of these equations is therefore equal to that of the right; and as we are now supposing x and y to have the same value in the one equation as they have in the other, these sums are

$$124x = 372; \text{ whence } x = 3.$$

Substituting this value for x in the first equation, it becomes

$$15 + 7y = 43,$$

which gives

$$y = 4.$$

It is impossible that any pair of values of x and y, other than 3 and 4, can satisfy both equations: for no values can satisfy both equations that do not also satisfy the equations

$$124x = 372, \text{ and } 15 + 7y = 43,$$

and we know that there is but one value that can satisfy a simple equation, containing only one unknown quantity. [111].

120. This furnishes us with a general rule for finding values for two unknown

* See art. 138.

quantities that will satisfy two equations containing them. *Clear the equations of fractions, if there be any, and in each of the equations collect the coefficients of each of the unknown quantities into one; fix on one of the unknown quantities, and multiply all the terms of each equation by the coefficient of that quantity in the other; that unknown quantity will now have the same coefficient in both equations, and by the addition or subtraction of their members, according as this coefficient has different or the same signs in the two equations, they will be reduced to one equation containing one unknown quantity; when the value of this unknown quantity is found, substitute it in one of the original equations, which will then contain the other unknown quantity only.*

121. I have a certain number of counters in each hand; if I put ten out of my left into my right, there will be twice as many in my right as remain in my left; if I put ten out of my right into my left, there will be three times as many in my left as remain in my right; how many are there in each hand? Call the number in the right x, and that in the left y. When I put ten out of my left into my right, these numbers become $x + 10$, and $y - 10$; and when I put ten out of my right into my left, they become $x - 10$, and $y + 10$. Now by the question

$$x + 10 = 2 (y - 10),$$

and

$$y + 10 = 3 (x - 10);$$

or, multiplying and collecting the terms,

$$2 y - x = 30,$$

$$3 x - y = 40.$$

Multiply the first of these equations by 3, it becomes

$$6 y - 3 x = 90.$$

To this add the second, and we find

$$5 y = 130,$$

whence $y = 26$. Substitute this value for y in the second equation, it becomes

$$3 x - 26 = 40,$$

whence $x = 22$. So that 22 in the right hand, and 26 in the left, are the numbers which satisfy the conditions given.

122. Just in the same way, when there are three unknown quantities, and three equations containing them, we can find one set of values that will satisfy every one of the three equations. This is done on the same principles as when there are two unknown quantities.

A, B, and C, sit down to play, every one with a certain number of shillings. A loses to B and C as many shillings as each of them has. Next, B loses to A and C as many as each of them now has. Lastly, C loses to A and B as many as each of them now has. After all, every one of them has 16 shillings. How much did every one gain or lose?

Call A's first sum x, B's y, and C's z.

First; A loses y to B, and z to C. He has remaining $x - y - z$. B has $2 y$, and C has $2 z$.

Secondly; B loses to A $x - y - z$, and to C $2 z$. He loses altogether $x - y - z + 2 z$, or $x - y + z$.

A now has $2 x - 2 y - 2 z$.

B has $2 y - (x - y + z)$,

or $3 y - x - z$

C has $4 z$.

Lastly; C loses to A $2 x - 2 y - 2 z$, and to B $3 y - x - z$. He loses in all $2 x - 2 y - 2 z + (3 y - x - z)$, or $x + y - 3 z$.

A now has $4 x - 4 y - 4 z$.

B has $6 y - 2 x - 2 z$.

C has $4 z - (x + y - 3 z)$,

or $7 z - x - y$.

Now, at last they all have 16. So that A's sum, or $4 x - 4 y - 4 z = 16$, whence

$$x - y - z = 4 \dots (a)$$

B's sum, or $6 y - 2 x - 2 z = 16$, whence

$$3 y - x - z = 8 \dots (b)$$

C's sum, or

$$7 z - x - y = 16 \dots (c)$$

Add (a) and (b), then

$$2 y - 2 z = 12 \dots (d)$$

Add (a) and (c), then

$$6 z - 2 y = 20 \dots (e)$$

Add (d) and (e), then

$$4 z = 32,$$

whence $z = 8$. Substitute 8 for z in (d), it becomes

$$2 y - 16 = 12,$$

whence $y = 14$. Substitute 14 and 8 for y and z in (a), it becomes

$$x - 14 - 8 = 4,$$

whence $x = 26$. So that A's original sum was 26, and he has lost 10. B's original sum was 14, and he has won 2. C's original sum was 8, and he has won 8.

123. We have seen [119], that we may find as many pairs of values of two unknown quantities as we please, that will satisfy one equation that contains them both. In the same way it may be shown, that we can find as many sets of values as we please of three unknown quantities that will satisfy one equation, or each of two equations containing them all; and so for a greater number of unknown quantities. It follows, that when we make use of two or more unknown quantities, to solve any question that admits of one value only for each unknown quantity, we must always obtain as many equations as there are unknown quantities.

Again, in solving such a question, if we obtained one equation between two unknown quantities, and formed another by multiplying all its terms by some quantity; or if we obtained two equations between three unknown quantities, and formed another by adding these two together, the new equation so formed would be of no service. It could give us no new information with respect to which of all the pairs of values of the two unknown quantities that satisfy the single equation, or of all the sets of values of the three unknown quantities that satisfy the two equations, is the pair of values, or the set of values required by our question. The new equation is not an *independent* one. The reader will find on trial, the impossibility of reducing two equations between two unknown quantities, to one equation containing one unknown quantity, when these equations are not independent. We shall return to this when we treat of *indeterminate problems.*

When there are more equations than there are unknown quantities, it may be impossible to find one set of values that will satisfy them all. Suppose, for instance, three equations between two unknown quantities. We can find, as we have seen [art. 120], a pair of values that will satisfy the first and second, and also a pair that will satisfy the first and third; but it is a mere chance if the two pairs of values so found be the same

Of Proportions.

124. Let there be two sets of numbers, such as

$$9,\ 21,\ 33,\ 40,\quad 60,\ 94\tfrac{1}{2},\ 297\ldots.$$

$$6,\ 14,\ 22,\ 26\tfrac{2}{3},\ 40,\ 63,\quad 198\ldots.$$

The numbers in the upper line are any whatever; those in the lower are so taken, that when a number in the upper line is divided by the one under it in the lower, the quotient shall always be the same, as in the present instance $\frac{3}{2}$. When this is the case, the numbers in the lower line are said to be *in direct proportion*, or *directly proportional*, to the corresponding ones in the upper. Since the quotient, when a number in the lower line is divided by the one above it, is the reciprocal [96] of the quotient when the number in the upper line is divided by the one below it, that quotient also is always the same; in the present instance it is $\frac{2}{3}$. Therefore the numbers in the upper line are also in direct proportion to the corresponding ones in the lower.

125. Daily experience furnishes us with sets of numbers that are in direct proportion to each other. Thus, if the lower line contains different numbers of yards of cloth of the same sort, and the upper their respective prices, the numbers will be in direct proportion; for the price divided by the number of yards, gives the price of one yard, which is the same, whatever be the number of yards. So if the one line were numbers of miles travelled at the same rate, and the other the respective numbers of hours spent in travelling, the numbers in the respective lines would be in direct proportion; for the number of miles divided by the corresponding number of hours, gives the number of miles travelled in one hour, or the rate of travelling, which is uniform.

To find out whether any two corresponding sets of numbers are in direct proportion, we must consider whether we are sure that the quotient of two corresponding numbers is the same for all. If we do not see a reason, as in the cases just referred to, why it cannot be otherwise, we must not conclude that they are in direct proportion. If one set of numbers, for instance, expressed the different sizes of barrels, every one with a hole in it of the same bore, we have no right to suppose that another set of numbers expressing the minutes that every barrel takes to run out, will be in direct proportion to the first; for we see no reason why all the quotients must be the same. In cases like this, where we do not see our way

clearly, other parts of the mathematics can be employed to find out whether the numbers are or are not in direct proportion; and if not, to find in what way they do depend on each other.

126. If we examine attentively what we mean when we speak of the proportion which one thing bears to another of the same kind, we shall find that we mean the magnitude of the quotient, when the number expressing the magnitude of the first thing is divided by that expressing the magnitude of the second. We call the proportion great when this quotient is great, and small when it is small. Thus, 12 bears to 4 a greater proportion than 18 does to 9; because $\frac{12}{4}$ is greater than $\frac{18}{9}$. So 2 bears to 11 a less proportion than 7 does to 23; because $\frac{2}{11}$ is less than $\frac{7}{23}$: and 15 bears to 5 the same proportion as 24 does to 8; because $\frac{15}{5}$ is equal to $\frac{24}{8}$.

The proportion that one quantity bears to another, is often called its *ratio* to that other. From our definition of direct proportion, it will be seen that when there are two sets of quantities, of which every pair of corresponding quantities have the same proportion or ratio to each other, the quantities in the two sets are in direct proportion to each other.

127. Suppose, now, that a and b are corresponding quantities in two sets that are in direct proportion, and that a' and b' are other two; then by art. [124]

$$\frac{a}{b} = \frac{a'}{b'}.$$

This relation between these quantities is often written in this way,

$$a : b :: a' : b';$$

and this is read, a is to b, as a' is to b'; that is, a has to b the same ratio or proportion as a' has to b'. From their situations in this way of stating a proportion, a and b' are called *extremes*; b and a' *means*, that is, middle ones.

128. If we multiply each member of the equation in the last article by $b\,b'$, it becomes

$$a\,b' = a'\,b.$$

The product of the extremes then is always equal to the product of the means. Thus, since

$$4 : 3 :: 12 : 9,$$

we have

$$4 \times 9 = 3 \times 12.$$

Again, if we multiply each member of the same equation by $\frac{b}{a'}$, it becomes

$$\frac{a}{a'} = \frac{b}{b'};$$

whence

$$a : a' :: b : b';$$

showing that the first has the same ratio to the third, as the second has to the fourth. Thus, since

$$5 : 15 :: 3 : 9,$$

we have

$$5 : 3 :: 15 : 9.$$

Again, if unity be added to each member of the same equation, it becomes

$$\frac{a}{b} + 1 = \frac{a'}{b'} + 1,$$

or

$$\frac{a+b}{b} = \frac{a'+b'}{b'}.$$

That is,

$$a + b : b :: a' + b' : b'.$$

And similarly by subtracting unity,

$$a - b : b :: a' - b' : b'.$$

So that the sum or difference of the first and second has to the second the same ratio, as the sum or difference of the third and fourth has to the fourth.

Thus, since

$$7 : 3 :: 35 : 15,$$

and since $7 + 3 = 10$ and $35 + 15 = 50$, we have

$$10 : 3 :: 50 : 15.$$

Also, since $7 - 3 = 4$ and $35 - 15 = 20$, we have

$$4 : 3 :: 20 : 15.$$

Quantities in direct proportion have many other properties, all of which can be easily deduced from the equation

$$\frac{a}{b} = \frac{a'}{b'}.$$

129. When we have

$$\frac{a}{b} = \frac{b}{c},$$

or

$$a : b :: b : c,$$

where the two means are the same, b is said to be a *mean proportional* between a and c, and c is said to be a *third proportional* to a and b. a, is also said to have to c the *duplicate ratio* of a to b. The last term is not a very well chosen one; the ratio of a to c is considered as if it were made up of the ratios of a to b and of b to c; now the ratio of b to c is the same as that of a to b, therefore the ratio of a to c is the ratio of a to b two-fold, or the duplicate ratio of a to b.

In this case a, b, and c are sometimes

said to be in *continued proportion*. When three quantities are in continued proportion, the product of the first and third is equal to the square of the second. This follows from the first property proved in art. [128], which becomes in this case

$$a\,c = b^2.$$

Similarly, the quantities a, b, c, d, e, &c. are said to be in *continued proportion*, when

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}, \text{\&c.};$$

and just as before, a is said to have to d the *triplicate ratio* that a has to b; a is said to have to e the *quadruplicate* ratio (or fourfold ratio) that it has to b, and so on.

Since

$$\frac{b}{c} = \frac{a}{b},$$

multiplying both members of this equation by $\frac{a}{b}$, we have

$$\frac{a}{c} = \frac{a^2}{b^2}.$$

Again, multiplying the first member of he last equation by $\frac{c}{d}$, and the second by $\frac{a}{b}$, which are equal quantities, we have

$$\frac{a}{d} = \frac{a^3}{b^3};$$

and in like manner

$$\frac{a}{e} = \frac{a^4}{b^4};$$

and so on.

130. Let us next have two lines of numbers, such as

$$1,\ 2,\ 3,\ 8,\ 16\frac{4}{11},\ 40,\ \ldots.$$

$$180,\ 90,\ 60,\ 22\frac{1}{2},\ 11,\quad 4\frac{1}{2},\ldots.$$

where the product of any two corresponding numbers is always the same, in this instance 180. When this is the case, the numbers in the one line are said to be *inversely* or *reciprocally* proportional to those in the other. The numbers in the one line are in fact directly proportional to the reciprocals [96] of those in the other. For let a and b be two corresponding numbers. The reciprocal of a is $\frac{1}{a}$, and this divided by b is $\frac{1}{a\,b}$. Now $a\,b$ is, by supposition, the same as the product of any other pair of numbers. Therefore on dividing the reciprocal of any number by the corresponding number, the quotient is always the same; that is, by [124], every number is directly proportional to the reciprocal of the corresponding one.

When two sets of numbers are in direct proportion, those in the one set increase when those in the other do, and at the same rate; if a number in one set be doubled, the one in the other set is also doubled. When they are in inverse proportion, those in the one set diminish as those in the other increase, and that too at the same rate; if a number in the one set be doubled, the one in the other is halved.

131. As in the case of direct proportion, we must never conclude that sets of corresponding numbers are in inverse proportion, unless we can prove that the product of every pair of them must be the same. If one set contains the different numbers of labourers that may be set about the same piece of work, and another set the numbers of days that the respective bands of labourers would take to finish it; the numbers in the one set are inversely proportional to those in the other: for the number of days multiplied by the corresponding number of labourers, gives for product the number of days' labour of one man required to do the work, which will be the same whatever be the number of labourers. So if one set of numbers contains the hours taken by different persons to travel the same distance, and another the number of miles that every person travels in an hour, the corresponding numbers will be in inverse proportion; for the number of hours multiplied by the space travelled in an hour, gives the whole distance travelled, the same for all.

132. If a and a' be two numbers, and b and b' two corresponding ones in inverse proportion, we have

$$a\,b = a'\,b'.$$

From this we find

$$\frac{a}{b'} = \frac{a'}{b},$$

or

$$a : b' :: a' : b.$$

They are in direct proportion, then, if b and b' change places. On the other hand, if there be two numbers, and other two in direct proportion to them, if the second two change places, they will now be in inverse proportion to the

first two; and this is the reason why the sort of proportion of which we are now speaking is called inverse.

133. When two numbers are given, and a third, it is the business of the *rule of three* in arithmetic to find a fourth, such that the second and fourth shall be either in direct proportion to the first and third, or in inverse, as the question may require. The first and third must both represent things of the same sort, as, for instance, sums of money or labourers; the second and fourth must also be of the same sort, as yards of cloth or hours. As in the former cases, let a and a' be the first and third, b and b' the second and fourth.

When the proportion is direct, we have [art. 127]

$$a : b :: a' : b';$$

and the equation in the same article gives

$$b' = \frac{a' b}{a},$$

which shows that *The fourth is the product of the second and third divided by the first.*

When the proportion is inverse, we have [art. 132]

$$a : b' :: a' : b,$$

and the equation in the same article gives

$$b' = \frac{a b}{a'},$$

which shows that *The fourth is the product of the first and second divided by the third.*

Whether the question furnish us with numbers that are in direct proportion or inverse, can, as we have seen, always be found out by a little consideration. For example, if a garrison of 800 men victualled for 90 days be reinforced by 300 men, for how many days is it now victualled? Here 800 men and 1100 men are the first and third quantities; 90 days and the number of days required, the second and fourth. The numbers of days are inversely proportional to the numbers of men; for the number of days must be such that the product of it, by the number of men, shall be the number of daily rations of food in the garrison for one man; and the number of these rations is the same after the reinforcement as it was before. Therefore by the second rule, we have

$$\text{Days required} = \frac{800 \times 90}{1100} = 65\frac{5}{11}.$$

134. Questions in compound proportion are those in which five quantities are given, and a sixth is required, or in which seven quantities are given, and an eighth required; and the like. Such questions can always be answered by reducing them to one of the two kinds of simple proportion. For instance; if 20 men weave 84 yards in 6 days, how many days will 12 men take to weave 100 yards? Call the number of days sought x. Since the number of yards woven in a given time is directly proportional to the number of weavers, and since 20 men weave 84 yards in six days, 12×20 or 240 men will weave 12×84 or 1008 yards in six days. Also, since 12 men weave 100 yards in x days, 20×12 or 240 men will weave 20×100 or 2000 yards in x days. So that our question, which stood at first thus,

Men.	Days.	Yards.
20	6	84
12	x	100,

will stand thus

240	6	1008
240	x	2000;

where the number of weavers is the same in both. This number, then, is now of no consequence to the question, and it may be put: If a certain number of men weave 1008 yards in 6 days, how many days will they take to weave 2000 yards? By the rule of direct proportion [133] the answer is

$$\frac{6 \times 2000}{1008} \text{ or } 11\frac{912}{1008} \text{ days. Observe that}$$

$$x = \frac{20 \times 100 \times 6}{12 \times 84}.$$

135. In every question of compound proportion there is one quantity given of the same nature as the quantity sought, x. Call this given quantity, for shortness, a. Of the other quantities given one half belongs to a and the other half to x. The rule of compound proportion is this: *Write* a, *and the quantities belonging to it, in one line, and* x, *and the quantities belonging to it, below, in another, observing to have quantities of the same nature one under the other; when two corresponding quantities have to each other the direct proportion of* a *to* x *change their places, writing the lower one in the upper line, and the upper one in the lower; divide the product of* a *the quantities in the upper line as it now stands, including* a,

by the product of all the given quantities in the lower line, the quotient is x.

Thus, in the question in art. [134], because the number of yards woven is in direct proportion to the number of days taken to weave them, we make 84 and 100 change places; and because the time required to do a piece of work is in inverse proportion to the number of men employed about it, we let 20 and 12 stand as they are. So that we find x, as before, to be

$$\frac{20 \times 100 \times 6}{12 \times 84}.$$

Take another example: If 10 men dig 8 acres in 6 days, working 8 hours a day, how many men will be able to dig 7 acres in 3 days, working 10 hours a day? Writing the quantities as the rule directs, they stand

	Men.	Acres.	Days.	Hours.
(a).	10	8	6	8
	x	7	3	10

The days and hours are in inverse proportion to the number of men, so that the numbers expressing these quantities must stand as they are. The acres are in direct proportion to the number of men, and therefore the numbers expressing acres must change places. Then, by the rule,

$$x = \frac{10 \times 7 \times 6 \times 8}{8 \times 3 \times 10}.$$

Before we multiply we can strike the common factors out of the dividend and divisor. We then find

$$x = \frac{7 \times 6}{3} = 14 \text{ men.}$$

The truth of the rule in any particular instance can easily be ascertained in the same way as in art. [134].

136. When the numbers in one set are directly proportional to those in another, if y stand for any number in the one set and x for the corresponding one in the other, and if m be the quotient always proceeding from the division of a number in the first set by the corresponding one in the other, then

$$y = m x.$$

For in that case we shall always have

$$\frac{y}{x} = \frac{m x}{x} = m.$$

This relation between y and x is sometimes expressed by saying that *y varies directly as x*. Thus, because the work done in a given time is proportional to the number of workmen, we say that the work varies directly as the number of workmen.

So, when the sets contain numbers in inverse proportion, if m be the product of two corresponding numbers we shall have

$$y = \frac{m}{x}.$$

For in that case

$$y x = \frac{m}{x} x = m.$$

Here we say, that *y varies inversely* as x; for instance, the time of doing a piece of work varies inversely as the number of men employed.

137. Besides being directly or inversely proportional, there are a great many other ways in which the numbers in one set may depend on those in another. For instance, take the sets

$$5,\ 20,\ 125,\ 245,\ 845, \ldots (y)$$
$$1,\ 2,\ 5,\ 7,\ 13, \ldots (x)$$

where the numbers in the upper set are directly proportional to the squares of those in the lower, and consequently increase when those in the lower set increase, but at a much faster rate. In this case, any number in the upper line, divided by the square of the one below it, will be found to be 5, so that $y = 5x^2$. In general, when $y = m x^2$, where m always remains the same whatever values are given to x and y, y is said to *vary directly as the square* of x. Similarly, if $y = \frac{m}{x^2}$, y is said to *vary inversely as the square* of x. Here y diminishes while x increases, and at a much faster rate.

The symbols $\propto$ and $\mathbin{\text{::}}$, but more usually the former, are sometimes used to denote variation. Thus, that y varies as the cube of x is sometimes expressed $y \propto x^3$. But it will always be found much more convenient to use the sign of equality, as we have done above. If this be attended to, such questions as the following can give no trouble.

If y varies inversely as x, and u varies as the square of y, in what way does u vary with respect to x? We have $y = \frac{m}{x}$. Also, $u = m' y^2$. In this expression for u substitute the value of y, and it becomes $u = m' \left(\frac{m}{x}\right)^2$ or $u = \frac{m'm^2}{x^2}$. Now $m' m^2$ is of the nature

of m or m', in respect that it does not vary, and therefore u varies inversely as the square of x.

Again, if y varies as the square of x, and u varies as x inversely, in what way does the product of y and u vary with respect to x? We have $y = m\,x^2$ and $u = \frac{m'}{x}$; therefore $y\,u = m\,x^2 . \frac{m'}{x}$ or $m\,m'x$. So that $y\,u$ varies as x directly.

138. When a question produces a simple equation of the form

$$a\,x = b,$$

that is, one in which x does not occur in the one member and is a factor in every term of the other, such a question can be answered by the arithmetical rule of *single position*. Substitute some number, such as s, for x, and suppose that we find that

$$a\,s = b'.$$

Now two fractions are equal when the numerator and denominator of the one are respectively equal to those of the other; therefore

$$\frac{a x}{a s} = \frac{b}{b'}$$

whence by [79]

$$\frac{x}{s} = \frac{b}{b'},$$

or by [127]

$$x : s :: b : b'.$$

The rule of single position is contained in this proportion, and is as follows: *Suppose some value for the unknown quantity, and find what the result would be on that supposition. The true value of the unknown quantity is to its supposed value as the true result is to the result found.*

For example: To find the number, of which the half, the third part, and the fourth part added together make 65. Suppose the number to be 48. The half, the third part, and the fourth part of 48, or 24, 16, and 12 added together make 52. Then we have by the rule

$$x : 48 :: 65 : 52.$$

And from this we find, by [133], that the number sought is 60.

139. When a question furnishes such an equation as

$$a\,x + b = c\,x + d,$$

it can be solved by the arithmetical rule of *double position*. As in art. [109], this equation may be put in the form

$$(a - c)\,x + b - d = 0.$$

Now suppose that s is substituted for x, and that instead of zero we find the expression to be equal to e. Then we have

$$(a - c)\,s + b - d = e,$$

and, subtracting the first of these equations from the second, as in [119],

$$(a - c)\,s - (a - c)\,x = e,$$

or

$$(a - c)\,(s - x) = e.$$

Again, make another substitution s' for x, and let e' be the result. Then, as before,

$$(a - c)\,(s' - x) = e'.$$

Dividing the members of the first of these two equations by those of the last, as in [138], we have

$$\frac{(a - c)\,(s - x)}{(a - c)\,(s' - x)} = \frac{e}{e'},$$

and, striking out the common factor $a - c$,

$$\frac{s - x}{s' - x} = \frac{e}{e'}$$

solving this equation, as is directed in [111], it becomes successively

$$e's - e'x = e s' - e x$$

and

$$(e - e')\,x = e\,s' - e'\,s\,;$$

whence

$$x = \frac{e\,s' - e'\,s}{e - e'}.$$

This expression contains the rule: *Make two suppositions for the unknown quantity and note the two false results; multiply the second supposition by the first result and the first supposition by the second result; subtract the second of these products from the first, and also the second result from the first; dividing the first of these differences by the second the quotient is the unknown quantity sought.*

For example: A doubles the money that B has got, and then B gives A 3*l.*; when this has been done three times, B finds that he has no money left; How much had he at first? First, suppose that B had 4*l.* at first. In this case, after the first transaction, he would have 5*l.*, after the second 7*l.*, and after

the third 11*l*. Next, suppose that B had 2*l*. at first. Then, after the first transaction, he would have 1*l*. After the second he would owe A 1*l*., that is, he would have $-1l$. Since A doubles B's money every time, the third transaction would be, that A should double B's debt [see art. 15.]; and then B ought to give A 3*l*. So that at the end of the third transaction, on this supposition, B owes A 5*l*., or B has $-5l$. We thus have

Suppositions	4,	2,
Results	11,	-5.

The product of the second supposition by the first result is 22; the product of the first supposition by the second result is -20, the difference of these products is $22-(-20)$ or 42. Again, the difference of the two results is $11-(-5)$ or 16. Then, by the rule, the number sought is the quotient $\frac{42}{16}$ or $2\frac{5}{8}$. This is equivalent to 2*l*. 12*s*. 6*d*., which will be found to satisfy the question.

The rule of double position is also applicable to solve equations of the form

$$ax + b = c.$$

But these rules of position are of little use, as all the questions to which they can be applied are much better answered by means of simple equations.

Of Arithmetical Progression.

140. Let there be a set, or *series*, of numbers, such as

$$4,\ 7,\ 10,\ 13,\ 16,\ 19,\ 22,\ \&c.,$$

where every one is formed by adding a certain number, in this case 3, to the one before it; such a set of numbers are said to be in *arithmetical progression*. The numbers themselves taken collectively are also said to constitute an *arithmetical series*, or *progression*. The constant number by which every one must be increased, so as to form the next, is called the common difference. The numbers

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \&c.,$$

for example, make an arithmetical progression in which the common difference is unity.

In like manner, when there is a series of numbers, such as

$$6,\ 4,\ 2,\ 0,\ -2,\ -4,\ -6,\ -8,\ \&c.,$$

where every one is formed by subtracting a certain number, in this case 2, from the preceding, such numbers also are said to be in arithmetical progression. In such a case the common difference is said to be negative, for every number is formed by adding a negative quantity to the one before it.

141. If a be the first term and b the common difference of any arithmetical progression where the common difference is positive,

$$a,\ a+b,\ a+2b,\ a+3b,\ \&c.\ a+nb$$

are the several terms of the progression. In like manner, when the common difference is negative,

$$a,\ a-b,\ a-2b,\ a-3b,\ \&c.,\ a-nb$$

are the terms. This last series is simply the first with $-b$ substituted in it for $+b$; so that by giving the proper values to a and b, the first series may be made to stand for any arithmetical progression, whether the numbers increase or decrease.

142. When we know the first term of an arithmetical progression and the common difference, we can always find any other term of it. The second term is the first with the common difference added to it; the third is the first with twice the common difference added to it, and so on; so that the tenth is the first with nine times the common difference added to it; and similarly for any other term. This may be stated generally by saying, that the n^{th} term is equal to the first, with $\overline{n-1}$ times the common difference added to it. Thus, if a be the first term, and b the common difference, the n^{th} term of the progression is

$$a + (n-1)\,b.$$

For example, the 100^{th} term of the progression 5, 11, 17, &c., where the first term is 5, and common difference 6, is

$$5 + (100-1)\,6,$$

or 599. As in the last article, this expression may be extended to decreasing progressions by making b negative. So that the 20^{th} term of the progression 105, 64, 23, -18, &c., where -41 is the common difference, is

$$105 + (20-1)\,(-41),$$

or -674.

143. When there are two quantities that are the first and last terms of an arithmetical progression, the other terms are said to be so many *arithmetical means* between them. Thus, 7 and 10 are two arithmetical means between 4 and 13; the progression, when completed, being 4, 7, 10, 13.

Let it be required to find five arithmetical means between 7 and 25. The five numbers required, together with 7 and 25, are to make an arithmetical progression of seven terms; therefore, by the last article, six times the common difference added to 7 must make 25. $25 - 7$, then, or 18, is six times the common difference, and the common difference therefore is 3. The five means sought are 10, 13, 16, 19, and 22.

In the same way we shall find, that if a be one quantity and l another, and it is required to find m arithmetical means between them, the common difference of the arithmetical series which these means will form will be

$$\frac{l-a}{m+1};$$

for, just as we divided $25 - 7$ by 6, which is $5 + 1$, in the former case, we shall find that in all cases we are to divide $l - a$ by $m + 1$.

This may also be deduced from the expression for the n^{th} term of an arithmetical series [art. 142] in this way: a and l and the m means are to form a series of $m + 2$ terms; therefore l is the $\overline{m+2}^{th}$ term of an arithmetical progression whose first term is a; if b be the common difference, substituting $m + 2$ for n for the expression in [142], it becomes

$$l = a + (m + 1)\,b,$$

hence,

$$(m + 1)\,b = l - a,$$

and, solving this equation, we find as before,

$$b = \frac{l-a}{m+1}.$$

When there is one arithmetical mean to be found, m is 1, and therefore we divide $l - a$ by $1 + 1$ or 2. One arithmetical mean between 10 and 40 is found by dividing $40 - 10$ by 2, and adding the quotient 15 to 10; the sum is 25; and 10, 25, and 40 are in arithmetical progression.

144. Let it be proposed to find the sum of any set of quantities in arithmetical progression, such as 4, 7, 10, 13, 16, 19. Call this sum s; then

$$s = 4 + 7 + 10 + 13 + 16 + 19,$$

or

$$s = 19 + 16 + 13 + 10 + 7 + 4;$$

taking the terms of the progression in an inverted order. Add these two equations together, and they become

$$2\ \ = 23 + 23 + 23 + 23 + 23 + 23.$$

Now all the terms in the second member of this equation are the same, and there are six of them. Therefore

$$2\,s = 6 \times 23 = 138,$$

and

$$s = \frac{138}{2} \text{ or } 69.$$

By treating in the same way the general expression in [141], we shall find a general expression for its sum, which may be applied to find the sum of any arithmetical progression. Taking n terms of it, and calling their sum s, we have

$$s = a \qquad + a + b \qquad + a + 2\,b \qquad + \&c. + a + (n-2)b + a + (n-1)\,b,$$

and

$$s = a + (n-1)\,b + a + (n-2)\,b + a + (n-3)\,b + \&c. + a + b \qquad + a,$$

inverting the order as before. Adding, we have

$$2\,s = 2\,a + (n-1)\,b + 2\,a + (n-1)\,b + \&c. + 2\,a + (n-1)\,b + 2\,a + (n-1)\,b.$$

In the second member of this equation we have $2\,a + (n - 1)\,b$ repeated n times, since there are n terms in the series. The sum of the terms of this member therefore is the product of $2\,a + (n - 1)\ b$ by n, whence

$$2\,s = n\,(2\,a + \overline{n-1}\,b),$$

and

$$s = \frac{n\,(2\,a + \overline{n-1}\,b)}{2}.$$

The expression in [141], the sum of the terms of which we have just found, became the general expression for a decreasing arithmetical series, by making b negative. Therefore the last expression for s becomes the sum of a decreasing progression, by making b negative. In that case

$$s = \frac{n\,(2\,a - \overline{n-1}\,b)}{2}$$

and this is the sum we should find for n terms of the progression a, $a - b$, $a - 2\,b$, &c. $a - (n - 1)\,b$, if we treated it in the same way as the increasing progression.

The expressions $2\,a + \overline{n-1}\,b$, $2\,a - \overline{n-1}\,b$ are respectively the first term of an increasing or decreasing progression added to the n^{th}; so that if l be this n^{th} term, we have

$$s = \frac{n\,(a + l)}{2}$$

This furnishes us with the general rule for finding the sum of any increasing or decreasing arithmetical progression. *Add the first term to the last, multiply this sum by the number of terms, and take half the product.*

The sum of the progression $1 + 2 + 3 + 4$ &c. up to 7541, is by this rule $\frac{7541 \times (7541 + 1)}{2}$, or 28437111. The sum of 40 terms of the progression $23 + 20 + 17 +$ &c. where the 40^{th} term is -94 is $\frac{40\,(23 - 94)}{2}$, or -1420.

145. The sum of n terms of the progression $1 + 3 + 5 +$ &c. is by our rule $\frac{n(2 + \overline{n - 1}.2)}{2}$, since the n^{th} term of the progression is $1 + \overline{n - 1}.2$. This expression for the sum reduces itself to $\frac{n \,.\, 2n}{2}$ or n^2. The sum of any number n of the consecutive odd numbers, beginning with unity, is therefore the square of n. So that $1 + 3 = 2^2$ or 4, $1 + 3 + 5 = 3^2$ or 9, &c., and $1 + 3 + 5 + 7 +$ &c. $+ 27 = 14^2$ or 196.

146. It is easy to see that when the first term and common difference are whole numbers, the expression for the sum $\frac{n\,.\,(2a + \overline{n-1}\,b)}{2}$ will also be a whole number as it ought; for if n be an odd number, $n - 1$ is an even number, and therefore $(2a + \overline{n - 1}\,b)$ is an even number. So that whether n be an even or an odd number, one of the two numbers n, or $2a + \overline{n - 1}\,b$, is even, and therefore their product can be always measured by 2. It is plain also, that when a and b are whole numbers, and n is measured by 3, the expression for s is measured by 3; for in that case $n\,(2a + n - 1\,b)$ is measured by 3 [art. 51]; and therefore $\frac{n\,(2a + \overline{n - 1}\,b)}{2}$ is measured by 3, [art. 62], since 2 and 3 are both prime factors of $n\,(2a + \overline{n - 1}\,b)$. It follows from this, that the sum of any three terms, of any six terms, of any nine terms, &c. in arithmetical progression is measured by 3. Thus, the sum of 12 terms of the series 100, 93, 86, &c. is 738, which is measured by 3.

We have seen [art. 76] that if any number N be put in the form

$$A_0 + A_1\,10^n + A_2\,10^{2n} + \text{\&c.};$$

and if

$$A_0 + A_1 + A_2 + \text{\&c.}$$

be measured by any number that also measures $10^n - 1$, then N is measured by the same number. Now $10^n - 1$ is always measured by 3, [art. 67], and we have just seen that if the numbers A_0, A_1, A_2, &c. be in arithmetical progression, and in number three, or six, or nine, &c., their sum is measured by 3. It follows, that when the digits of a number can be divided into three, or six, or nine, &c. periods, each of an equal number of digits, if the numbers then expressed by these periods be in arithmetical progression, the number is measured by 3.*

In like manner, the sum of any odd number m, of terms of an arithmetical progression is always measured by m, the odd number; thus, the sum of seven terms is always measured by seven, of nine terms by nine, and so on. Therefore, since $10^n - 1$ is always measured by 9 [art. 67], if the periods A_1, A_2, A_3, &c. be in number 9, or any multiple of 9, and in arithmetical progression, the number is measured by 9. Similarly, since $10^3 - 1$ is measured by 37; if the digits of any number can be divided into 37, or 74, or 111, &c. periods of three digits each, and these periods are in arithmetical progression, the number is measured by 37.

Of Geometrical Progression.

147. Next, let there be such a set of numbers as

5, 15, 45, 135, 405, 1215, 3645, &c

where every one is formed by multiplying the one before it by a certain number, in the present case 3; these numbers are said to be in *geometrical progression*. The number by which every one is successively multiplied to produce the next, is called the *common ratio*. This common ratio can always be found by dividing any term in the progression by the one before it. When the common ratio is greater than unity, the terms go on increasing; when it is less than unity, they go on diminishing, as in the series

* See *Preliminary Treatise*, p. 9, in the note, and the examples given there.

$$16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \text{\&c.},$$

in which the common ratio is $\frac{1}{2}$. When the terms increase, the series is called an *ascending* one; when they decrease, it is *descending*. When the common ratio is negative, the terms are alternately positive and negative, as in the series

$$1, -\frac{11}{10}, \frac{121}{100}, -\frac{1331}{1000}, \frac{14641}{10000}, \text{\&c.}$$

where the common ratio is $-\frac{11}{10}$.

148. If r be any number positive or negative, the general expression for a geometrical series will be

$$a, ar, ar^2, ar^3, ar^4 \ldots ar^n.$$

Here a is the first term, ar the second, ar^2 the third; and the n^{th} term is ar^{n-1}. The 7^{th} term of the series 3, 6, 12, &c. is $3 \times 2^6 = 3 \times 64$, or 192.

149. The intermediate terms of a geometrical progression are said to be so many *geometrical means* between the first term and the last. Let it be required to find m geometrical means between a and l. The means to be found, together with a and l, are to constitute a geometrical series of $m + 2$ terms. If r be the common ratio, (which as yet is an unknown quantity,) since l is the $(m + 2)^{th}$ term of the progression whose first term is a, we have, putting $m + 2$ for n in the expression in the last article,

$$l = ar^{m+1}.$$

Hence

$$r^{m+1} = \frac{l}{a}.$$

So that r is a number whose $(m + 1)^{th}$ power is $\frac{l}{a}$, and by [11] this number is expressed by $\sqrt[m+1]{\frac{l}{a}}$. We cannot as yet find this number, but supposing known, the means sought are

$$ar, ar^2, ar^3 \ldots ar^m$$

Since [art. 127]

$$a : ar :: ar : ar^2;$$

and since ar is a geometrical mean between a and ar^2; it follows that a mean proportional [129] between two quantities is also a geometrical mean between them. It follows from the same article that all the terms of a geometrical series are in continued proportion.

150. Let it be proposed to find the sum of the terms

$$8, 24, 72, 216, 648,$$

which form a geometrical progression where the common ratio is 3. Call the sum s; then

$$s = 8 + 24 + 72 + 216 + 648$$

and

$$3s = \quad 24 + 72 + 216 + 648 + 1944;$$

where the second equation is got by multiplying every term in each member of the first by 3, the common ratio. Now, subtract the first equation from the second, the terms in the second members destroy each other, and there remains

$$2s = 1944 - 8 = 1936,$$

whence

$$s = 968.$$

151. In the same way the sum of n terms of the general geometrical series in art. [148] may be found. Let the sum be s, then

$$s = a + ar + ar^2 + ar^3 + \text{\&c.}, \quad + ar^{n-2} + ar^{n-1},$$

and

$$rs = ar + ar^2 + ar^3 + \text{\&c.} \quad + ar^{n-2} + ar^{n-1} + ar^n,$$

just as before. Subtract the first equation from the second; then

$$rs - s = ar^n - a;$$

whence

$$(r - 1)s = ar^n - a,$$

and

$$s = \frac{ar^n - a}{r - 1}, \text{ or } a\frac{r^n - 1}{r - 1}.$$

If a be 1, these expressions become

$$1 + r + r^2 + \text{\&c.} + r^{n-1}$$

$$= \frac{r^n - 1}{r - 1},$$

and this last is the result we came to in art. [46].

The expression $s = \frac{ar - a}{r - 1}$ gives us the following rule for the sum of any number of terms in geometrical progression. *Subtract the first term from the last multiplied by the common ratio, and divide the difference by the common ratio diminished by unity; the quotient is the sum required.* For example, the

sum of five terms of the series 6, 12, 24, &c. where the last term is 96, and common ratio 2, is $\frac{2 \times 96 - 6}{2 - 1}$, or 186.

So the sum of 8 terms of the series 6, -18, 54, &c. where the common ratio is -3, and the 8^{th} term is -13122, is

$$\frac{(-13122).(-3)-6}{-3-1}=\frac{39366-6}{-4}$$

or

$$-9840.$$

When the series is a descending one r and r^n are both less than 1. In such a case it is better to put the expression for the sum in the form

$$s = a\frac{1-r^n}{1-r},$$

as in art. [81]. A corresponding change may be made in the arithmetical rule for summing a descending series.

152. Let the progression to be summed be

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\&c.$$

By the last expression the sum of n terms of this will be

$$s = 1.\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = \frac{1-\left(\frac{1}{2}\right)^n}{\frac{1}{2}}$$

or

$$s = 2\left(1-\frac{1}{2^n}\right),$$

or

$$s = 2 - \frac{1}{2^{n-1}},$$

[arts. 90, 93]. Now as we go on giving greater and greater values to n, 2^{n-1} becomes greater and greater; and, by making n sufficiently great, may be made greater than any quantity that can be named. But as 2^{n-1} becomes greater and greater, $\frac{1}{2^{n-1}}$ becomes less and less, and may, by making n sufficiently great, be made less than any quantity that can be named. It follows, then, that as n is made greater and greater, that is, as more and more terms of the series are taken, s differs from 2 by a less and less quantity, and may, by taking a sufficient number of terms, be made to differ from 2 by a quantity less than any that can be named. We conclude, then, that the sum of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$&c. if it be continued without limit to the number of terms, is *accurately* equal to 2; for though we do not show directly that it is so, yet if any one denies it, and states any quantity, however small it be, by which the sum differs from 2, we can show that it does not differ by a quantity so great. For instance, let it be said, that the sum differs from 2 by $\frac{1}{1000}$ of unity. Since $2^{10} = 1024$, and since s always differs from 2 by $\frac{1}{2^{n-1}}$, if $n-1 = 10$, that is, if $n = 11$ s differs from 2 by $\frac{1}{1024}$ of unity only. Therefore the sum of 11 terms of the series differs from 2 by a quantity less than the quantity named.

The truth of this may be illustrated in this way. A B is a line two inches long, and cut into two equal parts

at C; C B is cut into two equal parts at D, D B again at E, and so on. Then A C is one inch, C D half an inch, D E a quarter of an inch, and so on; so that A C + C D + D E + E F + &c. is the sum of the series

$$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\&c.\right) \text{ inches.}$$

Now we plainly see, that by making subdivisions enough, the sum of these lines may be made to differ from A B, or 2 inches, by a quantity as small as we please, while it can never exceed A B.

In these cases we take the sum of a number of terms of the series greater than any number that can be named, that is [art. 117] we suppose n to be infinite. But when n is infinitely great, 2^n is also infinitely great, and $\frac{1}{2^n}$ is infinitely small, or differs from nothing by a quantity infinitely small, or, in a word, is equal to nothing.

153. Such a series as the one whose sum we have just taken is called an *infinite series*, because the number of its terms is infinite. Every geometrical

series whose terms go on decreasing can be summed to infinity in the same way. In the expression

$$s = a\frac{1-r^n}{1-r}$$

when r is less than unity, by making n sufficiently great, r^n may be made less than any quantity that can be named. Therefore $1 - r^n$ may be made to differ from unity by a quantity less than any that can be named; and, as before, we shall have

$$s = \frac{a}{1-r},$$

in which we recognise the result in art. [46].

Hence the rule for summing a descending geometrical series to infinity: *Divide the first term by unity diminished by the common ratio, the quotient is the sum required.* The sum of

$$\frac{4}{5} + \frac{4}{25} + \frac{4}{125} + \text{\&c.},$$

continued to infinity, where the common ratio is $\frac{1}{5}$, is by this rule

$$\frac{\frac{4}{5}}{1-\frac{1}{5}} = \frac{\frac{4}{5}}{\frac{4}{5}} \text{ or } 1.$$ So the sum of

$\frac{2}{3} - \frac{4}{9} + \frac{8}{27}$ to infinity, where the common ratio is $-\frac{2}{3}$, is

$$\frac{\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{\frac{2}{3}}{1+\frac{2}{3}}, \text{ or } \frac{2}{5}.$$

Of Decimal Fractions.

154. We have seen [art. 74], that we can express all whole numbers by means of the nine digits and zero, by agreeing that every digit when it is written shall have only one-tenth of the value it would have had if it had held the next place towards the left. Let this agreement be pursued beyond the units' place, and let it be understood that when a digit is written next to the right of the units' digit, its value is one-tenth of what it would have been if it had held the units' place; when it is written in the second place to the right one-hundredth, in the third place one-thousandth, and so on. Proceeding from the units' place towards the left we shall then have tens, hundreds, thousands, &c.; and towards the right, tenths, hundredths, thousandths, &c. For example, 25.763 (where the units' place is marked by a point placed just after it) will mean

$$20 + 5 + \frac{7}{10} + \frac{6}{100} + \frac{3}{1000}$$

155. This way of expressing the part of a number that is less than unity is called a *decimal fraction.* A fraction written in the common form is in distinction called a *vulgar*, that is, a common fraction. Any vulgar fraction may be turned into a decimal. Take $\frac{3}{8}$ for instance,

$$\frac{3}{8} = \frac{1}{1000} \times \frac{3000}{8},$$

or

$$\frac{1}{1000} \times 375.$$

Therefore

$$\frac{3}{8} = \frac{300 + 70 + 5}{1000}$$

or

$$\frac{3}{10} + \frac{7}{100} + \frac{5}{1000};$$

and this written as a decimal fraction is .375, where a point is placed to show where the decimal digits begin, and is called the *decimal point.* In the same way we find, that

$$5\frac{7}{16} = 5 + \frac{1}{10000} \times \frac{70000}{16},$$

or 5.4375. Similarly,

$$\frac{1}{20} = \frac{1}{100} \times \frac{100}{20},$$

or $\frac{5}{100}$, or .05. Here zero is placed first on the left, because there are no tenths of unity in $\frac{1}{20}$.

To reduce vulgar fractions to decimals, the rule is this: *Add zeros to the right of the numerator and divide by the denominator; make the number of decimal places in the quotient equal to the number of zeros added to the numerator, and to make up this number, if necessary, add zeros to the left of the quotient.* A zero placed to the right of a decimal has no effect; .2 and .20, for instance, are respectively the same

as $\frac{2}{10}$ and $\frac{20}{100}$, both of which are equivalent to $\frac{1}{5}$.

156. If by this rule we try to reduce $\frac{1}{3}$ to a decimal fraction, the division will never come to an end, and we find for answer .333 &c., without limit. We shall find by the rule in art. [150], that the sum of the infinite geometrical series

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \text{\&c.}$$

which is equivalent to the decimal .333 &c., is $\frac{1}{3}$. It is easy on other grounds to see the reason of this result. When we multiply the numerator of a fraction by a number and divide the product by the denominator, it is necessary, in order that the division may leave no remainder, that the numerator be multiplied by some number that makes it a multiple of the denominator. Now, in our process, we multiply the numerator by some power of 10, and no power of 10 is measured by 3; therefore we may divide the product by 3 for ever without coming to an end.

The only prime factors [art. 61] of any power of 10 are 2 and 5; so that when a fraction is in its lowest terms [79], if there be any prime factor besides 2 and 5 in its denominator, it cannot be reduced into a decimal that will terminate. In that case, the numerator contains none of the prime factors of the denominator, and no power of 10 contains them all, therefore the product of the numerator by no power of 10 is measured by the denominator [61].

Thus, $\frac{5}{9}$ gives the decimal .555 &c., and $\frac{15}{22}$ gives .6818181, &c. Decimals such as these are called *repeating decimals;* those in which the repeating part consists of more than one digit are sometimes also called *circulating decimals.* .333 &c. is written $.\dot{3}$, so .68181 &c. is written $.6\dot{8}\dot{1}$, and .531531 &c. is written $.\dot{5}3\dot{1}$; the points being placed so as to mark out the repeating part.

157. When a proper fraction is in its lowest terms, it can therefore always be reduced to a terminating decimal if there be no prime factors in its denominator other than 2 and 5, and in that case only. The number of digits in the decimal to which it can be so reduced will be the number indicating the power of that one of the two factors 2 and 5, which is found in the highest power in the denominator. For example, $\frac{5}{16}$ is reduced to .3125 with four digits, because 16 is the fourth power of 2. So $\frac{17}{50}$ is reduced to .38, with two digits, because 5 is found in its second power in 50, while 2 is found only in its first. The reason is plain. Let $\frac{a}{2^m . 5^n}$ be the fraction, where a does not contain 2 or 5; and let m be greater than n. If we multiply a by 10^m, that is, by $2^m . 5^m$, it will be measured by $2^m . 5^n$; while if it be multiplied by any lower power of 10, as 10^{m-1}, it will not be measured by 2^m, and therefore not by $2^m . 5^n$. Now, when we multiply by 10^m we add m zeros, and by our rule the number of decimal places is equal to the number of zeros added.

158. When the fraction is in its lowest terms, and the denominator is prime to 10, we have seen that it will not produce a terminating decimal. It will always give a repeating decimal, which will begin to repeat from its first digit. Thus $\frac{7}{13}$ produces $.\dot{5}3846\dot{1}$, and $\frac{5}{7}$ produces $.\dot{7}1428\dot{5}$. The reason of this is as follows.

Let $\frac{a}{b}$ be a proper fraction where b is prime to 10. In turning it into a decimal we divide $10\,a$ by b; the whole part of the quotient, which we may call d_1, is the first digit of the decimal; let the remainder be r_1. We divide $10\,r_1$ by b for the second digit d_2, and have a remainder r_2. Proceeding thus we find digits d_1, d_2, d_3, d_4, &c., and remainders r_1, r_2, r_3, r_4, &c., corresponding. Now suppose that after a certain number of partial divisions we fall on a remainder, r_2 for instance, the same as one we have had before. The remainders that succeed r_2, the second time, must be the same as those which succeeded it at first; for the circumstances to produce them are the same. In both cases they will be r_3, r_4, &c.

But the remainder preceding r^2 in both cases will also be the same, *viz.* r

Suppose, for the sake of argument, that in the second case it may be different, and call it s. Then we have

$$10\,r_1 = d_2 b + r_2$$

in the first instance, and in the second

$$10\,s = q\,b + r_2.$$

Where q is some whole number less than 10, and different from d_2, else r_1 and s would be the same. Subtracting the second of these equations from the first

$$10\,(r_1 - s) = (d_2 - q)\,b,$$

whence

$$\frac{10\,(r_1 - s)}{b} = d_2 - q.$$

Now all the remainders r_1, r_2, s, r_3, &c. are less than b, and therefore $r_1 - s$ is less than b; therefore b cannot measure $r_1 - s$; and therefore since 10 is prime to b, b cannot measure $10\,(r_1 - s)$, which on our supposition it does. It thus leads to an absurdity to suppose that r_1 and s can be different, and therefore they are the same.

In the same way it may be shown, that the remainder preceding r_1 the second time it occurs, is a.

It follows, that if among the remainders any two are the same, all the remainders form periods exactly similar to each other, the second period beginning with a. But among the remainders some two must be the same, for they are all less than b, and therefore there can be but $b-1$ different ones, while their number is infinite. Now when we come to a at the beginning of the second period of remainders, the digits furnished will be d_1, d_2, d_3, &c., just as when we began from a at the beginning of the first period. Therefore the circulating part of the decimal will begin by repeating the first digit of the decimal. The number of digits in the repeating part cannot be more than $b - 1$.

159. When the fraction is in its lowest terms, and its denominator is not prime to 10, while it contains other factors besides 2 and 5, the decimal will, as we have seen, repeat, but will not begin from its first digit. If the m^{th} be the highest power of 2 or 5 in the denominator it will begin to repeat after the m^{th} digit. Thus, $\frac{5}{6}$ is reduced to $.8\dot{3}$, which repeats after the first digit; because 6 contains 2 in the first power. So $\frac{23}{120}$ or $.191\dot{6}$ repeats after the third digit, because 120 contains the third power of 2, and only the first power of 5. The reader will have no difficulty in deducing this from the principles just laid down.

160. To reduce any terminating decimal to a vulgar fraction: *Write the decimal as a whole number for the numerator, and unity followed by as many zeros as there are digits in the decimal for the denominator; reduce the fraction so formed into lower terms if possible.* The decimal .725, for instance, is equivalent to $\frac{7}{10} + \frac{2}{100} + \frac{5}{1000}$, or to

$$\frac{700}{1000} + \frac{20}{1000} + \frac{5}{1000}, \text{ or to } \frac{725}{1000}, \text{ or } \frac{29}{40}.$$

161. To reduce a repeating decimal to a vulgar fraction, is to take the sum of an infinite geometrical progression, [156]. Thus $.\dot{2}3\dot{1}$ is equivalent to

$$\frac{231}{1000} + \frac{231}{1000000} + \frac{231}{1000000000} + \&c.$$

where the common ratio is $\frac{1}{1000}$. The sum is by [153]

$$\frac{\frac{231}{1000}}{1 - \frac{1}{1000}} = \frac{\frac{231}{1000}}{\frac{999}{1000}}, \text{ or } \frac{231}{999}, \text{ or } \frac{77}{333}.$$

If P be the repeating period, and contain m digits, the value of the decimal is

$$\frac{P}{10^m} + \frac{P}{10^{2m}} + \frac{P}{10^{3m}} + \&c.$$

The common ratio is $\frac{1}{10^m}$ and the sum is $\frac{P}{10^m - 1}$, [art. 153]. Observe that $10^m - 1$ is a number composed of m nines.

Similarly $.17\dot{3}\dot{7}$, which does not repeat from its first digit, is equivalent to

$$\frac{17}{100} + \frac{37}{10000} + \frac{37}{1000000} + \frac{37}{100000000} + \&c.$$

that is, to

$$\frac{17}{100} + \frac{37}{100}\left\{\frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + \&c.\right\}$$

or

$$\frac{17}{100} + \frac{37}{100} \times \frac{1}{99},$$

or

$$\frac{17}{100} + \frac{37}{9900} = \frac{86}{495}.$$

From these examples we collect the following rule: *When the decimal repeats from its first digit, the vulgar fraction equivalent to it has for its numerator the repeating period, and for its denominator as many nines as there are digits in the repeating period; when it does not repeat from its first digit, the decimal fraction is equal to the sum of two vulgar fractions, the first of which is the value of the part not repeating considered as a terminating decimal, the second has for its numerator the repeating period, and for its denominator as many nines as there are digits in the repeating period, followed by as many zeros as there are digits in the part not repeating.*

162. To add or subtract terminating decimals, or mixed numbers containing them: *Write them one under the other, observing to have the decimal points all in the same column; then proceed as in whole numbers.* When they are written in this way, every column contains all the digits of some one denomination. The column to the left of the decimal points, for instance, contains all the units, the one to the right all the tenths, and so on. This being the case, the reasons in arts. [20] and [21] apply equally here.

163. To take the product of terminating decimals or mixed numbers containing them: *Multiply them together as if they were whole numbers, and point off from the product as many decimal places as there are in the multiplier and multiplicand together. If there are not so many digits in the product add zeros to the left.* Let 5.75 and 12.731, for instance, be the two decimals, $5.75 = 5 + \frac{75}{100}$ or $\frac{575}{100}$. Similarly, $12.731 = \frac{12731}{1000}$. Therefore

$$5.75 \times 12.731 = \frac{575}{100} + \frac{12731}{1000},$$

or

$$\frac{575 \times 12731}{100000}.$$

Taking this product it will be found to be $\frac{7320325}{100000}$ or $73 + \frac{20325}{100000}$, that is, 73.20325. We point off five places, because there is 100000 in the denominator of the product; that is, because there was 1000 in the denominator of one factor and 100 in that of the other; or because the original factors had respectively two and three decimal places.

Similarly $.0016 \times .023 = \frac{16}{10000} \times \frac{23}{1000}$ $= \frac{368}{10000000}$ or .0000368.

When the multiplier or the multiplicand is a whole number, we point off as many decimal places as the other factor contains. Thus, $18 \times 2.75 = 49.50$ or 49.5.

164. To divide one decimal fraction by another: *Make the number of digits after the decimal point equal in the divisor and dividend by adding zeros to the right of one of them, if necessary; then divide as if they were whole numbers, the quotient is the quotient required; the remainder gives a vulgar fraction which may be turned into a decimal by adding zeros to it, and dividing by the divisor.* This rule may be deduced from the following example:

$$127.54 \div 6.32 = \frac{12754}{100} \div \frac{632}{100} = \frac{12754}{100} \times \frac{100}{632} = \frac{12754}{632},$$

$$12.754 \div 63.2 = \frac{12754}{1000} \div \frac{632}{10} = \frac{12754}{1000} \times \frac{10}{632} = \frac{12754}{63200}$$

$$1275.4 \div .632 = \frac{12754}{10} \div \frac{632}{1000} = \frac{12754}{10} \times \frac{1000}{632} = \frac{1275400}{632}.$$

Here we have three cases in which the divisor and dividend consist of the same digits. In the first the number of decimal places is the same in both, and the quotient we find to be the same as if they were whole numbers. In the second there are three decimal places in the dividend, and one in the divisor; the quotient, then, is found by adding two zeros to the divisor. In the third case, where the dividend has one decimal place and the divisor three, we add two zeros to the dividend.

When there are more decimal places in the dividend than in the divisor, this rule amounts to dividing the dividend by the divisor, and pointing off as many decimal places in the whole part of the quotient as the number of decimal places in the dividend exceeds that in the divisor, or if there are not so many in the quotient, adding the requisite

number of zeros to the left. This is the form in which the rule is often stated, and may, perhaps, be found the more convenient.

Following our rule, when one of the numbers is an integer, before dividing we must add to it as many zeros as there are decimal places in the other.

165. If N be any number, and if a mixed number, with its fractional part expressed as a terminating decimal, have the same digits as N, and if there be n decimal digits, the mixed number is equal to $\frac{N}{10^n}$. Thus, $17.31 = \frac{1731}{100}$ or $\frac{1731}{10^2}$. Similarly, if all the digits be decimal and there be n of them, the value is $\frac{N}{10^n}$. On the other hand, any fraction such as $\frac{N}{10^n}$ is equivalent to a number consisting of the same digits as N, n of which are decimal. Thus $\frac{7243}{100} = 72.43$.

From this we may easily deduce the rules of multiplication and division given above. The product of $\frac{N}{10^n}$ and $\frac{N'}{10^{n'}}$ is $\frac{N\,N'}{10^{n+n'}}$, or the number N N′ with n and n' of its digits made decimal. But $\frac{N}{10^n}$ and $\frac{N'}{10^{n'}}$ are respectively the numbers N and N′ with n and n' decimal digits. Therefore to take the product of two numbers with n and n' decimal places respectively, we point $n + n'$ digits from their product considered as whole numbers.

Again the quotient of $\frac{N}{10^n}$ divided by $\frac{N'}{10^{n'}}$ is $\frac{N}{10^n} \cdot \frac{10^{n'}}{N'}$, [art. 92]. When n is greater than n', that is, when the number of decimal places in N exceeds that in N′, this becomes $\frac{N}{N'.10^{n-n'}}$. Therefore to N′ we must add $n - n'$ zeros, and $n - n'$ is the excess of the number of decimal places in N over that of those in N′. Similarly, when n' is greater than n, this quotient becomes $\frac{N\,.\,10^{n'-n}}{N'}$. Showing that in that case we must add to N $n' - n$ zeros.

166. With respect to repeating decimals, if perfect accuracy be necessary, they must in most cases be reduced to vulgar fractions before they are added, subtracted, multiplied, or divided. In almost all the applications of decimals, however, an approach to accuracy is sufficient, and this is attained by carrying the decimal only to a moderate number of digits, and omitting the rest. If, in converting a vulgar fraction into a decimal, we stop after the third digit, for instance, adding unity to that digit, if the next be 5 or upwards, it does not differ from its exact value by more than one five-thousandth part of the unit employed. Thus .172 differs from $.172\dot{4}3\dot{7}$ by $.000437\dot{2}$, which is less than .0005. Similarly, .983 differs from $.9827\dot{6}$ by $.00023\dot{1}\dot{7}$, which is also less than .0005.

Decimals are most frequently used to make calculations on numbers that have been obtained by observations of some kind, by measuring, for instance, or weighing; and it is very seldom indeed that the accuracy of these observations can be relied on to within one five-thousandth part of the unit employed. Now if we cannot rely on the measurement beyond three decimal places, it is needless to carry the result derived from it any farther. In all operations with decimals, then, whether terminating or repeating, we may usually stop at the third or fourth place, and need very rarely go beyond the fifth or sixth. We may, however, attain any degree of exactness that may be required, by carrying the decimal far enough.

Though the quantity thus neglected be very small, it is not less than any quantity that can be named. The accuracy of the result is therefore very different from that of such results as the one in art. [152], where the error was proved to be less than any that could be named. There the result was exact, here it only approaches to exactness.

167. The operations of multiplying and dividing decimals may be much shortened when this degree only of accuracy is required. For example, to multiply 2.753 by 2.31, carrying the product to three places of decimals,

```
    2.753        2.753           2.753
    2.313        3.313           2.313
    -----        -----           -----
    8259         5 506           5 506
   2753            8259            826
  8259               2753           28
 5506                  8259          8
 --------        --------        -----
 6.367689        6.367689        6.368
```

The first multiplication is in the usual way. The second is the same, but with this difference, that instead of multiplying first by the right hand digit of the multiplier, and shifting every partial product one step to the left, we multiply first by 2 the left hand digit of the multiplier, and shift every partial product one step to the right. The order of the partial products is thus inverted, as will be seen by comparing this multiplication with the first; but, except in the order of these partial products, the two processes are the same. In the third, which is the shortened form, we proceed as in the second, only we never write any digit to the right of the column that furnishes the third decimal place in the product, and we always add unity to this extreme digit when the next appears to be 5 or upwards. A trial or two will show how much more convenient it is to begin the shortened process from the left than from the right of the multiplier. The exact result would be found to be $6.368\dot{0}\dot{5}$, which differs from the approximate result by $\frac{1}{19800}$ of unity.

Again, to divide 49.782 by 2.7167 carrying the quotient to three places of decimals:

```
2.7167)49.7820(18.3244   2.7167)49.7820(18.324
       27167                    27167
       ------                   ------
       226150                   226150
       217336                   217336
       ------                   ------
         8814|0                   8814
         8150|1                   8150
         ----|--                  ----
          663|90                   664
          543|34                   543
          ---|--                   ---
          120|560                  121
          108|668                  109
          ---|---                  ---
           11|8920                  12
           10|8668                  --
```

The first operation is as the rule directs. The second is the shortened form, in which, instead of adding a zero to every remainder, we cut off one more digit from every successive product, increasing the last digit by unity when the next neglected one is 5 or upwards. On comparing the examples, the effects will be seen to be alike. These are examples of the shortened methods. The reader will have no difficulty in extending the same principles to other cases.

168. To reduce a number, whole or decimal, of a lower denomination [see art. 98, &c.] to a decimal of a higher: *Divide it by the number of times that the unit of the higher denomination contains that of the lower.* For example, 6.35 shillings $= 6.35 \times \frac{1}{20}$ of a pound, that is £$\frac{6.35}{20}$, or £.3175. So to reduce 6*s*. 4¾*d*. to a decimal of a pound;

$\frac{3}{4}d. = .75d.$

$4.75d. = \frac{4.75}{12}s.$, or $.3958\dot{3}s.$

$6.3958\dot{3}s. = £\frac{6.3958\dot{3}}{20}$, or $£.319791\dot{6}$.

And therefore this last is the decimal sought.

169. To reduce a fraction or a mixed number expressed as a decimal of a higher denomination into a number of a lower denomination: *Multiply it by the number of times that the unit of the higher denomination contains that of the lower.* Thus, 5.27 feet = 5.27 × 12 inches, or 63.24 inches. So to reduce £.1368 to shillings, pence, and farthings;

£.1368 = 20 × .1368, or 2.736*s*

.736*s*. = 12 × .736, or 8.332*d*.

.832*d*. = 4 × .832, or 3.328 *fars*.

The answer therefore is 2*s*. 8*d*. 3.328 farthings.

170. Only a small proportion of numbers have no other prime factors but 2 and 5, and therefore only a small proportion of vulgar fractions can be reduced to terminating decimals. If twelve were the base of our scale of notation [73] instead of ten, all those fractions whose denominators have no prime factors but 2 and 3, or one of these numbers, would terminate if presented in that scale in a form similar to decimal fractions. For instance, $\frac{1}{2}$ would become .6, $\frac{1}{3}$ would become .4, $\frac{1}{6}$ would become .2, $\frac{1}{27}$ would become .054, and so on. There are more numbers that have 2 and 3 only for their prime factors, than there are that have 2 and 5 only. Of the numbers between 1 and 100, for example, 18 belong to the first class, and only 13 to the second. As this way of expressing fractions is very convenient, and as terminating decimals are much more manageable than repeating ones, twelve, to the extent of this reason, would be a better base for a scale of notation than ten.

Of the Square and Cube Roots, and of Surds.

171. The square root of any number a is a number which, when multiplied by itself, produces a [art. 11]. When the square root of a number is an integer, the number is called a *square number*. Thus 4 is a square number, because 2 multiplied by itself produces 4. So 144 is a square number, for it is produced by the multiplication of 12 by itself.

172. The square root of a whole number that is not square, cannot be expressed by means of any fractional part of a whole number. Thus, if a be a number not square, its square root cannot be expressed in the form $r + \frac{p}{q}$, where r is a whole number, and $\frac{p}{q}$ an irreducible fraction. For if it could, we should have

$$\left(r + \frac{p}{q}\right)^2 = a,$$

or, multiplying $r + \frac{p}{q}$ by itself, [see art. 90]

$$r^2 + 2r\frac{p}{q} + \frac{p^2}{q^2} = a,$$

and, consequently,

$$\frac{p^2}{q^2} = a - r^2 - 2r\frac{p}{q}.$$

Multiply both members of this equation by q, and it becomes

$$\frac{p^2}{q} = aq - r^2q - 2rp.$$

Now p is prime to q, therefore p^2 is prime to q, and therefore it is not possible that $\frac{p^2}{q}$ can be equal to a whole number, as we have found it to be. It follows, that it is not possible that $\sqrt{a}$ can be expressed in the form $r + \frac{p}{q}$.

This is a remarkable property of such square roots. It amounts to this, that it is impossible to find any number, however small it be, that will at the same time measure unity and $\sqrt{a}$, when a is not a square number. If it were possible to find a number $\frac{1}{q}$ that measured both, then $\sqrt{a}$ might be presented in the form $r + \frac{p}{q}$, and that we have seen is impossible. This is usually expressed by saying that $\sqrt{a}$ is *incommensurable* with unity, when a is not a square number. Thus $\sqrt{2}$, $\sqrt{3}$, $\sqrt{15}$, &c. are all incommensurable with unity; there is no fraction, vulgar or decimal, that is exactly equivalent to them, or to any similar numbers.

173. In the same way, a is called a *cube number* when its cube root [art. 11] is a whole number. Thus 27 and 125, the cube roots of which are 3 and 5 respectively, are cube numbers. As before, it may be shown, that where a is not a cube number, its cube root is incommensurable with unity. So if $\sqrt[4]{a}$ [art. 11] is not a whole number, the fourth root of a is incommensurable with unity. All such incommensurable numbers as $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{9}$, &c. are called *irrational numbers*, or *surds*; while numbers commensurable with unity are called *rational numbers*. There are many

numbers besides surds that are incommensurable with unity.

174. We now proceed to explain the common methods of finding the square and cube roots of numbers, and in doing so we shall have occasion to refer to the following table of the squares and cubes of the first nine numbers.

Number.	Square.	Cube.
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	343
8	64	512
9	81	729

The square of any digit followed by any number of zeros, is the square of that digit in the table, followed by twice the number of zeros; and the cube is the cube of the digit in the table, followed by three times the number of zeros. Thus, the square of 700 is 490000, and the cube of 90 is 729000.

175. One way of finding the square root of a square number, would be to guess some number that might be near it, and try whether its square were greater or less than the number proposed. We might then correct the number guessed and try again, and so continually approach the square root sought till at last we should find it. The common way of finding the square root of a number is, like this, a succession of trials, only much less tedious; for we are able to find the digits in the square root one by one, and to correct them without the trouble of squaring at every step.

The square of every number between 1 and 10 is between 1 and 100, and has one or two digits. The square of every number between 10 and 100 is between 100 and 10000, and has three or four digits. Similarly, the square of every number between 100 and 1000 has five or six digits, and so on. On the other hand, every square number consisting of one or two digits has one digit in its square root, every square number of three or four digits has two in its square root, every square number of five or six digits three, and so on. It follows, that when a square number is proposed, if we divide it into periods consisting of two digits each, except the last, which will consist of two digits or one as the case may be, there will be as many of these periods as there are digits in the square root of the number. This division into periods, or *pointing*, as it is called, should begin from the right of the number, as will appear afterwards.

176. Let 4624 be a number of which the square root is wanted. In the first place, by pointing it thus, $4\dot{6}2\dot{4}$, it appears that the square root has two digits. Next, we observe by the table that the square of 70 is 4900, while the square of 60 is 3600, the one greater and the other less than 4624. The square root sought is thus between 60 and 70. Call it $60 + b$, where b is necessarily less than 10. Then $(60 + b)^2$ must be equal to 4624, or, multiplying $60 + b$ by itself,

$$60^2 + 2 \times 60\,b + b^2 = 4624.$$

From each member take away 60^2, or 3600; then, altering the first member a little, there remains

$$b\,(2 \times 60 + b) = 1024,$$

whence

$$b = \frac{1024}{2 \times 60 + b}, \text{ or } \frac{1024}{120 + b}.$$

Now b is less than 10, and therefore so much less than 120, that if we neglect it in the denominator, the result, which is $\frac{1024}{120}$, will not be very different from the true value of b. $\frac{1024}{120}$ is pretty nearly 8, so that we may try 8 for b. If we square $60 + 8$, or 68, we shall find the product to be 4624, and therefore 68 is the square root sought.

This example teaches the principle on which the rule for extracting the square root is founded. We first find the first part of the root. We then subtract the square of the part found, and the appearance of the remainder enables us to determine the second part. We then subtract the square of the first and second parts, and the appearance of the remainder enables us to determine the third, and so on.

177. Suppose that we find a to be the first part of the root, and that, subtracting a^2, we find a remainder from which it appears that b will be the second. To ascertain this we must as before subtract $(a + b)^2$ from the number proposed. But

$$(a + b)^2 = a^2 + 2\,a\,b + b^2,$$

or

$$(a + b)^2 = a^2 + b\,(2\,a + b),$$

and therefore if we subtract $b\,(2\,a + b)$ from the first remainder, it will be the

same thing as subtracting $(a+b)^2$ from the original number. In like manner, when we have determined $(a+b)$, supposing that c is the third part, we must subtract $(a+b+c)^2$ from the original number. But

$$(a+b+c)^2 = \{(a+b)+c\}^2,$$

and considering $a+b$ as one quantity, we have

$$(a+b+c)^2 = (a+b)^2 + 2(a+b)c + c^2,$$

or

$$(a+b+c)^2 = (a+b)^2 + c\left\{2(a+b)+c\right\},$$

as before; just as before then, to subtract $(a+b+c)^2$ from the original number, is equivalent to subtracting $c\left\{2(a+b)+c\right\}$ from the second remainder.

Observe how these properties shorten the following example: To extract the square root of 322624.

```
        3̇22̇62̇4 (500 + 60 + 8
   Subtract 250000 = 500²
1000 + 60) 72624
     Subtract 63600 = 60(1000 + 60)
     1120 + 8) 9024
       Subtract 9024 = 8(1120 + 8)
```

We first find, by pointing, that there are three digits in the square root. It next appears by the table, that the root is between 500 and 600. We subtract 500^2, and dividing the remainder as before by 2×500 or 1000, we should first find, as in last article, that 70 was the next part of the root. But we find that $70(2.500+70)$, or 74900, is greater than 72624, and therefore 570^2 must be greater than 322624, as has been shown. We then try 60 for the second part. We subtract $60(2 \times 500 + 60)$ which, with 500^2 already subtracted, makes, as we have seen, 560^2 in all. To find a number to try for the third part of the root, we divide the last remainder as before by 2×560, and find the quotient about 8. We then subtract $8(2 \times 560 + 8)$ equivalent to $c\left\{2(a+b)+c\right\}$ above, and making in all 568^2 subtracted, and as we then find that nothing remains, 568 must be the number sought. Omitting useless digits, this process may be put thus:

```
      3̇22̇62̇4 (568
      25
  106) 726
       636
 1128) 9024
       9024
```

And this agrees with the usual rule, which is as follows: *Point every second digit beginning with the units'; find by the table the greatest digit whose square is contained in the left hand period, and write it as the first digit in the root; subtract its square from the left hand period, and bring down the second period to the remainder to form a dividend; to the left of this dividend write twice the digit last found as an* imperfect divisor; *divide the dividend, omitting its last digit, by the imperfect divisor, and write the single digit, which is the whole part of the quotient, as the second digit in the root, and also to the right of the imperfect divisor to form the* perfect divisor; *multiply this divisor as it now stands by this digit, if the product be greater than the dividend the digit is too great, and a smaller one must be taken; when the digit is not too great, subtract the product last mentioned from the dividend, if the remainder be greater than twice the number composed of the two digits written in the root, the digit last found is too small, and a greater one must be taken; if the digit be not too small, to the remainder last mentioned bring down the third period to form a new dividend, and to the perfect divisor, as it now stands, add the digit of the root last found to form a new imperfect divisor, after which proceed as before.*

The reason why the second digit is too small, when, after the product of it by the perfect divisor is subtracted from the dividend, the remainder is greater than twice the part of the root found, may be explained thus. Let M be the value of the two first periods of the number, and P that of the part of the root found. The remainder in question is, as we have seen, the difference $M - P^2$. Now if $M - P^2$ be greater than $2P$, it follows that M is greater than $P^2 + 2P$, and since M and P are whole numbers, M must either be greater than $P^2 + 2P + 1$, or equal to it. But $P^2 + 2P + 1 = (P+1)^2$; therefore M is either equal to or greater than $(P+1)^2$. It follows that $P+1$ or some larger integer, and not P, ought to be the first part of the root, so that the second digit is too small. The same reasoning, of course, applies to any subsequent digit.

178. With respect to the square root of a number not square; observe that when we multiply any number by 100, we make its square root ten times as

great as it was before. The square root of $100\,a$ is $10\sqrt{a}$, or ten times the square root of a; since if we multiply $10\sqrt{a}$ by itself, the product is $10 \times 10 \times \sqrt{a} \times \sqrt{a}$, or $100\,a$. So, when we multiply any number by 10000, or by 1000000, we make its square root 100 times, or 1000 times as great as it was before.

Now, suppose we have to find the square root of 129. We find by the rule that 11 is the first part of it, but there is a remainder over after 11^2 is subtracted. If to this remainder we add two zeros, and go on another step, we shall proceed as if 12900 had been the number proposed, and we shall find that 113 is the whole part of its square root. But the square root of 12900 is, as we have seen, ten times as great as that of 129, and therefore $\frac{113}{10}$, or 11.3, is a nearer approach than before to the square root of 129. So if we again add two zeros to the remainder when $(113)^2$ is subtracted, we shall find 5 to be the next digit, showing that 1135 is the whole number next below the square root of 1290000. Therefore $\frac{1135}{100}$, or 11.35, is a still nearer approach to the square root of 129. If we add two more zeros we find the next digit to be 7, and the next in the same way to be 3, so that 11.3573 is a still nearer approach to the square root of 129. Proceeding in this way we can come as near to the square root of 129 as we please, though, for the reason in art. [172], it can never be expressed exactly in decimals.

We must make the following addition to our rule to meet this case: *When there is a remainder over, after the last period is brought down, add two zeros to it, and proceed, as before, placing the decimal point before the digit next found.*

179. The rule for finding the square root of decimals is on the same principle. The square root of 21.76 is one tenth of that of 100×21.76, or of 2176. So the square root of 97.968 is one hundredth part of that of 10000×97.968, or of 979680. Similarly, the square root of .034 is one hundredth part of that of $10000 \times .034$, or of 340. The rule then will be this: *Make the number of digits after the decimal place even, by adding a zero to the right if necessary; proceed as directed for whole numbers, making half as many decimal places in the root as there are in the number proposed as it now stands.* For example, to extract the square root of .07361: there are five digits, therefore by the rule we must add a zero to the right so as to make it .073610. We then point as if it were a whole number, thus $0\dot{7}\,6\dot{1}\,1\dot{0}$. There will be three decimal places in the root furnished by the three periods in the decimal as it now stands.

180. As in the case of the square root, and on the same principles, we can know at once how many digits there are in the cube root of any cube number. The cube of every number between one and ten has one, two, or three digits, of every number between ten and one hundred, four, five, or six digits, and so on. So, on the other hand, if a cube number consists of one, two, or three digits, its cube root has but one. If it consists of four, five, or six digits, its cube root has but two. If it consists of seven, eight, or nine digits, its cube root has but three, and so on. So that if any cube number be divided into periods of three digits each, except the last, which will contain one, two, or three, as the case may be, the number of these periods will be the number of digits in its cube root. This dividing into periods, as in finding the square root, should begin from the right of the number.

181. We find the cube root of a number part by part. When we have found the first part we subtract its cube, and from the appearance of the remainder we determine the second. We then subtract the cube of the first and second, and from the appearance of the remainder determine the third, and so on.

Now, let N be the cube number proposed. Suppose that we find that a is the first part of its cube root, we subtract a^3; let the remainder be R. Call the rest of the cube root x, then $(a+x)^3$ must be equal to N. $(a+x)$ is equal to

$$(a+x) \times (a+x)^2,$$

or to

$$(a+x)\,(a^2 + 2\,a\,x + x^2),$$

and multiplying, this becomes

$$a^3 + 3\,a^2\,x + 3\,a\,x^2 + x^3.$$

Therefore

$$a^3 + 3\,a^2\,x + 3\,a\,x^2 + x^3 = \mathrm{N},$$

and

$$3a^2x + 3ax^2 + x^3 = N - a^3 = R;$$

whence

$$x = \frac{R}{3a^2 + 3ax + x^2}.$$

But x is always less than a, and therefore $3a^2$ is usually much greater than $3ax + x^2$. As an approach then to b, the next part of the root, we may take a number near $\frac{R}{3a^2}$. [See art. 176.]

When we have found b, we subtract $(a+b)^3$, and calling the remainder R′ we find the next part by taking a number near $\frac{R'}{3(a+b)^2}$.

As will be seen, the operation is much shortened in consequence of the two following properties:

First,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

as before; or

$$(a+b)^3 = a^3 + b(3a^2 + 3ab + b^2);$$

or, finally,

$$(a+b)^3 = a^3 + b\{3a^2 + b(3a+b)\}.$$

Second, since

$$(a+b)^2 = a^2 + 2ab + b^2,$$

then

$$3(a+b)^2 = 3a^2 + 6ab + 3b^2,$$

or

$$3(a+b)^2 = 3a^2 + b(3a+b) + b(3a+b) + b^2.$$

These properties may be extended to the case of $a + b + c$, or further, as in art. [177].

Let us now proceed to find the cube root of the number 98611128.

		$a \quad b \quad c$
$3a = 1200$		$9\dot{8}\ 61\dot{1}\ 12\dot{8} \mid 400+60+2$
$b = 60$		$64\ 000\ 000 = a^3$
$3a+b = 1260$	$3a^2 = 480\ 000$	$34\ 611\ 128 = R$
$2b = 120$	$b(3a+b) = 75\ 600$	
$3(a+b) = 1380$	$3a^2 + b(3a+b) = 555\ 600$	$33\ 336\ 000 = b\{3a^2 + b(3a+b)\}$
$c = 2$	$b^2 = 3\ 600$	
$3\overline{a+b} + c = 1382$	$3(a+b)^2 = 634\ 800$	$1\ 275\ 128 = R'$
	$c\{3(a+b)+c\} = 2\ 764$	
	$637\ 564$	$1\ 275\ 128 = c\{3.\overline{a+b}^2 + c(3.\overline{a+b} + c)$

Pointing the number, it appears that there are three digits in its cube root. Next, the root appears to be between 400 and 500, since by the table the cubes of these numbers are respectively 64000000 and 125000000. We subtract the cube of 400, and find the remainder, which we have called R. To the left of this remainder we set down 3×400^2, or 480000, as the *imperfect divisor*, and to the left of that 3×400, or 1200. Dividing R by the imperfect divisor, the quotient is about 60, which, as we have seen, will be a near approach to the next part of the root. We add 60 to 1200 on the left, and multiply the sum 1260 by 60. We then write the product 75600 under the imperfect divisor, and add these numbers for a *perfect divisor*. We multiply the sum 555 600 by 60, and write the product 33 336 000 under the first remainder R and subtract.

Now we shall find from its composition, that the number last subtracted is

$$60\{3.400^2 + 60(3 \times 400 + 60)\}$$

and this with 400^3 already subtracted, makes up $(400 + 60)^3$, by the first property. The remainder 1275128 is therefore what we have denoted above by R′.

For a new imperfect divisor we write 60^2 under the last perfect divisor. We then add 60^2, and the two numbers above it together. The sum from its composition is

$$3 \times 400^2 + 60(3 \times 400 + 60) + 60(3 \times 400 + 60) + 60^2$$

or, by the second property, 3×460^2. Lastly, to the number on the left we add 2×60, or 120, the sum 1380 is 3×460.

Dividing by the new imperfect divisor, we find that 2 is the next digit of the root, with which we proceed as before. We add 2 to 1380, multiply the sum by 2, and add the product to the imperfect divisor for a new perfect divisor. We then multiply this perfect divisor by 2. When we have subtracted the product, we have, as has been already shown, subtracted 462^3,

and as there is no remainder, 462 must be the root sought.

This operation may be shortened, as in the next example, by omitting useless digits. To extract the cube root of 625245708151.

```
245                              625  245  708  151 | 8551
 10                              512
----                           ------
2555          .         19200 | 113  245
  10                     1225 |
-----                   ----- |
25651                   20425 | 102  125
                           25 |
                        ------------------
                      2167500 |11   120  708
                        12775 |
                      ------- |
                      2180275 |10   901  375
                           25 |
                      --------------------------
                    219307500 | 219  333  151
                        25651 |
                    --------- |
                    219333151 | 219  333  151.
                    ----------------------------
```

182. When the product of the perfect divisor, and the digit assumed is greater than the preceding dividend, that digit is of course too great, and a smaller one is to be taken. Again, if the remainder, when this product is subtracted from the dividend, be greater than $3P^2 + 3P$, where P is the part of the root found, the digit last assumed is too small, and a greater must be taken. For let M be the value of the periods already brought down, then the remainder in question is, as we have seen, $M - P^3$. If this be greater than $3P^2 + 3P$, it must follow that M is greater than $P^3 + 3P^2 + 3P$, and that M is either equal to, or greater than

$$P^3 + 3P^2 + 3P + 1.$$

Now this last expression will be found equal to $(P + 1)^3$, and if M be equal to or greater than $(P + 1)^3$, $P + 1$, or some larger integer, and not P, ought to be the part of the root found, and therefore the last digit must be increased by unity.

When there is a remainder after all the periods are brought down, the number is not a perfect cube. But, for the same reason as in extracting the square root [art. 178], if we add three zeros to this remainder and proceed, the digit next found will be the first decimal digit in the root, and by continuing to add zeros three at a time, we may approach to the cube root as nearly as we please.

183. The rule may be stated thus: *Point every third digit in the number proposed, beginning with the units; find by the table the greatest digit* (a) *whose cube is contained in the left hand period, and write it as the first in the root; subtract* a^3 *from the first period, and to the right of the remainder write the second period to form a dividend; to the left of this write* $3a^2$, *to which annex two zeros to form an imperfect divisor; still farther to the left write* 3a; *find a second digit* (b) *by dividing the dividend by the imperfect divisor, and annex it to the right of* 3a *in the column farthest to the left; multiply the number in this column as it now stands by* b, *and write the product under the imperfect divisor for a correction; add the imperfect divisor to the correction, and write their sum, which is the perfect divisor, under them; multiply the perfect divisor by* b, *and subtract the product from the dividend, and to the right of the remainder annex the next period; for a new imperfect divisor write* b^2 *under the last perfect divisor, and take the sum of* b^2 *and the two numbers above it; then add* 2b *to the lowest number in the column farthest to the left, and proceed as before; the several digits so found are the digits of the root.*

When there is a remainder at last, add three zeros to it and proceed, the digits afterwards found are decimal parts of the root.

184. When part of the number proposed is decimal, the rule is founded on the same reasons as for the corresponding case in the square root, [art. 179.] It is this, *Make the number of digits after the decimal point three, or some multiple of three, by adding one or two zeros to the right if necessary; then proceed as is directed for whole numbers, placing the decimal point before that digit in the root which is first*

found after taking down the first decimal period in the number proposed.

For example, to extract the cube root of 21.7698, that is of 21.769800. This root is [art. 179] one hundredth part of the cube root of 21769800. Pointing the number thus, 21769800, and extracting its cube root, we must mark off two decimal digits from the whole part of the result, and this amounts to what is directed by the rule.

Under these rules for pointing decimals before extracting the square and cube roots, a point always falls on the units' digit, and consequently on every second, or on every third digit to the right and left of it.

185. The operations for finding the square and cube roots of numbers admit of being shortened in a way similar to that adopted in [art. 167]. For instance, let it be proposed to find the square root of 329.1 to five places of decimals. Adding a zero, as directed by the rule, and pointing, we have 329.10.

```
       32910(18.14111
       1
    28)229
       224
     361)510
         361
    3624)14900
         14496
      3628)4040
           3628
            412
            363
             49
             36
```

Here we proceed by the rule till four digits are found. We then add zero to 404 the remainder, and proceed to divide by the divisor 3628, as in the division of decimals in its shortened form. The first six decimal places in the nearest approximation to the root, [art. 167,] are 141113, so that our result is true for five places.

So; to find the cube root of 869243.13. Pointing it becomes 869243.130.

```
                                   869243130 | 95.436335591
275                                729
 10
2854               24300          |140243
   8               1375           |
28623              25675          |128375
    6                 25          |
286296             2707500        |11868130
                     11416        |
                   2718916        |10875664
                        16        |
                   273034800      |992466000
                       85869      |
                   273120669      |819362007
                           9      |
                   27320654700    |173103993000
                       1717776    |
                   27322372476    |163934234856
                            36    |
      a  . . .     27324090288    |91697581440  . .  b
                         85889    |
                   27324176177    |81972528531
                   27324262066    | 9725052909
                            86    |
                   273242706      | 819728118
                   273242792      | 152777173
                                    13662140
                                     1615577
                                     136621
                                      24937
                                      2459
                                         35
                                          2
```

Here we proceed as the rule directs, till five digits are found; we then add zero to the remainder, and continue to divide the number so formed and marked b by the new imperfect divisor marked a. This division is in the shortened method [art. 167.] To come to an end the sooner and have less cumbrous numbers to deal with, we cut one digit off from every successive remainder, and diminish the divisor by two digits, and the correction by three digits, at every division, instead of diminishing the divisor at every step by one digit, and the correction by one. This result is accurate to the last decimal place.*

186. Similar rules may be given for finding the fourth, fifth, and other roots of numbers. The operations, however, become extremely laborious, and the result can always be found with sufficient accuracy by means of a table of logarithms.

Rules for finding the square, cube, and other roots of algebraical expressions are founded on the same principles as those just laid down for numbers; the same purposes are however much better served by the binomial theorem, so that these rules are of no practical use.

187. By the rules we have laid down we can find a number consisting of as many decimal places as we please, which number is always less than the square root or cube root or other surd sought. Now, by increasing the last digit of this number by unity, we find a number which is greater than the root sought; for the process by which we obtain each digit assures us that it is the greatest digit that is not too great. We can thus always present two numbers, one greater and the other less than the surd sought, while these two numbers may be made to differ from each other by a quantity as small as we please. For instance, we have found [art. 185], that 95.436335 is less than the cube root of 869243.13, while 95.436336 is greater than the same root, and these numbers differ only by $\frac{1}{1000000}$ of unity. If we wish to find two numbers which differ only by $\frac{1}{10000000}$ of unity, and which include between them the same root, we do it by carrying the process one step farther, and find the numbers 95.4363355, and 95.4363356.

188. We can now explain what is meant by the product of two or more surds. Take, for instance, $\sqrt{a}$ and $\sqrt[3]{b}$. Let s be a number greater, and s' less than $\sqrt{a}$; and let t be a number greater, and t' a number less than $\sqrt[3]{b}$. The product $\sqrt{a}\,.\sqrt[3]{b}$ means a number which is always intermediate between $s\,t$ and $s'\,t'$, however small the difference made between s and s', and between t and t', or, in other words, however small the difference be made between $s\,t$ and $s'\,t'$.

189. We may extend the proposition in art. [23], which was proved in art. [89] to be true for fractions, to the case of surds. Retaining the notation of the last article, the product $\sqrt{a}\,.\sqrt[3]{b}$ is a number that is always intermediate between $s\,t$ and $s'\,t'$; and $\sqrt[3]{b}\,.\sqrt{a}$ is a number intermediate between $t\,s$ and $t'\,s'$. But $s\,t$ and $s'\,t'$ are respectively equal to $t\,s$ and $t'\,s'$, since s, t, s', and t' are all fractional numbers, [89]. It follows that $\sqrt{a}\,.\sqrt[3]{b}$ and $\sqrt[3]{b}\,.\sqrt{a}$ are always intermediate between the two nnmbers $s\,t$ and $s't'$, while the difference of these two numbers may be made as small as any number that can be named. Therefore the difference between $\sqrt{a}\,.\sqrt[3]{b}$ and $\sqrt[3]{b}\,.\sqrt{a}$ may be made less than any number that can be named, and hence, on the reasoning in art. [152], these quantities are accurately equal. The same reasoning may plainly be extended to the product of more surds than two.

190. The root of any number made up of factors, is the same as the product of the roots of tho separate factors. For example, $\sqrt{a\,b} = \sqrt{a}\,.\sqrt{b}$. For multiplying $\sqrt{a}\,.\sqrt{b}$ bv itself, the product is

$$\sqrt{a}\ \ \sqrt{b}\,.\sqrt{a}\,.\sqrt{b};$$

and, as we have just seen [art. 189], this is equal to

$$\sqrt{a}\,.\sqrt{a}\,.\sqrt{b}\,.\sqrt{b},$$

or $a\,b$. Therefore $\sqrt{a}\,.\sqrt{b}$ is a num ber which, when multiplied by itself, produces $a\,b$, and is therefore equal to $\sqrt{a\,b}$, [art. 11]. Thus $\sqrt{144}$ or $\sqrt{9 \times 16} = \sqrt{9}\,.\sqrt{16} = 3 \times 4$, or 12. Similarly,

$$\sqrt[m]{a\,b\,c} = \sqrt[m]{a}\,.\sqrt[m]{b}\,.\sqrt[m]{c}.$$

191. When $\sqrt{a}$ and $\sqrt[3]{b}$ are surds,

* The rule we have given for the cube root is much less laborious than the one commonly used. It was first given in a Treatise on the Numerical Solution of Equations, by Mr. Holdred.

the expression $\frac{\sqrt{a}}{\sqrt[n]{b}}$ means a quantity, such that its product by $\sqrt[n]{b}$ is always included between the same numbers as include between them $\sqrt{a}$, however small the difference between these numbers be made. [See art. 10.]

192. The root of any fraction, is the root of its numerator divided by the root of its denominator. For, by art. [90], the n^{th} power of $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ is $\frac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n}$, or $\frac{a}{b}$. Therefore $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ is the number which when multiplied by itself produces $\frac{a}{b}$, that is [art. 11] the n^{th} root of $\frac{a}{b}$. Thus the square root of $\frac{9}{16}$ is $\frac{3}{4}$.

193. The n^{th} root of a^{mn} is a^m. This follows from art. [24], where it was shown that $(a^m)^n = a^{mn}$. So that in general when p is a multiple of n,

$$\sqrt[n]{a^p} = a^{\frac{p}{n}}.$$

Thus, $\sqrt[3]{3^6}$ is equal to $3^{\frac{6}{3}}$ or 3^2, or 9; accordingly we shall find that the cube of 9, and the sixth power of 3 are both 729.

194. When, in the last article, p is not a multiple of n, let $p = n\,q + r$. Then we shall have

$$\sqrt[n]{a^p} = a^q \,.\, \sqrt[n]{a^r}.$$

For since

$$a^{nq+r} = a^{nq} \,.\, a^r$$

by art. [24]; it follows that

$$\sqrt[n]{a^{nq+r}} = \sqrt[n]{a^{nq}.\,a^r}.$$

But, by art. [190],

$$\sqrt[n]{a^{nq}.\,a^r} = \sqrt[n]{a^{nq}} \,.\, \sqrt[n]{a^r},$$

and we have just seen [art. 93], that $\sqrt[n]{a^{nq}} = a^q$, therefore

$$\sqrt[n]{a^{nq}.\,a^r} = .\; a^q \,.\, \sqrt[n]{a^r}.$$

Now

$$\sqrt[n]{a^{nq}.\,a^r} = \sqrt[n]{a^{nq+r}} = \sqrt[n]{a^p},$$

and therefore

$$\sqrt[n]{a^p} = a^q\,\sqrt[n]{a^r}$$

Thus

$$\sqrt[4]{5^9} = 5^2 \,.\, \sqrt[4]{5},$$

because

$$9 = 2 \times 4 + 1.$$

195. We can sometimes use the property proved in the last article to reduce two different surds to two expressions that have the same irrational part. Thus $\sqrt{8}$ and $\sqrt{18}$ are severally equal to $\sqrt{2 \times 4}$ and $\sqrt{2 \times 9}$, or to $2\sqrt{2}$ and $3\sqrt{2}$. It follows that $\sqrt{8} + \sqrt{18} = (2 + 3)\sqrt{2} = 5\sqrt{2}$; and this again is equivalent to $\sqrt{25} \,.\, \sqrt{2}$ or $\sqrt{50}$.

196. The m^{th} power of $\sqrt[n]{a}$ is $\sqrt[n]{a^m}$. For this m^{th} power is the product

$$\sqrt[n]{a} \,.\, \sqrt[n]{a} \,.\, \sqrt[n]{a} \text{ \&c.}$$

repeated m times, and by [190] this product is the same as

$$\sqrt[n]{a\,.\,a\,.\,a\,[m \text{ times}]},$$

or as $\sqrt[n]{a^m}$. Thus the fifth power of $\sqrt{3}$ is $\sqrt{3^5}$, or by [art. 185] $3^2 \,.\, \sqrt{3}$.

197. The m^{th} root of $\sqrt[n]{a}$ is $\sqrt[mn]{a}$. For let $\sqrt[mn]{a} = b$, then since the power of one quantity is equal to the same power of an equal quantity, $a = b^{mn}$. Again, the n^{th} roots of equal quantities are equal; and therefore

$$\sqrt[n]{a} = \sqrt[n]{b^{mn}} = b^m$$

[art. 193]. It follows that b or $\sqrt[mn]{a}$ is a quantity such that its m^{th} power is equal to $\sqrt[n]{a}$, which is what we proposed to prove. For instance

$$\sqrt[3]{\sqrt{2}} = \sqrt[6]{2}.$$

198. The product $\sqrt[m]{a} \,.\, \sqrt[n]{a}$ is $\sqrt[mn]{a^{m+n}}$. For, observing that b and $\sqrt[n]{b^n}$ are the same, and then putting $\sqrt[m]{a}$ for b, we have

$$\sqrt[m]{a} = \sqrt[n]{(\sqrt[m]{a})^n}.$$

Again, by [art. 196],

$$(\sqrt[m]{a})^n = \sqrt[m]{a^n},$$

therefore

$$\sqrt[m]{a} = \sqrt[n]{\sqrt[m]{a^n}},$$

or by [art. 197]

$$\sqrt[m]{a} = \sqrt[mn]{a^n} \,\ldots\, (A),$$

similarly

$$\sqrt[n]{a} = \sqrt[mn]{a^m} \,\ldots\, (B).$$

Therefore, multiplying these equations together,

$$\sqrt[m]{a} \,.\, \sqrt[n]{a} = \sqrt[mn]{a^n} \,.\, \sqrt[mn]{a^m}.$$

But by [art. 190] the second member of the last equation is equal to

$$\sqrt[mn]{a^m.\,a^n};$$

therefore

$$\sqrt[m]{a} \,.\, \sqrt[n]{a} = \sqrt[mn]{a^m.\,a^n},$$

or

$$\sqrt[mn]{a^{m+n}}.$$

Thus

$$\sqrt[3]{3} \,.\, \sqrt[4]{3} = \sqrt[4\cdot3]{3^{4+3}} = \sqrt[12]{3^7}$$
$$= \sqrt[12]{2187}.$$

199. Similarly dividing the two equations (A) and (B) in the last article by each other, we have

$$\frac{\sqrt[m]{a}}{\sqrt[n]{a}} = \frac{\sqrt[mn]{a^n}}{\sqrt[mn]{a^m}},$$

or by [art. 192]

$$\frac{\sqrt[m]{a}}{\sqrt[n]{a}} = \sqrt[mn]{\frac{a^n}{a^m}}$$

Thus

$$\frac{\sqrt{3}}{\sqrt[3]{3}} = \sqrt[6]{\frac{3^3}{3^2}} = \sqrt[6]{3}.$$

These results seem to be extremely complicated. We shall find however, when we come to explain fractional and negative exponents, that they are capable of being expressed very generally and simply.

200. By [30], the square of $-a$ is a^2, the fourth power of $-a$ is a^4, its sixth power is a^6, and so on. Therefore a and $-a$ are square roots of a^2, fourth roots of a^4, sixth roots of a^6, and so on. Every number, in like manner, has two square roots, two fourth roots, two sixth roots, and in general two of every root indicated by an even number, and these two roots differ only in their sign. In writing such roots, when there is nothing to determine us which of them is to be taken, we ought to write them both; and this is done by prefixing both the positive and negative sign, or the *double sign*, to the positive root. Thus $\sqrt{9}$ is ± 3; the fourth root of a is $\pm \sqrt[4]{a}$. $\pm$ is read *plus or minus*

201. A positive quantity multiplied by itself, and a negative quantity multiplied by itself, both give a positive product; there is therefore no quantity positive or negative, which when multiplied by itself gives a negative product. It follows that a negative quantity cannot have for square root a quantity either positive or negative. The square root of a negative quantity such as $\sqrt{-a}$ is therefore called an *imaginary*, or an *impossible quantity*. Whenever such a quantity appears in a result, it is certain that there is something contradictory or absurd in the conditions of the question. In the same manner, since every power indicated by an even number of a negative quantity is positive, no negative quantity can have any root indicated by an even number that is not impossible. The quantity $\sqrt[2n]{-a}$ is therefore always imaginary. Quantities not imaginary, such as a, $\sqrt{a}$, &c. are sometimes called *possible* or *real quantities*.

202. Since $\sqrt{-a} = \sqrt{a.(-1)}$, we have by art. [190]

$$\sqrt{-a} = \sqrt{a} \times \sqrt{-1}.$$

In the same way, any impossible square root may be written in the form $m\sqrt{-1}$ where m is possible.

$\sqrt{-1}$ means an expression which when multiplied by itself produces -1. Therefore

$$(\sqrt{-1})^2 = -1,$$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 . \sqrt{-1}$$
$$= -\sqrt{-1},$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^3 . \sqrt{-1}$$
$$= -\sqrt{-1} . \sqrt{-1} = +1,$$

and so on. In like manner we have

$$(m\sqrt{-1})^2 = -m^2,$$

$$(m\sqrt{-1})^3 = -m^3 . \sqrt{-1},$$

$$(m\sqrt{-1})^4 = +m^4,$$

$$(m\sqrt{-1})^5 = +m^5 . \sqrt{-1},$$

$$(m\sqrt{-1})^6 = -m^6,$$

$$(m\sqrt{-1})^7 = -m^7 . \sqrt{-1},$$

and so on. So that all the powers of an impossible square root can be written in the form $n\sqrt{-1}$, where n is possible.

In the same way it may be shown that $\sqrt[4]{-a}$, or any other impossible fourth root and all its powers can be written in the form

$$n . \sqrt[4]{-1}.$$

It may be observed, that $\sqrt[4]{-1}$ is essentially different from $\sqrt{-1}$, since if we raise both these expressions to the fourth power the results are different, namely, -1 and $+1$.

203. If we cube $\frac{-1+\sqrt{-3}}{2}a$, or $\frac{-1-\sqrt{-3}}{2}a$, the result in either case will be a^3. These two expressions, then, as well as a, are cube roots of a^3. We have already seen, that a^2 has two square roots, a and $-a$. It will be shown afterwards, that every quantity has four fourth roots, five fifth roots, and so on. But of all the roots of any quantity indicated by an

odd number, one only is real, and all the others imaginary. So of all the roots indicated by an even number, all are imaginary but two, and these two differ only in their sign.

Of Quadratic Equations.

204. Let it be proposed to find two numbers such that their difference is 8, and their product 48. Calling the one number x, the other will be $8 + x$, and taking their product we have

$$x \times (8 + x) = 48,$$

or

$$x^2 + 8x = 48.$$

It will be found, that the circumstance of x^2 occurring in this equation prevents us from solving it by the rules laid down for the solution of simple equations [111.] To each member of it let us add 16. It then becomes

$$x^2 + 8x + 16 = 48 + 16 = 64.$$

Now observe, that the first member is here the square of $x + 4$, as may be proved by multiplication. Therefore

$$(x + 4)^2 = 64 = 8^2.$$

When two quantities are equal, their square roots are also equal, and therefore

$$x + 4 = \pm 8,$$

[art. 201], whence

$$x = 8 - 4, \text{ or } 4,$$

if the positive sign be taken; and

$$x = -8 - 4, \text{ or } -12,$$

if the negative sign be taken. Either of these numbers will satisfy the question. If 4 be taken the other number is $4 + 8$ or 12; we find that the difference of 12 and 4 is 8, and their product is 48. Similarly, if -12 be taken, the other number is $-12 + 8$ or -4; the difference of -4 and -12 is 8, and their product also is 48.

In taking the square roots in this example, the double sign should have been affixed to each member, and we should have had

$$\pm (x + 4) = \pm 8.$$

This expression furnishes four equations; but of these, the two produced by giving the first member the negative sign, namely,

$$-(x + 4) = 8,$$

and

$$-(x + 4) = -8,$$

are equivalent to the two produced by giving it the positive sign. The double sign therefore need be prefixed to one member only.

205. Every equation which, like the one just solved, can be put into the form

$$x^2 + px + q = 0,$$

[see art. 109], where p and q may be any quantities positive or negative, is called a *quadratic equation.* Quadratic is derived from a Latin word meaning a *square,* and the equation is so called because the square of x occurs in it. All equations in which x is found in its first power and its square only, can be put into this form. Thus the equation

$$ax + bx^2 + c = x^2 + d,$$

by removing x^2 to its first member and c to its second, becomes

$$(b - 1)x^2 + ax = d - c,$$

and, dividing both members by $b - 1$,

$$x^2 + \frac{a}{b-1}x = \frac{d-c}{b-1},$$

or

$$x^2 + \frac{a}{b-1}x - \frac{d-c}{b-1} = 0;$$

and this last is of the form required.

The equation $x^2 + q = 0$ is a quadratic equation; for, algebraically speaking, it is of the form given above when p, the coefficient of x, is made zero.

206. To solve generally the quadratic equation

$$x^2 + px + q = 0,$$

remove q into the second member, and we have

$$x^2 + px = -q.$$

Now $x^2 + px + \frac{p^2}{4}$ is in every case the square of $x + \frac{p}{2}$. To each member of the equation then add $\left(\frac{p}{2}\right)^2$, or $\frac{p^2}{4}$, and it becomes

$$x^2 + px + \left(\frac{p}{2}\right)^2 = \frac{p^2}{4} - q,$$

or

$$\left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} - q.$$

As, in art. [204], the square roots of these equal quantities are equal; the square root of $\left(x + \frac{p}{2}\right)^2$ is $\pm\left(x + \frac{p}{2}\right)$,

and that of $\frac{p^2}{4} - q$ is $\pm\sqrt{\frac{p^2}{4} - q}$, therefore

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q},$$

whence

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q} \dots (A).$$

When the equation is of the form $x^2 + q = 0$, or $x^2 = -q$, taking the square root of each member as before we have

$$x = \pm\sqrt{-q} \dots (B).$$

These expressions (A) and (B) are quite general, and contain in themselves the rules for solving quadratic equations, whatever the values of p and q may be. For instance, the equation in art. [204] is the same as

$$x^2 + 8x - 48 = 0,$$

where $p = 8$ and $q = -48$. Therefore $\frac{p}{2} = 4$ and $\frac{p^2}{4} = 16$, so that the expression (A) becomes

$$x = -4 \pm \sqrt{16 - (-48)},$$

or

$$x = -4 \pm \sqrt{64} = -4 \pm 8,$$

as before.

Observe, that when zero is put for p in the expression (A), that expression is reduced to $x = \pm\sqrt{-q}$ which is the expression (B).

207. The values of the unknown quantity which solve, or satisfy, a quadratic equation are called *roots*, because they are found by the extraction of a root. The expressions just found show us that every quadratic equation has got two of these roots: *viz.*

$$-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q},$$

and

$$-\frac{p}{2} - \sqrt{\frac{p^2}{4} - q},$$

in the case of the form (A); and $+\sqrt{-q}$, and $-\sqrt{-q}$ in the case of the form (B). We may show that it can have but two. Take the equation

$$x^2 + px + q = 0,$$

and let a be one root of it; then

$$a^2 + pa + q = 0$$

Again, let a' be another root, then as before,

$$a'^2 + pa' + q = 0.$$

Subtracting the second of these equations from the first we have

$$a^2 - a'^2 + p(a - a') = 0,$$

or

$$a^2 - a'^2 = -p(a - a').$$

Divide both members of this equation by $a - a'$, and it becomes

$$a + a' = -p,$$

or

$$a' = -(p + a).$$

It is quite plain that there is only one quantity equal to $-(p+a)$, and therefore the equation has only one root besides a.

Learners sometimes find it difficult to understand how the same quantity can have two different values in the same equation. This will be fully explained afterwards. It is enough to say at present, that either of these roots is always found to satisfy the equation, and therefore if our expression did not furnish us with both of them, it would not solve the equation completely.

208. The sum of the roots of the equation $x^2 + px + q = 0$ is

$$-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}$$
$$+ \left(-\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}\right),$$

which reduces itself to $-p$. Their product is

$$\left\{-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right\}$$
$$\times\left\{-\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}\right\}.$$

Comparing this with the second example in art. [28] we find it equivalent to

$$\left(-\frac{p}{2}\right)^2 - \left(\sqrt{\frac{p^2}{4} - q}\right)^2,$$

or

$$\frac{p^2}{4} - \left(\frac{p^2}{4} - q\right)$$

which reduces itself to q.

Thus the sum of 4 and -12, the two roots of the equation in art. [195], is -8, and their product is -48. We have seen [art. 197] that the values of p and q in that equation are 8 and -48 respectively.

We can thus form a quadratic equa-

tion whose roots shall be any given numbers, 2 and 3 for instance. We must make $-p = 2 + 3$, or $p = -5$, and $q = 2 \times 3$, or 6. The equation then is

$$x^2 - 5x + 6 = 0.$$

209. The rule for solving quadratic equations is as follows: *Clear the equation of fractions if there be any* [110]; *remove all the terms containing the unknown quantity into one member, and all those that do not contain it into another; collect into one the coefficients of the square of the unknown quantity if more than one, and also the coefficients of the simple power of the unknown quantity; divide every term in both members by the coefficient of the square of the unknown quantity; to each member of the equation as it now stands add the square of half the coefficient of the simple power of the unknown quantity; extract the square root of each member, and write the results as equal to each other, prefixing the double sign to the one which does not contain the unknown quantity; the quadratic equation is then reduced to a simple one, and may be solved accordingly.*

210. The following are examples of questions producing quadratic equations.

The sum of two numbers is 6, and the sum of their reciprocals is $\frac{3}{4}$, what are the numbers? Call the one of them x; then, as in art. [115], the other is $6 - x$, and taking the sum of their reciprocals [96] we have by the question

$$\frac{1}{x} + \frac{1}{6-x} = \frac{3}{4}$$

Multiplying both members of this equation by $x(6 - x)$ it becomes

$$6 - x + x = \frac{3}{4}(6x - x^2),$$

or

$$\frac{3}{4}(6x - x^2) = 6$$

and multiplying both members by $-\frac{4}{3}$,

$$x^2 - 6x = -8.$$

To each member add $\left(\frac{6}{2}\right)^2$, or 9, and we have

$$x^2 - 6x + 9 = 9 - 8,$$

or

$$(x - 3)^2 = 1,$$

whence

$$x - 3 = \pm 1;$$

and this gives us

$$x = 3 \pm 1, \text{ or } x = 4, \text{ or } 2.$$

If 4 be taken for the one number, the other is $6 - 4$, or 2; and 2 and 4 are the numbers that satisfy the question since $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

It is observable, that in this case the two roots of the equation 2 and 4 are the two numbers sought. This is necessarily the case; for whichever of the numbers is called x, the other is $6 - x$, and there is no way of distinguishing that x is to stand for one of the numbers rather than the other. Every reason, then, that exists to make one of the numbers satisfy the equation applies equally to the other. It is therefore impossible that the equation can be satisfied by one of the numbers and not by the other.

211. To find two numbers such that their sum is 10, and the sum of their squares 58. Call the one number x, then the other is $10 - x$. By the question

$$x^2 + (10 - x)^2 = 58,$$

or

$$2x^2 - 20x + 100 = 58.$$

Carrying 100 to the other member of the equation and dividing by 2 this becomes

$$x^2 - 10x = -21.$$

Add 25 to each member and we have

$$x^2 - 10x + 25 = 25 - 21,$$

or

$$(x - 5)^2 = 4;$$

whence

$$x - 5 = \pm 2,$$

which gives $x = 3$, or 7. We find $3^2 + 7^2 = 9 + 49 = 58$. For the same reason as in the last example, the two roots are the two numbers sought.

Suppose that it had been required to find two numbers such that while their sum is 10, the sum of their squares should be 20. As before, we should have

$$2x^2 - 20x + 100 = 20,$$

and

$$x^2 - 10x = -40.$$

Adding 25 to each member, this becomes

$$x^2 - 10x + 25 = -15,$$

which gives us

$$x = 5 + \sqrt{-15},$$

or

$$x = 5 - \sqrt{-15}.$$

Here the values of x are impossible [art. 201], which shows that there are no numbers that can satisfy the conditions given, but that the question contains something absurd and contradictory. This is plainly the case; for we cannot divide 10 into two parts such that the sum of their squares shall be less than 50. The imaginary values of x however satisfy the question, and, as before, the two roots of the equation are the two expressions sought.

Returning to [206] the roots of a quadratic equation will be impossible whenever $\frac{p^2}{4} - q$ is a negative quantity. Now $\frac{p^2}{4}$ is in its nature positive, since every square number is always positive. Therefore the roots are impossible only when q is positive, and greater than $\frac{p^2}{4}$. When q is a negative quantity, such as $-m$, the expression under the radical sign becomes $\frac{p^2}{4} - (-m)$, or $\frac{p^2}{4} + m$, a quantity essentially positive, and therefore when q is negative the roots are always possible. If one root be impossible the other must be impossible also, for the two roots always consist of the same terms connected by different signs.

212. A company of persons spend 3*l.* 10*s.* at a tavern. Four of them go away without paying, in consequence of which each of the others has to pay 2*s.* more than his share. How many persons were there in the company, and what was the proper share of each? Call the number of persons x. They spend 70 shillings, so that the proper share of each is $\frac{70}{x}$ shillings. But there are only $x - 4$ who pay, so that every one of these pays $\frac{70}{x-4}$ shillings. Now each of these pays 2*s.* more than his proper share, therefore

$$\frac{70}{x-4} = \frac{70}{x} + 2.$$

Multiply each member by $x \cdot (x - 4)$ and this becomes

$$70x = 70x - 280 + 2x^2 - 8x,$$

or, when reduced to the general form

$$x^2 - 4x = 140.$$

Adding 4 to each member we have

$$x^2 - 4x + 4 = 144,$$

which gives

$$x - 2 = \pm 12,$$

whence

$$x = 14, \text{ or } -10.$$

The positive value of x, 14, satisfies the question; for when the number in company is 14 the proper share of each is $\frac{70}{14}$ or 5 shillings, while what each of those who remain actually pays is $\frac{70}{10}$, or 7 shillings, which is 2 shillings more.

With respect to the negative value, -10, it also satisfies the equation, since

$$\frac{70}{-10-4} = \frac{70}{-10} + 2.$$

As we have stated the question, the sums $\frac{70}{x-4}$ and $\frac{70}{x}$ are to be paid by the company, when they are positive numbers. Therefore when they become negative numbers, that is, when x becomes negative, they are sums to be received by the company. But when we introduced x into the equation, we introduced it quite generally, as any quantity either positive or negative that would satisfy the equation; and therefore the equation, as we stated it, necessarily applied as much to the case in which the company were to receive, as to that in which they were to pay the money. When they are to receive the money the question must be altered to this: A company of persons are entitled to have 70 shillings distributed among them, their number is increased by four, in consequence of which the share of each is diminished by two shillings; how many persons were there at first? This question is, algebraically speaking, the same as the former, with the signs of the different numbers changed; the answer to it is 10, the negative answer to the former question with its sign changed. Not only, then, do the two

roots satisfy the original equation numerically, but they both satisfy the conditions of the question in the way in which these conditions must be translated into the language of algebra.

213. When there are two or more quadratic equations involving two or more unknown quantities, there is no general rule that can be laid down for their solution, and we must treat them differently, according to the circumstances of each case. For instance, let it be required to find two numbers whose product is $\frac{1}{9}$, and the sum of their squares $\frac{17}{36}$. Call the one x, and the other y. Then

$$x y = \frac{1}{9} \dots . (A),$$

$$x^2 + y^2 = \frac{17}{36} \dots . (B).$$

Multiply (A) by 2 and it becomes

$$2 x y = \frac{2}{9}.$$

Adding the members of this last equation to those of (B) we obtain

$$x^2 + 2 x y + y^2 = \frac{17}{36} + \frac{2}{9} = \frac{25}{36};$$

and, in like manner, subtracting them,

$$x^2 - 2 x y + y^2 = \frac{17}{36} - \frac{2}{9} = \frac{9}{36}.$$

Now, as we have seen,

$$x^2 + 2 x y + y^2 = (x + y)^2,$$

and similarly

$$x^2 - 2 x y + y^2 = (x - y)^2.$$

Therefore

$$(x + y)^2 = \frac{25}{36}$$

$$(x - y)^2 = \frac{9}{36},$$

and extracting the square roots of both members of each of these last equations

$$x + y = \pm \frac{5}{6}$$

$$x - y = \pm \frac{3}{6}.$$

Adding these equations, and attending to all the signs, we have

$$2 x = \pm \frac{4}{3}, \text{ or } \pm \frac{1}{3},$$

and

$$x = \pm \frac{2}{3} \text{ or } \pm \frac{1}{6}.$$

Similarly by subtracting,

$$y = \pm \frac{1}{6}, \text{ or } \pm \frac{2}{3}$$

The numbers sought then are $\frac{2}{3}$ and $\frac{1}{6}$, or $-\frac{2}{3}$ and $-\frac{1}{6}$; either of these pairs satisfies the question.

The values of x and y are the same, because the equations (A) and (B) are symmetrical with respect to x and y. [See art. 112.]

214. Find two numbers such that their sum, their product, and the difference of their squares, shall be all equal. As before, let the one be x and the other y. Their sum is $x + y$, their product $x y$, and the difference of their squares $x^2 - y^2$. Therefore by the question

$$x + y = x^2 - y^2,$$

$$x + y = x y.$$

Dividing each member of the first of these equations by $x + y$ we find

$$1 = x - y,$$

whence

$$y = x - 1.$$

Substitute this value for y in the second equation, and it becomes

$$x + x - 1 = x (x - 1),$$

or

$$x^2 - 3 x = - 1,$$

and the roots of this quadratic, determined by the rule, are

$$x = \frac{3 + \sqrt{5}}{2} \text{ and } x = \frac{3 - \sqrt{5}}{2}.$$

The corresponding values of y, found by putting these values for x in the equation

$$y = x - 1$$

are

$$y = \frac{1 + \sqrt{5}}{2} \text{ and } y = \frac{1 - \sqrt{5}}{2}.$$

Accordingly, either of the pairs of numbers $\frac{3 + \sqrt{5}}{2}$ and $\frac{1 + \sqrt{5}}{2}$, or $\frac{3 - \sqrt{5}}{2}$ and $\frac{1 - \sqrt{5}}{2}$ will be found to satisfy the conditions given.

We may observe, that the two equations which we have just solved are not symmetrical; because, though they contain the same powers of x and y, the sign of y^2 is different from that of x^2. Accordingly, the values of x and y are not the same, as they always are when the equations are symmetrical.

215. Again let us have the two general equations

$$a x^2 + b y^2 + c x y = d$$
$$a' x^2 + b' y^2 + c' x y = d' \text{ .. (A)}$$

and let it be required to find the values of x and y. Suppose that

$$y = z x$$

that is, instead of y write $z x$, where z is an unknown quantity. Then the equations become

$$a x^2 + b z^2 x^2 + c z x^2 = d$$
$$a' x^2 + b' z^2 x^2 + c' z x^2 = d'.$$

Divide the members of the first equation by those of the other, as in art. [139], and we have

$$\frac{a x^2 + b z^2 x^2 + c z x^2}{a' x^2 + b' z^2 x^2 + c' z x^2} = \frac{d}{d'},$$

and striking the common factor x^2 out of the numerator and denominator of the left-hand equation,

$$\frac{a + b z^2 + c z}{a' + b' z^2 + c' z} = \frac{d}{d'}.$$

Getting rid of the fractions in this last equation, it assumes the general form of a quadratic equation [art. 205], and may be solved accordingly.

When we have found the value of z as we substitute it in equation (A), x will be the only unknown quantity in this equation, and it may therefore be solved. When we know x and z we find y from the equation

$$y = z x.$$

If we observe the two original equations, we shall see that the sum of the indices of x and y is the same in every term where they occur. It was for this reason that the substitution $y = z x$ was had recourse to, which would otherwise have been of no use, as the success of the method depends upon the same power of x appearing in the several terms, so that x may disappear on the reduction of the fraction. In all cases where the same condition is fulfilled whatever be the number of equations and unknown quantities, we may by a similar method of substitution reduce the number of both by one. Thus take the three equations involving x, y, and z.

$$a x^2 + b y z = \text{A},$$
$$c y^2 + d z x = \text{B},$$
$$e z^2 + f x y = \text{C}.$$

Let $\quad y = u x,$

and $\quad z = v x.$

Substituting $a x^2 + b u v . x^2 = \text{A},$

$$c u^2 . x^2 + d v . x^2 = \text{B},$$
$$e v^2 . x^2 + f u . x^2 = \text{C}.$$

Dividing the second and third by the first, and striking out x^2, which will be common to the numerators and denominators of both fractions, we have

$$\frac{c u^2 + d v}{a + b u v} = \frac{\text{B}}{\text{A}},$$
$$\frac{e v^2 + f u}{a + b u v} = \frac{\text{C}}{\text{A}},$$

and getting rid of the fractions by multiplication we have two quadratic equations involving u and v, which can be found accordingly. Substituting, then, for y and z, $u x$, and $v x$, respectively, in the first of our equations, we shall have a quadratic equation involving only x.

Of Negative Exponents.

216. We have seen [art. 35] that if we subtract unity from the exponent of any power of a, we form the quotient of that power of a divided by a itself. Thus

$$a^{m-1} = \frac{a^m}{a},$$
$$a^{3-1} \text{ or } a^2 = \frac{a^3}{a},$$
$$a^{2-1} \text{ or } a^1 = \frac{a^2}{a}.$$

Now, from the nature of the expression a^m, the subtraction of unity from its index, as long as that index is greater than 1, is the same as dividing it by a. In fact, the diminution of the index by 1 is the result of the division by a. But when the index is no longer greater than 1, there is no necessary connection between division and subtraction from the index. We may, however, still continue to subtract 1 from the index instead of dividing; considering it, however, not as a result but as a matter of notation. We, in fact, represent the operation of dividing by a, by the subtraction of 1 from its index. Proceeding upon these principles we obtain

$$a^{1-1} = \frac{a^1}{a}, \text{ or } 1;$$

so that since $1 - 1 = 0$, a^0 will represent unity. We are always to regard a^0 as the result of some such fraction

as $\frac{a^n}{a^n}$, in the numerator and denominator of which a is raised to the same power, and such a fraction is always equal to unity. It is immaterial what the quantity a is; $\left(\frac{1}{x}\right)^0$, and $(1+x)^0$, have both the same value as a^0, or 10^0.

217. Again, subtracting unity from the successive exponents, we have

$$a^{0-1}, \text{ or } a^{-1} = \frac{a^0}{a}, \text{ or } \frac{1}{a},$$

$$a^{-1-1}, \text{ or } a^{-2} = \frac{\frac{1}{a}}{a}, \text{ or } \frac{1}{a^2},$$

$$a^{-2-1}, \text{ or } a^{-3} = \frac{\frac{1}{a^2}}{a}, \text{ or } \frac{1}{a^3};$$

[see art. 87.] and, generally,

$$a^{-m-1}, \text{ or } a^{-(m+1)} = \frac{\frac{1}{a^m}}{a}, \text{ or } \frac{1}{a^{m+1}}.$$

We are to consider a^{-m} as the result of such a fraction as $\frac{a^n}{a^{n+m}}$, where the exponent of a in the denominator exceeds by m its exponent in the numerator, and which is, consequently, equivalent to $\frac{1}{a^m}$.

We may thus form two series,

$$a^m, a^{m-1} .. a^2, a^1, 1, \frac{1}{a}, \frac{1}{a^2}, .. \frac{1}{a^m},$$

$$a^m, a^{m-1} .. a^2, a^1, a^0, a^{-1}, a^{-2} .. a^{-m},$$

of which the corresponding terms are the same. In the first of these we find each term by dividing the preceding one by a, agreeably to the notation that we have hitherto used. In the second we form each term by subtracting unity from the exponent of the preceding.

All the powers of any quantity a, which have positive numbers for their exponents, such as a^m, a^2, a, and the like, are called the *direct powers* of a. The reciprocals [art. 96] of these direct powers, such as $\frac{1}{a^m}$, $\frac{1}{a^2}$, $\frac{1}{a}$, are called the *inverse powers* of a [art. 130]. So that the direct powers of every quantity have for exponents the numbers 1, 2, 3, &c. taken positively, and its inverse powers the same numbers taken negatively.

The definition given of exponents in art. [8] confined them to positive numbers. The extension of this definition, which admits negative numbers, may be compared to what we did in arithmetic, when we carried the decimal notation to the right of the units' digit, and so arrived at the method of decimal fractions [art. 154]. In the propriety of these extensions of our notation, and a great many others of a similar kind, we have examples of the law of continuity [art. 116].

218. On the supposition that m and n are whole positive numbers, and consequently indicate direct powers, we have arrived at the four results following:

First [art. 24],

$$a^m \times a^n = a^{m+n}; \ldots [a].$$

Secondly [art. 35],

$$a^m \div a^n = a^{m-n}, \ldots [b]$$

which by art. [217] is equally true, whether m be greater or less than n.

Thirdly [art. 24],

$$(a^m)^n = a^{mn}; \ldots . [c].$$

And fourthly, when m is a multiple of n [art. 193],

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} \ldots . [d].$$

We are now to prove that these four properties hold, whatever the signs of m and n are.

219. First, when m is greater than n [art. 35],

$$a^m \times \frac{1}{a^n} = a^{m-n},$$

that is,

$$a^m \times a^{-n} = a^{m-n} \ldots [a];$$

and, when m is less than n [art. 79],

$$a^m \times \frac{1}{a^n} = \frac{1}{a^{n-m}},$$

that is,

$$a^m \times a^{-n} = a^{-(n-m)}, \text{ or } a^{m-n} \ldots [a].$$

Again, since [art. 88]

$$\frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^m \cdot a^n} = \frac{1}{a^{m+n}},$$

we have

$$a^{-m} \times a^{-n} = a^{-m-n} \ldots [a].$$

The four expressions marked [a] prove the first property to be true, whatever are the signs of m and n.

220. Secondly, [art. 91]

$$a^m \div \frac{1}{a^n} = a^m \cdot a^n = a^{m+n},$$

that is,

$$a^m \div a^{-n} = a^{m+n}, \text{ or } a^{m-(-n)} \ldots . [b].$$

Also [art. 87],

$$\frac{1}{a^m} \div a^n = \frac{1}{a^{m+n}},$$

that is,

$$a^{-m} \div a^n = a^{-m-n} \ldots . [\text{b}].$$

Again [art. 91],

$$\frac{1}{a^m} \div \frac{1}{a^n} = \frac{a^n}{a^m},$$

that is [art. 217],

$$a^{-m} \div a^{-n} = a^{n-m}, \text{ or } a^{-m-(-n)} \ldots [\text{b}].$$

The four expressions marked [b] prove the second property to be true, whether m or n be positive or negative.

221. Thirdly, since [art. 90]

$$\left(\frac{1}{a^m}\right)^n = \frac{1}{a^{mn}},$$

it follows that

$$(a^{-m})^n = a^{-mn} \ldots . [\text{c}];$$

and, since

$$\frac{1}{(a^m)^n} = \frac{1}{a^{mn}},$$

it follows that

$$(a^m)^{-n} = a^{-mn} \ldots . [\text{c}].$$

Again, since [art. 93]

$$\frac{1}{\left(\frac{1}{a^m}\right)^n} = \frac{1}{\frac{1}{a^{mn}}} = a^{mn},$$

it follows that

$$(a^{-m})^{-n} = a^{mn}, \text{ or } a^{(-m)\cdot(-n)} \ldots . [\text{c}].$$

The four expressions marked [c] prove the third property to be true, whatever the signs of m and n are.

222. Fourthly, since [art. 192]

$$\sqrt[n]{\frac{1}{a^m}} = \frac{1}{\sqrt[n]{a^m}},$$

and [art. 193]

$$\frac{1}{\sqrt[n]{a^m}} = \frac{1}{a^{\frac{m}{n}}},$$

when m is a multiple of n; it follows that

$$\sqrt[n]{a^{-m}} = a^{-\frac{m}{n}} \ldots . [\text{d}].$$

The two results marked [d] prove the fourth property to be true, whether m be positive or negative.

Of Fractional Exponents.

223. We have now seen that we can express the different powers, direct and inverse, of any quantity a, by annexing to it different exponents. Let us next consider the different roots of a, and of the powers of a, and let us try to express these quantities in a similar way.

By art. [241], when we multiply the exponent of a quantity by 2, we square the quantity. Now if we represent $\sqrt{a}$ by $a^{\frac{1}{2}}$, such a notation will resemble the other in this at least, that when we multiply the exponent of $a^{\frac{1}{2}}$ by 2, we produce a^1 or a, which is the square of $\sqrt{a}$. A similar reason leads us to represent $\sqrt[3]{a}$ by $a^{\frac{1}{3}}$, $\sqrt[4]{a}$ by $a^{\frac{1}{4}}$, and generally $\sqrt[n]{a}$ by $a^{\frac{1}{n}}$, since if we multiply the exponents of these expressions respectively by 3, by 4, and by n, we obtain in each case a^1 or a, which is the cube of $\sqrt[3]{a}$, the fourth power of $\sqrt[4]{a}$, and the n^{th} power of $\sqrt[n]{a}$. In like manner, $a^{\frac{3}{2}}$ may be used to express $\sqrt{a^3}$, because if we multiply its exponent by 2, we obtain a^3, the square of $\sqrt{a^3}$; and in general, $a^{\frac{m}{n}}$ may be used to express $\sqrt[n]{a^m}$, since $a^{n \cdot \frac{m}{n}} = a^m$, which is the n^{th} power of $\sqrt[n]{a^m}$. Before, however, adopting this notation of fractional exponents, as part of the same system with integral exponents, we must show that it agrees equally well in all other respects with that system.

224. As in art. [198], result [B], we have

$$\sqrt[n]{b} = \sqrt[np]{b^p}.$$

Now instead of b put a^m and we obtain

$$\sqrt[n]{a^m} = \sqrt[np]{a^{mp}};$$

and, if we use the notation of the last article, this becomes

$$a^{\frac{m}{n}} = a^{\frac{mp}{np}}.$$

It follows, that if $\sqrt[n]{a^m}$ be expressed by $a^{\frac{m}{n}}$, it will also be expressed by a with any fraction equivalent to $\frac{m}{n}$ for its exponent. Thus if $\sqrt{a}$ be expressed by $a^{\frac{1}{2}}$, it is also expressed by $a^{\frac{2}{4}}, a^{\frac{3}{6}}, a^{\frac{5}{10}}$, &c.

225. In art. [198] it was shown that

$$\sqrt[n]{a} \cdot \sqrt[n']{a} = \sqrt[nn']{a^{n+n'}};$$

and if we use fractional exponents, this becomes,

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n'}} = a^{\frac{n+n'}{nn'}}, \text{ or } a^{\frac{1}{n}+\frac{1}{n'}}.$$

Sinee

$$\frac{n+n}{n\,n'} = \frac{1}{n} + \frac{1}{n'}.$$

In like manner, since, as in art. [223],

$$\sqrt[n]{a^m} = \sqrt[n'n]{a^{mn'}},$$

and

$$\sqrt[n']{a^{m'}} = \sqrt[nn']{a^{nm'}},$$

we have

$$\sqrt[n]{a^m} \times \sqrt[n']{a^{m'}} = \sqrt[n'n]{a^{mn'}} \times \sqrt[nn']{a^{nm'}};$$

or, as in art [198],

$$\sqrt[n]{a^m} \times \sqrt[n']{a^{m'}} = \sqrt[mm'nn']{(a^{mm'})^{m'n+mn'}}$$
$$= \sqrt[mm'nn']{a^{mm'^2n+m^2m'n'}}.$$

If we use fractional exponents, this last expression becomes,

$$a^{\frac{m}{n}} \times a^{\frac{m'}{n'}} = a^{\frac{mm'^2n+m^2m'n'}{mm'nn'}};$$

or

$$a^{\frac{m}{n}} \times a^{\frac{m'}{n'}} = a^{\frac{m'}{n'}+\frac{m}{n}}$$

In a way very similar to what we have just used we may show that if $a^{-\frac{m}{n}}$ be employed to express $\frac{1}{\sqrt[n]{a^m}}$, then the same expression will be true if one or both of the exponents $\frac{m}{n}$ and $\frac{m'}{n'}$ be negative. It therefore follows, that the first property in art. [218] is true, whether the exponents be positive or negative, whole or fractional.

226. Using the expression in art. [199] in the same way as in the last article we used the expression in art. [198], we should find that,

$$a^{\frac{m}{n}} \div a^{\frac{m'}{n'}} = a^{\frac{m}{n}-\frac{m'}{n'}};$$

and that whether the fractional exponents be positive or negative, and whether $\frac{m'}{n'}$ be a less or a greater fraction than $\frac{m}{n}$. The second property in art. [218], therefore, is also true, whether the exponents be positive or negative, whole or fractional.

227. By art. [196] we have

$$(\sqrt[n]{b})^p = \sqrt[n]{b^p};$$

and, if for b we put a^m, this becomes

$$(\sqrt[n]{a^m})^p = \sqrt[n]{a^{mp}}.$$

Consequently, if we use fractional exponents, we find

$$\left(a^{\frac{m}{n}}\right)^p = a^{\frac{mp}{n}}$$

This can easily be extended to include those cases in which one or both of the quantities $\frac{m}{n}$ and p are negative, and therefore the third property in art. [218] is true for fractional exponents.

228. Again, since by art. [197],

$$\sqrt[p]{\sqrt[n]{b}} = \sqrt[pn]{b},$$

if for b we put a^m, we have

$$\sqrt[p]{\sqrt[n]{a^m}} = \sqrt[pn]{a^m};$$

or, using fractional exponents,

$$\sqrt[p]{a^{\frac{m}{n}}} = a^{\frac{m}{np}},$$

whence

$$\left(a^{\frac{m}{n}}\right)^{\frac{1}{p}} = a^{\frac{m}{np}}.$$

This may, as before, be proved to be true, if one or both of the exponents $\frac{m}{n}$ and $\frac{1}{p}$ be negative. The fourth property in art. [218], therefore, holds in the case of fractional exponents.

229. We have now shown that, so far as we have yet seen, there is no difference between the effect of an operation performed on an integral exponent, and the effect of the same operation performed on a fractional exponent. Integral and fractional exponents are therefore parts of the same system of notation; and in future we shall always use $a^{\frac{m}{n}}$ to express the n^{th} root of the m^{th} power of a, or, what is the same thing, [art. 196], the m^{th} power of the n^{th} root of a.

Observe, that in the first instance a^m was an arbitrary symbol for the m^{th} power of a, that is to say, any other symbol for the same quantity would have been equally proper. But when once we have adopted a^m in this sense, then, proceeding as we have done, a^{-m} and $a^{\frac{m}{n}}$ are not arbitrary symbols, but their signification results from our process, or it would have been improper to have used them to express any thing but $\frac{1}{a^m}$ and $\sqrt[n]{a^m}$.

230. The subject of exponents is thus reduced to the three very simple rules following:

First; *To find the product of any number of powers and roots, direct and inverse, of the same quantity, affix to the quantity, as its exponent, the sum*

of the exponents of the factors, attending to their signs. Thus the product of $\sqrt{2}$, $\sqrt[3]{2}$, $\frac{1}{2^2}$, 2^3 and $\sqrt{2^3}$, where the sum of the exponents is

$$\frac{1}{2} + \frac{1}{3} - 2 + 3 + \frac{3}{2},$$

or $\quad 3\frac{1}{3}, \quad$ or $\frac{10}{3}, \quad$ is $2^{\frac{10}{3}}$,

or $\sqrt[3]{2^{10}}$, or $\sqrt[3]{1024}$, or $1024^{\frac{1}{3}}$.

Secondly; *To divide any power or root, direct or inverse, of a quantity, by any other power or root, direct or inverse, of the same quantity, subtract the exponent of the divisor from that of the dividend, the result is the exponent of the quotient.* Thus the quotient of $\sqrt[3]{3}$, divided by $\sqrt{3}$, where the difference of the exponents is $\frac{1}{3} - \frac{1}{2}$, or $-\frac{1}{6}$, is $3^{-\frac{1}{6}}$, or $\frac{1}{3^{\frac{1}{6}}}$.

Thirdly; *To find any power, or root, of any power, or root of any quantity, multiply the exponent of the quantity by the number which is the exponent of the power or root required.* Thus the cube of $a^{\frac{2}{3}}$ is $a^{\frac{6}{3}}$, or a^2. The square root of $\frac{1}{a^{\frac{4}{3}}}$, or of $a^{-\frac{4}{3}}$, is $a^{-\frac{4}{3}\cdot\frac{1}{2}}$, or $a^{-\frac{2}{3}}$, or $\frac{1}{a^{\frac{2}{3}}}$.

231. These rules contain every thing that it is of much importance to attend to in the arithmetic of surds, but they apply to those surds only in which the same quantity is affected with different exponents. When different surds are to be added to or subtracted from each other, or when the product or quotient of surds in which the quantities affected with fractional indices are not the same is to be taken, we cannot, in most instances, do more than indicate these operations by the proper signs. Sometimes, however, the results of these operations admit of being simplified a good deal.

If we take any whole number N, and reduce it into its prime factors [art. 62], we get a result either of the form a^m, where a is a prime number and m either unity, or some other whole number, or of the form $a^m\, a'^{m'}$, where a and a' are different primes, and m and m' whole numbers, or of the form $a^m . a'^{m'} . a''^{m''}$, or some similar form. Now, if we take the n^{th} root of N, we shall always find it to be a surd, unless all the exponents m, m', m'', &c. are multiples of n. Thus $360^{\frac{1}{2}}$ is a surd, because $360 = 2^3 . 3^2 . 5$; and 3 and 1, two of the exponents, are not multiples of 2.

This may be proved as follows: let us suppose that m, m', m'', &c. are not multiples of n, and still that $N^{\frac{1}{n}}$ is a whole number. Let us reduce this whole number into its prime factors, so that we find

$$N^{\frac{1}{n}} = b^p . b'^{p'} . b''^{p''} \ldots$$

Raising both members of this equation to the n^{th} power, we have

$$N = b^{np} . b'^{np'} . b''^{np''} \ldots$$

But by our supposition,

$$N = a^m . a'^{m'} . a''^{m''} \ldots;$$

where m, m', m'', &c. not being all multiples of n, are some of them different from np, np', np'', &c. which are all multiples of n. We, therefore, have reduced N into two different sets of prime factors, which [art. 61] cannot be done. It follows that $N^{\frac{1}{n}}$ cannot be a whole number.

When m, m', m'', &c. are not multiples of n, we have, by [art. 190],

$$N^{\frac{1}{n}} = a^{\frac{m}{n}} . a'^{\frac{m'}{n}} . a''^{\frac{m''}{n}} \ldots,$$

and if $m = q\,n + r$, $m' = q'\,n + r$, &c. we have, by [art. 194],

$$N^{\frac{1}{n}} = a^{q+\frac{r}{n}}\, a'^{q'+\frac{r'}{n}}\, a''^{q''+\frac{r''}{n}}.$$

$$= a^q . a'^{q'} . a''^{q''} \times \ldots a^{\frac{r}{n}} . a'^{\frac{r'}{n}} . a''^{\frac{r''}{n}} \ldots$$

Thus

$$360^{\frac{1}{2}} = 2^1 . 3^1 . 2^{\frac{1}{2}} . 5^{\frac{1}{2}}, \text{ or } 6 . 10^{\frac{1}{2}}.$$

232. In this way, when we have a number affected with a fractional exponent, we can, at once, find whether it be a surd, and if so we can reduce it into its simplest form. When surds are to be added or subtracted, if, when they are reduced into their simplest forms, the surd parts are the same in each, their sum or difference can be reduced to one surd, as in art. [195]. Again, when surds are to be multiplied or divided by each other, by reducing them into their simplest forms, we can per-

ceive at once whether the result will admit of any material simplification. Thus, to take the product of $216^{\frac{1}{6}}$ by $108^{\frac{1}{4}}$. We find $108 = 2^2 . 3^3$, and therefore $108^{\frac{1}{4}} = 2^{\frac{1}{2}} . 3^{\frac{3}{4}}$. Again, $216 = 2^3 . 3^3$, and therefore $216^{\frac{1}{6}} = 2^{\frac{1}{2}} . 3^{\frac{1}{2}}$. The product sought is therefore $2^{\frac{1}{2}} . 3^{\frac{3}{4}} . 2^{\frac{1}{2}} . 3^{\frac{1}{2}}$, or $2 . 3^{\frac{5}{4}}$, or $2 \times 3 \times 3^{\frac{1}{4}}$, or $6 \times 3^{\frac{1}{4}}$. Similarly to divide $48^{\frac{1}{2}}$ by $72^{\frac{1}{4}}$, we find $48 = 2^4 . 3$, and therefore $48^{\frac{1}{2}} = 2^2 . 3^{\frac{1}{2}}$; so $72 = 2^3 . 3^2$ and $72^{\frac{1}{4}} = 2^{\frac{3}{4}} . 3^{\frac{1}{2}}$. The quotient, therefore, is $\dfrac{2^2 . 3^{\frac{1}{2}}}{2^{\frac{3}{4}} . 3^{\frac{1}{2}}} = \dfrac{2^2}{2^{\frac{3}{4}}} =$ $2^{2-\frac{3}{4}}$ [art. 230] $= 2^{\frac{5}{4}} = 2 . 2^{\frac{1}{4}}$.

233. Since [art. 224] if the exponent of any quantity be a fraction we may substitute for it any equivalent fraction, and since there is a decimal fraction equivalent to every vulgar fraction, we may always use decimal fractions to express the fractional exponents of quantities. Thus $a^{.5}$ is the same as $a^{\frac{5}{10}}$ or $a^{\frac{1}{2}}$. So $a^{.\dot{1}8\dot{5}}$ is the same as $a^{\frac{5}{27}}$, $a^{.\dot{7}1428\dot{5}}$ is the same as $a^{\frac{5}{7}}$, and so on.

When we find for the exponent of a quantity a repeating decimal, or one consisting of many places, we may cut it short, as in art. [166], without introducing any sensible or material error into our result. Thus, if instead of $a^{.\dot{7}1428\dot{5}}$ we write $a^{.714286}$ we make the exponent too great by $.00000029$ nearly. Instead of being equal to $a^{\frac{5}{7}}$ then, our expression is equal to $a^{\frac{5}{7} + .00000029}$ nearly, or to $a^{\frac{5}{7}} \times a^{.00000029}$. Now $a^0 = 1$ [art. 216], and therefore $a^{.00000029}$ differs from unity by a very small quantity, so small that it may be neglected,* and if it be, we have $a^{\frac{5}{7}} . a^{.00000029} = a^{\frac{5}{7}} \times 1$ or $a^{\frac{5}{7}}$.

* At a future part of this treatise we shall inquire into the actual magnitude of such quantities as that in the text, and shall prove rigorously that they may be neglected.

Of Logarithmic Arithmetic.

234. When we have the equation $a^{\frac{m}{n}} = \mathrm{N}$, we have seen that $\frac{m}{n}$ is called the exponent of a. When we speak of $\frac{m}{n}$ however with reference to N, as indicating the power of a that is equal to N, it is called the *logarithm* of N. The exponent of a then, and the logarithm of N, mean the same quantity, and we use the one or the other of these names for it, according as we speak of it with reference to a or to N.

To express what the quantity a is, with respect to which $\frac{m}{n}$ is the logarithm of N, we say that $\frac{m}{n}$ is the logarithm of N *to the base a*. Similarly if $a'^{\frac{m'}{n'}} = \mathrm{N}$, $\frac{m'}{n'}$ would be the logarithm of N to the base a'. Thus, since $9^{\frac{1}{2}} = 3$, $\frac{1}{2}$ or $.5$ is the logarithm of 3 to the base 9. So since $10^2 = 100$, 2 is the logarithm of 100 to the base 10.

Most of the purposes of logarithms are best served when the number 10 is the base. Logarithms to this base are therefore the most frequently used, and are called *common logarithms*; they are always expressed by decimals. When we write

$$\log. 5 = 0.69897,$$

we mean that the common logarithm of 5 is 0.69897; that is, that $5 = 10^{0.69897}$, or $5 = 10^{\log. 5}$.

235. The word logarithms is composed of two Greek words, that respectively mean *ratio* and *number*, so that it literally signifies *the ratios of numbers*. We have seen [art. 129] that a^2 is said to have to unity the duplicate or double ratio of a to unity, a^3 the triplicate or threefold ratio, and so on. The numbers 2, 3, &c. (which are the logarithms of a^2, a^3, &c. to the base a) thus actually express the ratios of a^2, a^3, &c. to unity, as compared with the ratio of a to unity, and are therefore, with propriety, called the logarithms of these quantities.

236. Our present purpose is to explain how logarithms are used to abridge the operations of arithmetic, and we shall reserve till afterwards the

explanation of the way in which the logarithm of any number to any base may be found. We shall therefore suppose that we have got ready formed a table of logarithms of the usual extent; that is a table containing the logarithms of all whole numbers less than 100,000, or that consist of less than six digits. Very few of these can be expressed in terminating decimals, but this causes little inconvenience, since a logarithm carried to six or seven decimal digits is sufficiently exact for all common purposes. [art. 233.]

237. That part of any logarithm which stands to the left of the decimal point, is called the *characteristic* of the logarithm. Thus in the table we find

$$\log. 75293 = 4.8767546,$$

in which, consequently, 4 is the characteristic.

Since $10^0 = 1$, by art. [216], and since $10^1 = 10$, the logarithms of 1 and 10 are respectively 0 and 1. The logarithm of every number between 1 and 10 is therefore between 0 and 1, and so has zero for its characteristic. Thus

$$\log. 2 = 0.3010300.$$

So since $10^2 = 100$, the logarithm of 100 is 2; and the logarithm of every number between 10 and 100, or which has two digits to the left of the decimal point, has unity for its characteristic. Similarly, the characteristic of the logarithm of every number between 100 and 1000 is 2; and generally n is the characteristic of the logarithm of every number between 10^n and 10^{n+1}, or which has $n+1$ digits to the left of the decimal point. To find the characteristic of the logarithm of any number greater than unity we have, therefore, the following rule: *If the number be an integer, the characteristic of its logarithm is the number of digits of which it consists, diminished by unity; if part of the number be a decimal, the characteristic is the number of digits to the left of the decimal point diminished by unity.*

238. Let us suppose that $10^{\frac{m}{n}} = N$. Then, by art. [225], $10^{\frac{m}{n}+1} = 10N$, $10^{\frac{m}{n}+2} = 100N$, and generally $10^{\frac{m}{n}+p} = 10^pN$. That is to say, that $\frac{m}{n}$, which is the logarithm of N, becomes that of 10N by adding unity to it, of 100N by adding 2 to it, and so on. The logarithms of the products of a number by 10 and all the powers of 10, therefore, differ from the logarithm of the number itself only in their characteristics. Thus, since

$$\log. 24 = 1.3802112,$$

it follows that

$$\log. 240 = 2.3802112,$$
$$\log. 2400 = 3.3802112,$$

and so on.

239. From the last two articles it follows, first, that a table of logarithms need not contain their characteristics, since the rules for finding these are so simple; and, secondly, that a table of the logarithms of all the whole numbers, from 10,000 to 100,000, is sufficient to furnish the logarithms of all the whole numbers less than 100,000. Thus the logarithm of 2598 is found by prefixing 3, the proper characteristic, to the fractional part of the logarithm of 25980. So, since

$$\log. 80000 = 4.9030900,$$

we have

$$\log. 8 = 0.9030900.$$

Accordingly, the tables of logarithms most in use contain the fractional parts only of the logarithms of all the whole numbers between 10,000 and 100,000, or of all the whole numbers that consist of five digits, leaving the characteristics to be supplied.

240. If we examine several consecutive logarithms in any part of the table, we shall find that they consist of numbers that are very nearly in arithmetical progression. For example, we find

$$\log. 41337 = 4.6163390,$$
$$\log. 41338 = 4.6163495,$$
$$\log. 41339 = 4.6163600,$$
$$\log. 41340 = 4.6163705;$$

where each logarithm is formed by adding .0000105 to the preceding one. This goes on till we come to log. 41387, which is formed by adding .0000104 only to the preceding one. Thus 49 successive logarithms, from log. 41337 up to log. 41386, are in arithmetical progression. The difference of any two successive logarithms, from log. 41387 up to log. 41777, is either .0000105 or .0000104, being sometimes the one of these numbers, and sometimes the other. From log. 41337 then to log. 41777, or for 330 numbers in succession, the dif-

ferences of the logarithms are either .0000105 or .0000104.

This regularity in the table enables us to use it for finding the logarithms of all whole numbers between 100,000 and 1,000,000, consisting of six digits. For example, if we add unity to the characteristics of log. 41337 and log. 41338 we find

$$\log. 413370 = 5.6163390,$$

and

$$\log. 413380 = 5.6163495,$$

where, as before, the difference is .0000105. Now, according to the law which we have just observed, the logarithms of the numbers between 413370 and 413380 must be in arithmetical progression, and must, consequently, consist of nine arithmetical means between 5.6163390 and 5.6163495: and by art. [143] the common difference of the progression which these means form will be

$$\frac{.0000105}{10}, \text{ or } .00000105,$$

since .0000105 is the difference of the first and last terms. Accordingly, if we add this quantity to log. 413370, we obtain log. 413371; if we add twice this quantity we obtain log. 413372, and so on. Thus, to find log. 413375, we have

$$\log. 413375 = 5.6163390 + 5 \times 00000105$$
$$= 5.6163443,$$

cutting down the result to seven decimal digits.

In the example just given, if D be the difference between the two logarithms, the number to be added to the least of them to produce the logarithm required is $\frac{5\,D}{10}$. Similarly in any other case where n is the units' digit of the number of six digits, $\frac{n\,D}{10}$ is the number to be added to the logarithm of the first five digits.

241. In the same way we may use the table to find the logarithms of numbers consisting of seven digits. If, as before, D be used to denote the difference between the logarithm of the first five digits of such a number and the logarithm next above it in the table, and if n be the number consisting of the last two digits of the number proposed; then, just as before, the number to be added to the logarithm of the first five digits is $\frac{n\,D}{100}$. Thus, if the number proposed be 8561427, we find

$$\log. 85614 = 4.9325448,$$
$$\log. 85615 = 4.9325499.$$

The difference of these logarithms is .0000051. The last two digits of our number are 27, and $\frac{27 \times .0000051}{100}$ $= .000001377$. Reducing this result to seven digits it becomes .0000014, and adding this last number to the first logarithm, and prefixing the proper characteristic, we find

$$\log. 8561427 = 6.9325462.$$

We may observe, that in like manner we should find that the number to be added to the first logarithm to form the logarithm of 8561428 ought to be .0000014 also; so that

$$\log. 8561428 = 4.9325462,$$

the same as before. It follows that the first seven decimal digits of the logarithms of the numbers, 8561427 and 8561428, are the same; so that if we wish to distinguish them, we must use a table that contains a greater number of decimal digits than seven.

242. The rule for finding the logarithm of a number that consists of six or seven digits is therefore this: *Find in the table the logarithm of the first five digits; find the difference between this logarithm and the next greater one in the table; if the number proposed consist of six digits, multiply the difference by the units' digit of the number, and divide the product by 10; or if the number proposed consist of seven digits, multiply the difference by the last two digits of the number, and divide the product by 100; adding the quotient in either case to the logarithm first found: the sum with its proper characteristic is the logarithm required.*

In the margin of all tables of logarithms, the difference of the successive logarithms in that part of the tables is set down. In some tables also there are little tables in the margin, called *tables of proportional parts.* These are placed under every successive difference; and contain for that difference the number to be added in respect of each units' digit, so as to form the logarithms of numbers of six digits. These numbers are found as is directed by the rule given above.

243. We can now find the logarithms of all whole numbers of six or seven digits or under. It is easy to deduce from them the logarithms of all numbers greater than unity that consist of six or seven or fewer digits, of which any part is a decimal. For instance, since $873043 = 10000 \times 87.3043$, the logarithm of 873043 differs from that of 87.3043 only in its characteristic, [art. 238]; also the characteristic of the logarithm of 87.3043 is unity [art. 237]. Now

$$\log. 873043 = 5.9410356,$$

and therefore

$$\log. 87.3043 = 1.9410356.$$

In the same way the logarithm of any other number greater than unity, and consisting of six or seven digits, may be found; the rule is this: *Find in the table the logarithm of the number proposed considered as a whole number; prefix the proper characteristic to the decimal part of this logarithm; the result is the logarithm sought.*

244. With respect to the logarithms of numbers less than unity, we must observe that the logarithm of every such number is negative. For as we have seen [art. 216] the logarithm of unity is zero, and since $.1$ or $\frac{1}{10} = 10^{-1}$, the logarithm of $.1$ is -1. Therefore the logarithm of every number between unity and $\frac{1}{10}$, or between unity and $.1$, is between zero and -1. Again, since $.01$, or $\frac{1}{100} = 10^{-2}$, the logarithm of $.01$ is -2, and, therefore, the logarithm of every number between $\frac{1}{10}$ and $\frac{1}{100}$, that is, between $.1$ and $.01$, is between -1 and -2. Similarly, the logarithm of every number between $.01$ and $.001$ is between -2 and -3; and so on.

Let us now propose to find the logarithm of such a number as 0.7434. This number is equal to $\frac{7434}{10000}$, or $\frac{7434}{10^4}$. Now we find

$$\log. 7434 = 3.8712226,$$

and therefore

$$7434 = 10^{3.8712226};$$

consequently

$$.7434 = \frac{10^{3.8712226}}{10^4}.$$

By art. [226] this last number becomes $10^{3.8712226-4}$, or $10^{-0.1287774}$. Since $3.8712226 - 4 = -.1287774$, it follows that $\log. .7434 = -.1287774$, a negative number.

We shall soon find that it is much more convenient to express negative logarithms, such as this, in a way similar to that used in art. [22]. Thus

$$\begin{aligned} 3.8712226 - 4 &= .8712226 - 4 + 3 \\ &= .8712226 - 1 \\ &= \bar{1}.8712226. \end{aligned}$$

So that

$$10^{3.8712226-4} = 10^{\bar{1}.8712226},$$

whence

$$\log. .7434 = \bar{1}.8712226,$$

where the characteristic alone is negative. Similarly to find the logarithm of $.0263573$. Since $.0263573 = \frac{263573}{10000000}$, or $\frac{263573}{10^7}$, its logarithm must be that of 263573, with 7 subtracted from its characteristic. Now we find

$$\log. 263573 = 5.4209010,$$

and therefore

$$\log. .0263573 = \bar{2}.4209010.$$

If we always bear in mind that such an expression as $\bar{2}.4209010$ means $.4209010 - 2$, this notation cannot cause any confusion.

245. The logarithm of every number between unity and $.1$ being, as we have seen, between zero and -1, is equal to -1 with some number less than unity added to it. Its negative characteristic is therefore $\bar{1}$. Similarly, since the logarithm of every number between $.1$ and $.01$ is between -1 and -2, it is equal to -2 with some number less than unity added to it; its negative characteristic is therefore $\bar{2}$. Similarly, the negative characteristic of every number between $.01$ and $.001$ is $\bar{3}$, and so on. In general, *The characteristic of the logarithm of a number less than unity is found by affixing the negative sign to the number which is greater by unity than the number of zeros which are between the decimal point and the first*

significant digit of the number proposed.

246. To find the logarithm of any number which consists of six, or seven, or fewer digits, and which is less than unity; *find in the table the logarithm of the digits of which the number is composed, as if they formed a whole number, and affix to this logarithm the proper characteristic.* Thus, to find the logarithm of 0.001865, we have

$$\log.\ 1865 = 3.2706788,$$

therefore

$$\log..\ 0001865 = \bar{4}.2706788.$$

This rule follows from art. [244].

247. All that has gone before is on the supposition that it is a positive number whose logarithm is sought. If we now endeavour to find the logarithm of a negative number we are led into a curious inquiry. In art. [200] it was shown that every root indicated by an even number ought to have the double sign affixed to it, thus $a^{\frac{1}{2}}$ and $10^{\frac{1}{24}}$, are each of them equal to two quantities, of which the one is positive and the other negative. In like manner, the number represented by $a^{\frac{m}{2n}}$, or $(a^m)^{\frac{1}{2n}}$ ought to have the double sign; so that, in general, whenever the exponent of a number is a fraction whose denominator is an even number, there result two numbers the one positive and the other negative. Thus, in the common logarithms, $\frac{1}{2}$, or $.5$ is the logarithm of the positive square root of 10, or of 3.162278, it is therefore the logarithm of -3.162278.

But as the subject of the logarithms of negative numbers is of no practical importance, we shall not pursue it further here.*

248. Having now explained how the logarithm of any number, whole or decimal, consisting of six, or seven, or fewer digits may be found in the table, it remains to show how the number corresponding to any given logarithm containing seven decimal digits is to be found.

If we find the given logarithm in the table, nothing is necessary further than to take the corresponding number, and place in it properly the decimal point. This is done by means of the characteristic of the logarithm, as is explained in arts. [237] and [245]. Thus if the logarithm be $\bar{2}.6976826$, we find in the table

$$\log.\ 49852 = 4.6976826\ ;$$

the characteristic $\bar{2}$ shows that there are to be two zeros to the left; the number sought, therefore, is 0.049852.

Let us suppose, however, that the given logarithm is 3.8447592, a number which is not in the table. We find in the table

$$\log.\ 6994.5 = 3.8447567,$$

and

$$\log.\ 6994.6 = 3.8447629.$$

These two logarithms contain between them the logarithm proposed and their difference is $.0000062$. The difference between the number proposed and the least of these logarithms is $.0000045$. Now, if we call the first of these differences D, and the second d, we have seen [art. 241] that

$$d = \frac{n\,\mathrm{D}}{100},$$

where n is the last two digits of the number sought. Therefore

$$n = 100\,\frac{d}{\mathrm{D}}.$$

In our example we have

$$n = 100\,\frac{0000045}{0000062} = 7258.$$

We thus find 7 and 3 for the sixth and seventh digits of the number required, so that

$$\log.\ 6994.573 = 3.8447592.$$

The rule deduced from this example is as follows: *Find in the table two successive logarithms which include between them the given logarithm; find the difference between these two logarithms, and also the difference between the least of them and the given logarithm; mul-*

* The existence of the logarithms of negative numbers was strenuously denied by many eminent mathematicians of the last century, and as strenuously asserted by others. Every discussion on this subject appears to depend entirely upon the definition of a logarithm made use of; an arithmetical one excluding the notion of the logarithm of a negative number, and an algebraical one admitting it. Thus if $10^{\frac{1}{2}}$ means only that number which results from extracting arithmetically the square root of 10, it is evident that no negative number can arise; but if $10^{\frac{1}{2}}$ represents that quantity which, when squared, is equal to 10, then it is equally evident that we shall have a negative as well as a positive number. We may also observe that, by the same method of reasoning, though we cannot enter into it here, we may establish the existence of logarithms of impossible as well as possible quantities.

tiply the second of these differences by 100, and divide the product by the first of them; place the first two digits of the quotient to the right of the number corresponding to the least of the two logarithms found in the table; the result, with the decimal point, if necessary, properly placed as indicated by the characteristic of the given logarithm, is the number sought.

When the characteristic of the given logarithm requires a greater number of digits to the left of the decimal point than there are in the number found as the rule directs, we must make up the deficiency by adding a sufficient number of zeros to the right. Thus, suppose that the given logarithm were 10.7543756, we find

$$\log.\ 5.680357 = .7543756.$$

Now the characteristic 10 shows that the number sought has 11 digits; it therefore is 56803570000.

249. By means of a table of logarithms the operations of arithmetic are very much abridged. Multiplication and division are performed by addition and subtraction, the powers and roots of numbers are found by multiplication and division.* This use of the tables is what we are now to explain.

By the first rule in art. [230] the exponent of the product of two or more factors is found by taking the sum of the exponents of the factors. In other words, the logarithm of the product of any numbers is the sum of the logarithms of the numbers. Thus, by our definition [art. 224] we have

$$b = 10^{\log. b}$$

and

$$b' = 10^{\log. b'}.$$

Therefore

$$b\,.\,b' = 10^{\log. b + \log. b'},$$

so that, again, by our definition,

$$\log.\ b\,b' = \log.\ b + \log.\ b'.$$

This gives us the rule for taking the product of numbers by means of a table of logarithms: *Find the logarithms of the several factors taken positively, and add them together, attending to the signs of their characteristics; find in the table the number corresponding to this sum, and affix to it the negative or the positive sign, according as an odd or an even number of the factors are negative; the result is the product sought.* Thus to find the product of 17.934, -0.077692, and 0.3257; we have

$$\begin{aligned}
\log.\ 17.934 &= 1.2536772,\\
\log.\ \ .077692 &= \bar{2}.8903763,\\
\log.\ \ .3957 &= \bar{1}.5973661,\\
\hline
\log.\ \ .5513401 &= \bar{1}.7414196.
\end{aligned}$$

In adding these three logarithms we consider $\bar{2}.8903763$ as if it were written $.8903763 - 2$ [art. 243]. The product must be negative, since one of the factors is negative; accordingly $-.5513401$ is the product sought, for it is the number corresponding to the sum of the logarithms taken negatively.

250. By the second rule in art. [230], the exponent of the quotient of two powers of the same quantity is found by subtracting the exponent of the divisor from that of the dividend. The logarithm of a quotient is therefore the difference of the logarithms of the dividend and divisor. For, since

$$b = 10^{\log b},$$

and

$$b' = 10^{\log b},$$

we have

$$\frac{b}{b'} = 10^{\log. b - \log. b'};$$

that is

$$\log. \frac{b'}{b'} = \log. b - \log. b'.$$

It follows that to divide one number by another we have the following rule: *Find the logarithms of the two numbers taken positively, and subtract that of the divisor from that of the dividend; the number in the table corresponding to the difference, with the proper sign affixed to it, is the quotient sought.* Thus to divide -0.077698 by -0.13976 we have

$$\begin{aligned}
\log.\ .077698 &= \bar{2}.8904098,\\
\log.\ .13976 &= \bar{1}.1453829,\\
\hline
\log.\ .5559384 &= \bar{1}.7450269.
\end{aligned}$$

Here we subtract $(.1453829 - 1)$ from $(.8904098 - 2)$, and we find for their difference $(.7450269 - 1)$, that is $\bar{1}.7450269$. This is the logarithm of $.5559384$, which is the quotient sought. The sign is positive, because both dividend and divisor were negative [art. 38].

* See the Preliminary Treatise, p. 9.

The logarithm of a vulgar fraction is found by subtracting the logarithm of its denominator from that of its numerator. We can find the decimal corresponding to this logarithm, and, where the numbers are large, this is the best way of reducing a vulgar fraction into a decimal.

We may here remark, that since

$$\log.\frac{a}{b} = \log.a - \log.b,$$

we have, arranging the terms differently,

$$\log.\frac{a}{b} = -(\log.b - \log.a)$$
$$= -\log.\frac{b}{a}.$$

251. By combining the last two rules we can perform at once an operation, that, by the common rules of arithmetic, would require both multiplication and division, as, for instance, the rule of three. Thus, when we have the proportion

$$a : b :: a' : b',$$

which gives [art. 133] the equation

$$b' = \frac{a'b}{a};$$

by [art. 250] we find

$$\log.b' = \log.(a'b) - \log.a,$$

or

$$\log.b' = \log.a' + \log.b - \log.a.$$

For example, let it be required to find a number which shall have to 93 . 7624 the same proportion as 272 . 396 has to 357 . 698, we have

	log. 272 . 396	= 2 . 4352007
add	log. 93 . 7624	= 1 . 9720287
		4 . 4072294
sub.	log. 357 . 698	= 2 . 5535165
	log. 71 . 40241	= 1 . 8537129

The answer is 71 . 40241.

252. By the third rule in art. [230], any given power of a quantity is found by multiplying its exponent by the number indicating the given power. In other words, the logarithm of any given power of a quantity is found by multiplying the logarithm of the simple quantity by the number indicating the power in question. Thus, since

$$b = 10^{\log.b},$$

it follows that

$$b^n = 10^{n(\log.b)};$$

so that

$$\log.(b^n) = n\log.b.$$

To raise a number to any given power, then, the rule is this: *Find the logarithm of the given number, and multiply it by the number indicating the given power; the number corresponding to the product, with its proper sign, is the power sought.* Thus, to find the twentieth power of . 996, we have

$$\log..996 = \bar{1}.9982593$$
$$\underline{\quad 20}$$
$$\log..9229666 = \bar{1}.9651860.$$

We multiply (.9982593 − 1) by 20, we find for product $\bar{1}$. 9651860, and this is the logarithm of . 9229666, which is the number sought.

So to find the thirtieth power of 2;

$$\log.2. = .3010300$$
$$\underline{\quad 30}$$
$$\log.1073741000 = 9.0309000,$$

the result is 1073741000, which of course is only an approach to the truth, and is not correct beyond the first six digits.

253. By the third rule in art. [230], the root of any quantity is found by dividing its exponent by the number indicating the root in question; that is to say, the logarithm of any root of a quantity is found by dividing the simple quantity by the number indicating the root. As in the last article, we should find

$$\log.\left(b^{\frac{1}{n}}\right) = \frac{\log.b}{n}.$$

Therefore, to find any root of a given number, we have this rule: *Divide the logarithm of the given number by the number indicating the given root; the number corresponding to the quotient is the root sought.* Thus to find the cube root of 10, we have

$$3)1.0000000 = \log.10$$
$$.3333333 = \log.2.15442;$$

so that 2 . 15442 is the number sought.

When the characteristic of the logarithm is negative, and is not a multiple of the number indicating the given root, there is a slight change to be made in the logarithm before this division is performed. Let it be required, for instance, to find the square root of 0.006543. We find

$$\log..006543 = \bar{3}.8157769.$$

Before dividing this logarithm by 2, we

must observe that it is equivalent to $.8157769-3$, that is, to $1.8157769-4$. The half of this is $.9078885-2$, or $\bar{2}.9078885$, and we find

$$\log.\ .0808888 = \bar{2}.9078885.$$

So that $\pm .0808888$ is the root sought. If we had divided the original logarithm by 2, without this preparation, we should have had our result disturbed by a negative decimal digit proceeding from the -3.

In like manner, to find the value of $.2^{\frac{5}{6}}$, we must multiply the logarithm of $.2$ by 5, and divide the product by 6;

$$\begin{array}{r} \log.\ .2 = \bar{1}.3010300 \\ 5 \\ \hline \bar{4}.5051500. \end{array}$$

This last logarithm is the same as $.5051500-4$, or as $2.5051500-6$, the sixth part of which is $.4175250-1$, or $\bar{1}.4175250$. Now

$$\log.\ .261532 = \bar{1}.4175250.$$

So that $\pm .261532$ is the number sought.

254. An equation, such as

$$a^x = b,$$

where the unknown quantity is an exponent, is called an *exponential equation*. It is solved at once by means of a table of logarithms; for since the logarithms of equal quantities are equal, we have

$$\log. (a^x) = \log. b;$$

but [art. 242]

$$\log. (a^x) = x \log. a;$$

therefore

$$x \log. a = \log. b;$$

whence

$$x = \frac{\log. b}{\log. a}.$$

Thus, to solve the equation

$$2^x = 1976,$$

or to find the number indicating that power of 2 that is equal to 1976, we have

$$x = \frac{\log. 1976}{\log. 2};$$

or

$$x = \frac{3.2957869}{.3010300};$$

that is,

$$x = 10.9483669.$$

We may observe that, since $2^{10.9483669} = 1976$, the number 10.9483669 is the logarithm of 1976 to the base 2. Multiplying then the logarithm of the number by $\frac{1}{\log. 2}$ (log. 2 being calculated to base 10) the result is the logarithm of the number to base 2. This we shall see is only a particular case of a general rule. We may, in like manner, by means of a table of common logarithms, find the logarithm of any number to any base whatever. This is done by solving, as above, the equation

$$a^x = b,$$

where a is the given base, and b the number.

It will be observed that some of the rules which we have given for the use of logarithmic tables are deduced merely by observing the way in which the logarithms succeed each other. This is not a strict method of proof. A more regular demonstration of the accuracy of these rules will be given in a subsequent chapter, in which we shall treat further of exponential quantities, such as a^x, and show how a table of logarithms may be formed.

Of Permutations and Combinations.

255. The *combinations* of any number of things are the different parcels that can be taken, each consisting of a certain number of those things, without regard to the order in which they stand in the parcels. Thus,

ab, ac, ad, bc, bd, cd,

are different combinations of the four letters a, b, c, and d, taken two at a time. Similarly,

abc, abd, acd, bcd,

are different combinations of the same letters, taken three at a time, since every two of these parcels consist of different letters. But the parcels

cba, dba, dca, dcb,

are respectively the same combinations as the last, for they consist respectively of the same letters, though these letters are differently arranged.

256. The *permutations*, again, of any number of things, are the different ways in which those things may be presented by varying the order in which they stand; or the different ways in which the different combinations, each consisting of a certain number of those things, may be presented by varying the order in which the things stand in each

combination. Thus,

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba,$$

are permutatious of the three letters a, b, and c, taken all at once. Similarly,

$$ab,\ ba,\ ac,\ ca,\ bc,\ cb,$$

are different permutations of the same letters, taken two at a time, formed by varying the order of the letters in each of the combinations

$$ab,\ ac,\ bc.$$

257. Let us propose to find how many permutations can be made of any number of different things, taken two at a time. Let the things, for instance, be the six letters $a, b, c \ldots f$. It is plain that if we write a, followed by each of the other five letters, b followed by each of the other five letters, and so on, we shall have formed the number of permutations required, and no more; since every two of the permutations so formed are different. Now, when we have done this, we have five permutations beginning with a, five with b, five with c, and so on; that is, five with each of the six letters. The whole number, therefore, is 6×5, or 30.

In exactly the same way we may show that the number of permutations of five things, taken two at a time, is 5×4, or 20; and that the number of permutations of m things, taken two at a time, is

$$m.(m-1).$$

Again; let us propose to find the number of permutations of any number of things, taken three at a time. Taking the six letters as before, if we write a, followed by all the permutations of the other five letters, taken two at a time, then do the same for b, then the same for c, and so on, it is plain, as before, that we shall have formed all the permutations required. Now, considering the class of these permutations that begins with a, we have a followed by all the permutations of five letters taken two at a time, that is, as we have seen, by 5×4 permutations. The number of permutations in the class beginning with b is in like manner 5×4, and so on. There are therefore six classes, each consisting of 5×4 permutations, and therefore the whole number of permutations is $6.5.4$, or 120.

Similarly, the number of permutations of five things, taken three at a time, is $5.4.3$, or 60; and of m things taken three at a time, is

$$m.(m-1).(m-2).$$

Once more, to find the number of permutations of any number of things, taken four at a time. Taking the six letters; in the class beginning with a, that letter is followed by all the permutations of the other five letters taken three at a time, therefore the number of permutations of this class is, as we have seen, $5.4.3$. As before, there are six classes, and therefore the number sought is $6.5.4.3$, or 360.

In like manner the number of permutations of m things, taken four at a time, is

$$m.(m-1).(m-2).(m-3).$$

We may carry on this process, step by step, as far as we please. From the instances already given, however, we may conclude, that the number of permutations of m different things, taken n at a time, is

$$m.(m-1).(m-2)\ldots(m-n+2).\\(m-n+1);$$

an expression in which the blank must be filled up with all the numbers, (if any,) between $m-2$ and $m-n+1$. Thus, if m be six, and n four, this expression gives us, as above, $6.5.4.3$, or 360.

For examples: the number of permutations of 12 things, taken 5 at a time, is

$$12.11.10.9.8,$$

or 95040. The number of permutations of the 26 letters of the alphabet, taken 6 at a time, is

$$26.25.24.23.22.21,$$

or 165765600.

258. When n is made equal to m, this expression becomes

$$m.(m-1).(m-2)\ldots\\(m-m+2).(m-m+1),$$

or

$$m.(m-1).(m-2)\ldots3.2.1;$$

since

$$m-m+1=1$$
$$m-m+2=2,$$

and so on. This, therefore, is the number of permutations of m things taken m at a time; that is, the number of ways in which the whole number of things may be arranged. For example, the number of ways in which the eight letters in the word *Scotland* can be written is

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

or 40320.

259. The number of things remaining the same, when we increase the number of them taken at a time we increase the number of permutations. This appears from the expression art. [257]. There is, however, one exception, and that is, that the number of permutations of m things taken $m-1$ at a time is the same as when they are taken m at a time. In these respective cases the expression becomes

$$m \cdot (m-1) \cdot (m-2) \ldots 4 \cdot 3 \cdot 2,$$

and

$$m \cdot (m-1) \cdot (m-2) \ldots 3 \cdot 2 \cdot 1,$$

and these expressions are equal. For example, the number of permutations of 9 things taken 8 at a time is 362880, and the number when they are taken 9 at a time is the same. The reason of this is plain. Suppose that we have formed all the permutations of m letters taken $m-1$ at a time. A certain class of these does not contain the letter a, to all these let us prefix that letter. In like manner, let us prefix the letter b to the class that does not contain it, and so on. When we have done this, we have, without increasing their number, changed the permutations of m things taken $m-1$ at a time into the permutations of the same things taken m at a time.

260. Let us now propose to find how many different combinations can be made of any number of things of which a certain number are taken at a time. In the first instance, let us suppose that there are eight things, and that they are to be taken four at a time. Let us call the nnmber of their combinations, which, as yet, we do not know, x. Now, if we take any one of these x combinations, and arrange the four things, of which it consists, in every possible way, we shall form, as we have shown in art. [258], $4 \cdot 3 \cdot 2 \cdot 1$ different permutations of them. Again, if we take any other of the x combinations and treat it in the same way, we shall form of it, in like manner, $4 \cdot 3 \cdot 2 \cdot 1$ permutations; and these are all different from the former ones, since they are composed of a different set of things. In the same way we may form $4 \cdot 3 \cdot 2 \cdot 1$ permutations of every one of the x combinations, and when we have done so, we shall have formed

$$x \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

permutations in all. But these are plainly all the permutations that can be made of the eight things taken four at a time; and we know [art. 257] that the number of permutations of eight things taken four at a time is

$$8 \cdot 7 \cdot 6 \cdot 5.$$

We, therefore, have

$$x \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8 \cdot 7 \cdot 6 \cdot 5;$$

whence

$$x = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1680}{24}, \text{ or } 70.$$

261. Just in the same way we may find the number of combinations of m things taken n at a time. Calling the number, as before, x; every one of these x combinations furnishes us with

$$n \cdot (n-1) \ldots 3 \cdot 2 \cdot 1$$

different permutations of the n things of which it consists, taken n at a time [art. 258]; and, therefore, the whole x combinations furnish us with

$$x \cdot n \cdot (n-1) \ldots 3 \cdot 2 \cdot 1$$

such permutations. As before, these are all the possible permutations of the m things taken n at a time, and are, therefore, in number [art. 257]

$$m \cdot (m-1) \ldots (m-n+2) \cdot (m-n+1),$$

As before, we have the equation

$$x \cdot n \cdot (n-1) \ldots 3 \cdot 2 \cdot 1$$
$$= m \cdot (m-1) \ldots (m-n+2) \cdot (m-n+1);$$

And from this we find

$$x = \frac{m \cdot (m-1) \ldots (m-n+2) \cdot (m-n+1)}{n \cdot (n-1) \ldots \quad 2 \quad \cdot \quad 1},$$

or inverting the order of the factors in the denominator

$$x = \frac{m}{1} \cdot \frac{m-1}{2} \cdot \frac{m-}{3} \ldots$$
$$\frac{m-n+2}{n-1} \cdot \frac{m-n+1}{n}$$

Thus the number of combinations of 12 things taken 5 at a time is, by this expression,

$$\frac{12}{1} \cdot \frac{11}{2} \cdot \frac{10}{3} \cdot \frac{9}{4} \cdot \frac{8}{5},$$

or 792.

Observe that this expression consists of the product of a set of fractions, the denominators consisting of the natural numbers, and each numerator formed by subtracting the denominator from $m+1$.

262. The expression in the last article, being the number of combinations of a certain number of things taken a certain number at a time, must always furnish us with a whole number. This also appears from the nature of that expression. Thus, since either m, or $m-1$, must be an even number, $\frac{m.(m-1)}{1.\ 2}$ must be a whole number. [art. 62]. Again, either m, $m-1$, or $m-2$, must be measured by 3, and one, at least, of these numbers is even, and, therefore, $\frac{m.(m-1).(m-2)}{1\ .\ 2\ .\ 3}$ is a whole number. Similarly, either m, $m-1$, $m-2$, or $m-3$, is measured by 4, and some one of them different from the one measured by 4 is measured by 2; also, some one of them is measured by 3, and, therefore, their product is measured by $1.2.3.4$. In the same way it can be shown, that the numerator of the general expression is measured by $1.2.3\ldots(n-1).n$.

263. Suppose that we have got ten things, and proceed to form all the combinations of those things taken seven at a time. To form one of these combinations we select seven of the ten things, and, consequently, reject three; the three rejected ones form one of the combinations of the ten things, taken three at a time. In like manner every one of the combinations of the ten things taken seven at a time has corresponding to it a combination of the same things taken three at a time; and when, by selecting seven of the things in every possible way, we have formed all the combinations of them taken seven at a time, we have also rejected three of the things in every possible way, and so have formed all the combinations of them taken three at a time. It follows, that the number of combinations of ten things taken seven at a time is the same as the number of combinations of the same things taken three at a time.

Let us see how this property follows from the expression in art. [261]. By that the number of combinations of ten things taken three at a time is

$$\frac{10.9.8}{1.2.3},$$

and taken seven at a time is

$$\frac{10.9.8.7.6.5.4}{1.2.3.4.5.6.7}.$$

Now in the last of these expressions, $4.5.6.7$, is a factor, both in the numerator and denominator, we may, therefore, strike it out of both, and we find

$$\frac{10.9.8}{1.2.3},$$

which is the first expression.

264. Similarly, since, in forming every one of the combinations of m things taken n at a time, we reject $m-n$ of the things, it follows that the number of combinations of m things taken n at a time is the same as the number of combinations of the same things taken $m-n$ at a time. To find the latter of these numbers, we must observe that the last factor in the numerator of the expression in art. [261] becomes, in this case, $m-(m-n+1)$, or $n+1$, so that that expression becomes

$$\frac{m.(m-1)\ldots.(m-n+1).(m-n)\ldots(n+2).(n+1)}{1.\ \ 2\ \ \ldots.\ \ \ n\ \ .(n+1)\ldots(m-n-1).(m-n)}.$$

Here all the numbers, from $n+1$ to $m-n$ inclusive, are factors, both in the numerator and denominator. Striking these factors out, the expression becomes

$$\frac{m.(m-1)..(m-n+1)}{1.\ \ 2\ \ ..\ \ \ n};$$

and this is the expression for the number of combinations of m things taken n at a time.

265. There is but one combination of m things taken m at a time, since there is but one parcel that can be formed of them all. Again, strictly speaking, there is one combination of m things taken none at a time, since there is one way of rejecting them all. Attending to these observations, we may form the following table, showing, with respect to any number of things from one to ten inclusive, how many combinations can be made of that number of things, any number of them being taken at a time.

1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

The first line has reference to one thing, the second to two things, the third to

three, and so on. The first column contains the combinations when the things are taken none at a time, the second when they are taken one at a time, the third two at a time, and so on. This table is sometimes called the *arithmetical triangle*. The numbers contained in it are possessed of many remarkable properties.

266. In each line of the table the numbers equally distant from the ends are the same. When the line has reference to an even number, there is a single number in the middle of it greater than any of the rest. Thus, in the middle of the sixth line we find 20, the number of combinations of six things taken three at a time; it is different from the numbers on each side of it, which are the numbers of combinations of six things taken two at a time and four at a time, and which are equal by art. [264]. When the line has reference to an odd number, we find a pair of equal numbers in the middle of it. Thus, in the middle of the ninth line we find 126 and 126, the numbers of combinations of nine things taken four at a time and five at a time. All this follows directly from art. [264].

In general, if $2r$ stands for any even number, there will be $2r+1$ columns in the line that has reference to $2r$ things. Now $2r+1$ is an odd number, so that there will be a middle term in the line, and this middle term will be the number of combinations of $2r$ things taken r at a time, or it will be [art. 261]

$$\frac{2r.(2r-1).(2r-2)\ldots(2r-r+1)}{1\ .\ 2\ .\ 3\ .\ r}.$$

or

$$\frac{2r.(2r-1).(2r-2)..(r+1)}{1\ .\ 2\ .\ 3\ ..\ r}.$$

Again, if $2r+1$ stands for any odd number, there will be $2r+2$ columns in the line that has reference to $2r+1$ things, and as $2r+2$ is an even number there will be no middle term in this line, but there will be two terms equidistant from the extremities, and equal to each other, and each of these will represent the number of combinations of $2r+1$ things taken r at a time, and taken $r+1$ at a time. The expression for the first of these numbers is [art. 261]

$$\frac{(2r+1).2r.(2r-1)\ldots(2r+1-r+1)}{1\ .\ 2\ .\ 3\ \ldots\ r},$$

or

$$\frac{(2r+1).2r.(2r-1)\ldots(r+2)}{1\ .\ 2\ .\ 3\ \ldots\ r}.$$

The expression for the other number is found by inserting $r+1$ at the end of the decreasing factors in the numerator, and $r+1$ at the end of the increasing factors of the denominator. These two factors destroy each other and leave the expression as it was.

267. The p^{th} number of the m^{th} line in the table is the number of combinations of m things taken $p-1$ at a time. Thus, the fourth number in the eighth line is 56, the number of combinations of 8 things taken 3 at a time. Now the number of combinations of m things taken $p-1$ at a time is found by multiplying the number of combinations of m things taken $p-2$ at a time by $\frac{m-p+2}{p-1}$. It follows that the p^{th} number in the m^{th} line is formed by multiplying the number before it by $\frac{m-p+2}{p-1}$. Thus 210, the seventh number in the tenth line, is formed by multiplying 252, the number before it, by $\frac{10-7+2}{6}$, or by $\frac{5}{6}$.

268. The $\overline{p-1}^{th}$ number in the $\overline{m-1}^{th}$ is, as we have seen,

$$\frac{(m-1)\ (m-2)\ldots(m-p+2)}{1\ \ 2\ \ ..\ \ (p-2)},$$

and the p^{th} number in the same line is

$$\frac{(m-1).(m-2)..(m-p+2).}{1\ \ 2\ \ (p-2)}\ .\ \frac{(m-p+1)}{(p-1)}.$$

The sum of these two expressions is

$$\frac{(m-1)\ldots.(m-p+2)}{1\ \ldots.\ (p-2)}\left\{1+\frac{m-p+1}{p-1}\right\},$$

or

$$\frac{(m-1)\ldots(m-p+2)}{1\ \ldots\ (p-2)}\left\{\frac{m}{p-1}\right\},$$

or, removing the factor m from the right of the numerator to its left,

$$\frac{m.(m-1)..(m-p+2)}{1\ \ 2\ \ ..\ \ (p-1)}.$$

This last expression is the p^{th} number in the m^{th} line which is thus the sum of

the $\overline{p-1}^{th}$ and p_{th} numbers in the $\overline{m-1}^{th}$ line. In other words, any number in the table is the sum of the number above it, and the number above it and next to the left. Thus 20, the fourth number in the sixth line, is equal to $10 + 10$.

Again, for the same reason 10 in the fourth column is equal to $6 + 4$, and 4 in the fourth column is equal to $3 + 1$; so that 20 in the fourth column is equal to $10 + 6 + 3 + 1$, the sum of all the numbers above it in the column next to the left. The same is manifestly true for any other number in the table.

269. In this table let us write the vertical columns horizontally, so that we have

1	1	1	1	1	1
1	2	3	4	5	6
1	3	6	10	15	21
1	4	10	20	35	56
1	5	15	35	70	126

Here the second line contains the *natural numbers;* the third line the *first order of figurate numbers;* the fourth line the *second order of figurate numbers;* the fifth line the *third order of figurate numbers;* and so on. The property proved in art. [268] is therefore this, that the m^{th} number of the p^{th} order of figurate numbers is equal to the sum of the first m figurate numbers of the $\overline{p-1}^{th}$ order.

270. Observe, that all of these expressions for the permutations and combinations of any number of things are true, only on the supposition that no two of the things in question are the same. If two or more of them are the same, that makes a material difference in the result. Thus, let us inquire in how many ways the seven letters of the word *Barbara* may be written. Let us suppose that there are seven blank spaces arranged in a line, and that we first file three of these, then other three, and so on, with the three letters a in the word *Barbara*. The number of ways in which we can do this is plainly the number of combinations of seven things taken three at a time, that is $\frac{7.6.5}{1.2.3}$, or 35 ways. When three of the seven blanks are filled, there remain four; let us fill two of these with the two letters b, this can be done in $\frac{4.3}{1.2}$ or 6 ways. Now for every one of the 35 ways in which three of the seven blanks can he filled with the three letters a, we can fill two of the four remaining blanks in 6 ways with the two letters b. We can, therefore, fill five of the seven blanks with the three letters a, and the two letters b in 6×35, or 210 ways. And for every one of these 210 ways we can fill the two remaining blanks with the two letters r. The number sought is, therefore, 210, or $\frac{7.6.5.4.3.2.1}{1.2.3.1.2.1.2}$. If all the things had been different the number of these permutations would have been 5040.

In the same way if we have m things, whereof n are alike, n' alike, n'' alike, and so on, the number of ways in which they can be arranged is

$$\frac{m.(m-1).(m-2)\ldots.3.2.1}{1.2\ldots n.1.2\ldots n'.1.2\ldots n''\ldots}.$$

Thus the number of ways in which the eleven letters in the word *Mississippi* can be written is

$$\frac{11.10.9.8.7.6.5.4.3.2.1}{1.2.3.4.1.2.3.4.1.2},$$

or 34,650 instead of 39,916,800, which it would have been had the letters been different.

We have here supposed all the letters taken together, and there is, therefore, clearly only one way of combining them. Had we proposed to find the number of permutations of m things, of which p are alike, taken n at a time, we should have proceeded somewhat differently. We should have separated the computation by finding, 1st, the number of permutations taken n at a time, including *none* of the p things which are alike; 2dly, the number involving one of the p things, and so on. The sum of all these is evidently equal to the number of permutations required. The reader will find no difficulty in the computation of the several parts referring to art. [264] and the commencement of the present article. Thus, take the term where q of the like quantities enter. Proceeding similarly as before, let us suppose that we have n blank spaces, q of which are to be filled by the things which are alike, and the rest $n-q$, by the things which are unlike. Now, by the same reasoning as that made use of at the beginning of this article, it appears immediately, that q of the spaces may be occupied by like things in

$$\frac{n(n-1)\ldots\ldots(n-q+1)}{1\,.\,2\quad\ldots\quad q}$$

different ways. Again, there are $m-p$ things which are unlike, and the $n-q$ vacant spaces may be filled by the unlike things in as many different ways as there are permutations of $m-p$ things taken $n-q$ at a time, or art. [264] in $(m-p)(m-p-1)\ldots(m-p-(n-q)+1)$ different ways. Hence, since for every arrangement of the like things there are $(m-p)(m-p-1)\ldots(m-p-(n-q)+1)$ arrangements of the unlike, and there are

$$\frac{n(n-1)\ldots\ldots(n-q+1)}{1\,.\,2\quad\ldots\quad q}$$

different arrangements of the like things, it follows, that the whole number of different arrangements is

$$\frac{n(n-1)\ldots(n-q+1)}{1\,.\,2\quad\ldots\quad q}\times$$
$$(m-p)(m-p-1)\ldots(m-p-(n-q)+1).$$

We will apply this to one example. What is the number of permutations of 20 things of which 18 are alike taken 5 at a time?

It is evident, in the first place, that at least 3 of the like things must enter into each permutation. There are then 3 sets of arrangements. 1st, Where none but like things enter, and there is evidently only one way of arranging these. 2dly, Where there are 4 like things, and one of the unlike. The 4 like can be arranged $\frac{5\,.\,4\,.\,3\,.\,2}{1\,.\,2\,.\,3\,.\,4}$, or 5 different ways ($n=5$ and $q=4$ in the expression $\frac{n(n-1)\ldots n-q+1}{1\,.\,2\quad\ldots\quad q}$), and either of the 2 unlike can be joined with them; so that we have 2×5, or 10, for this set of arrangements. 3dly, Where there are 3 of the like quantities, and the 2 unlike. To find this we have only to put in the general expression

$n=5,\quad q=3,\quad m=20,\quad p=18,$

and it becomes

$$\frac{5\,.\,4\,.\,3}{1\,.\,2\,.\,3}\times 2\,.\,1,\text{ or } 20.$$

Hence, adding the permutations in the 3 sets of arrangements, we have 31 for our result.

Had we been finding the number of combinations in the same case, the process would have been similar; taking the number of combinations of the unlike things, instead of their permutations, and considering that whatever number of like things enter, there is only one way of combining them.

In the same way, separating them into different sets of arrangements, we may find the number of permutations of m things, of which p are of one kind, q of another, and so on taken n at a time.

Of the Binomial Theorem.

271. It will be necessary now to extend the definition of a *coefficient*, which was confined in art. [9] to mean the numerical factor in any product. When we have a product, and are, for the time, considering any factor or factors in it as its principal and distinguishing part, we call its other factor, or factors, the coefficient of that part. Thus, if we have the product $m\,.\,\frac{m-1}{2}\,.\,a^2\,x^{m-2}$, and are considering it chiefly with reference to x, $m\,\frac{m-1}{2}\,a^2$ is called the coefficient of x^{m-2}. Again, if we are considering it with reference to a, $m\,.\,\frac{m-1}{2}\,x^{m-2}$ may be called the coefficient of a^2; and if with reference to a and x, $m\,\frac{m-1}{2}$ is the coefficient of $a^2\,x^{m-2}$. So that a coefficient may consist of more than one term; thus in the expression $3(a+b+c)\,x^2$, $3(a+b+c)$ is the coefficient of x^2.

272. An expression, such as $a+b$, or $1-x$, that consists of two terms is called a *binomial* expression. An expression that consists of three or more terms, is, sometimes, called a *polynomial* expression. The *binomial theorem* is the algebraical rule, or formula, for expressing any power, or root, of a binomial expression in a series consisting of single terms.

273. If we wish to raise a binomial expression to any power indicated by a whole number, we can do so only by the rules of multiplication. In this way we have found [art. 176] that

$$(x+a)^2=x^2+2\,a\,x+a^2,$$

and [art. 181] that

$$(x+a)^3=x^3+3\,a\,x^2+3\,a^2\,x+a^3.$$

Again, if we take the example in art.

[29], and for each of the quantities b, c, and d there, substitute a, the expression $(x+a).(x+b).(x+c).(x+d)$ becomes $x+a$ repeated four times, or $(x+a)^4$; also the coefficient of x^3 in the product becomes a repeated four times, or $4a$, the coefficient of x^2 becomes a^2 repeated six times or $6a^2$, the coefficient of x becomes $4a^3$, and the term $abcd$ becomes a^4; so that we find $(x+a)^4$
$= x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$. In like manner if we form the product of five or six binomial factors we may deduce from it the expression for $(x+a)^5$, or $(x+a)^6$; and if we can discover the general law according to which the continued product of m binomial factors is formed, we can deduce from it the general expression for $(x+a)^m$.

274. Let us take the expression

$$\left.\begin{array}{l} x^m \\ +(a+b+c+d+e+\ldots)x^{m-1} \\ +(ab+ac+ad+\ldots+bc+bd\ldots+cd+\ldots+de+\ldots)x^{m-2} \\ +(abc+abd+\ldots+acd+\ldots+bcd+\ldots+cde+\ldots)x^{m-3} \\ +(abcd+abce+\ldots+acde+\ldots+bcde+\ldots)x^{m-4} \\ +\ldots\ldots \\ +abcde\ldots \end{array}\right\}[A].$$

Here we suppose that the quantities a, b, c, d, &c. are in number m; and that the coefficient of x^{m-1} is the sum of all these quantities, the coefficient of x^{m-2} the sum of all the products of every two of them, the coefficient of x^{m-3} the sum of all the products of every three of them; so that if there were a term containing x^{m-r}, its coefficient would be the sum of all the products of every r of the quantities. This being understood, it is easy to see how the blanks in our expression are to be filled up when any particular value is given to m.

Now if in this expression we make m equal to 3, we find the same result as was found in art. [29] for the continued product of three binomial factors. If we make m equal to 4, the expression becomes the same as was found in the same article for the continued product of four binomial factors. Similarly, if the reader will, by multiplication, form for himself the products of five and six binomial factors, he will find that the results are, respectively, the same as if he had substituted the numbers 5 and 6 for m in the expression above, and so for any number of factors. We may, therefore, conclude, that the expression [A] truly expresses the continued product of m binomial factors.

It may be objected to this conclusion, that we have not proved it, but have raised a presumption only in favour of it, by showing that it is true in a number of particular cases, and many instances may be brought of the fallaciousness of such reasoning. We may, however, make our proof quite strict as follows. Let us multiply the expression [A] by $x+i$, i not being one of the quantities a, b, c, d, &c. already found in it. To do this we multiply every term in the expression by x, and also every term by i. x^m multiplied by x, becomes x^{m+1}, which is therefore a term in the product. x^m multiplied by i becomes ix^m, and the second term of [A] multiplied by x becomes $(a+b+c+\ldots)x^m$; the sum of these, or $(a+b+c+\ldots+i)x^m$, is therefore another term in the product. Again, the second term of [A] multiplied by i becomes $(ai+bi+ci+..)x^{m-1}$, and the third term multiplied by x becomes $(ab+ac+ad+..)x^{m-1}$; the sum of these therefore, or $(ab+ac+ad+\ldots+ai+bc+bd+\ldots+bi+..)$ is a term in the product. Similarly x^{m-2} multiplied by the sum of all the products of the quantities $a, b, c, d\ldots i$, taken three at a time, is another term, and so on; the last term in the product becoming $a, b, c, d\ldots i$. It follows that the product of the expression [A] by $x+i$ is of the same form as [A]; $m+1$ being substituted for m, and the quantities $a, b, c, d\ldots$ having i placed among them. Therefore if [A] truly expresses the continued product of m binomial factors, an expression similar to [A] will truly express the continued product of $m+1$ binomial factors. Now we have seen [art. 29] that [A] truly expresses the product of four factors, it therefore, by what we have just proved,

truly expresses the continued product of $4+1$, or five factors. Again, since it truly expresses the product of five factors, it truly expresses the product of $5+1$, or six factors. In like manner it truly expresses the product of seven, eight, or nine factors, and so, counting upwards, of any number of factors. It therefore truly expresses the product of m factors.

Owing to the great importance of the subject, we will show that this must be the case from somewhat different considerations. Glancing at the products $(x+a)\ (x+b)$, and $(x+a)(x+b)(x+c)$ as we have written them,

$$\begin{array}{ll} x^2+ax+ab & x^3+ax^2+abx+abc \\ \quad +bx & \quad +bx^2+acx \\ & \quad +cx^2+bcx, \end{array}$$

it is clear, from the nature of the process, that if we multiply together m factors of the form $(x+a)(x+b)\ldots\ldots(x+h)$, the first term of the result will be simply the product of all the first terms of the binomials, and the last simply the product of all their second terms; that is, the first term will be x^m, and the last $ab\ldots h$. It is also clear that every intermediate power of x will appear in the result. Again, no numbers will enter into the coefficients, since the multiplication cannot produce 2 terms exactly alike. If then A, B, &c. be the sums of all the coefficients of x^{m-1}, x^{m-2}, &c. in the result of the multiplication as above written, we shall have it represented by

$$x^m + A x^{m-1} + B x^{m-2}, \&c. \ldots + abc..h.$$

Now it follows from the multiplication that, x^{m-1} being considered composed of $m-1$ factors and so on, every term in the above product must have exactly m factors. Hence the coefficients of x^{m-1}, of which A is the sum, have each of them but one factor, and that is a, or b, or c, &c. So every coefficient of x^{m-r} has exactly r factors, and is some product arising from multiplying together r of the m quantities $a, b, c \ldots h$. Now all the quantities $a, b, c \ldots h$ enter into the product in precisely the same way, since the result would have been the same in whatever order we had multiplied the binomial factors together. Hence every possible product of r factors out of the m quantities $a, b, c \ldots h$ must appear in the result as the coefficient of x^{m-r}; and we therefore see, as before, that it will be the sum of all the products of every r of the same quantities.

275. The expression [A] being thus the product of the m different binomial factors $x+a$, $x+b$, $x+c$, &c.; let us suppose that each of the quantities b, c, d, &c. is made equal to a. The expression $(x+a).(x+b).(x+c)\ldots$ in that case becomes $x+a$ repeated m times, that is, it becomes $(x+a)^m$. As to the expression [A], its first term remains the same, x^m. The coefficient of the second term becomes a repeated m times, or ma; the second term therefore becomes $m a x^{m-1}$. The coefficient of the third term becomes a^2, repeated as often as there are products of m quantities taken two at a time, that is, as often as there are combinations of m things taken two at a time [art. 261]; this coefficient is therefore $m\,.\,\dfrac{m-1}{2}\,a^2$; the third term therefore is $m\,.\,\dfrac{m-1}{2}\,.\,a^2 x^{m-1}$. Similarly the fourth term becomes $m\,.\,\dfrac{m-1}{2}\,.\,\dfrac{m-2}{3}\ a^3\,.\,x^{m-3}$; $m\,.\,\dfrac{m-1}{2}\,.\,\dfrac{m-2}{3}$ being the number of combinations of m things taken three at a time. Proceeding in this way, the $r+1^{th}$ term has for coefficient a^r repeated as often as there are combinations of m things taken r at a time, or is [art. 261]

$$m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\ldots\frac{m-r+1}{r}\,a^r x^{m-r}.$$

The last term of the expression [A] becomes $aaa\ldots$ repeated m times, or a^m. The last but one becomes a^{m-1} repeated as often as there are combinations of m things taken $m-1$ at a time, or $m a^{m-1} x$. The last term but two becomes $m\ \dfrac{m-1}{2}\ a^{m-2}\,x^2$. Collecting these terms together we find

$$(x+a)^m = x^m + m\,a\,x^{m-1} + m\,\frac{m-1}{2}\,a^2\,x^{m-2} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,a^3\,x^{m-3} + \&c.$$

$$+\,m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,\ldots\,\frac{m-r+1}{r}\,a^r\,x^{m-r} + \&c.$$

$$+\,m\,.\,\frac{m-1}{2}\,a^{m-2}\,.\,x^2 + m\,a\,x^{m-1} + a^m.$$

276. The expression which we obtained in the last article, gives us a series of single terms together, equal to $(a+x)^m$, when m is a whole positive number, and is, therefore, [art. 272] the binomial theorem. It is also, sometimes, called the *developement*, or *expansion* of $(x+a)^m$, and is of the most important use. We may observe with respect to it:

First, that it consists of $m+1$ terms;

Secondly, that the exponent of a, added to that of x, always gives m for sum;

Thirdly, that the coefficients for different values of m are the numbers contained in the table in art. [265]; and are, like them, the same for any 2 terms equidistant from the beginning and the end of the sines, so that the coefficient of $x^{m-p}\,a^p$ is the same as that of $x^p\,a^{m-p}$. See art. [264].

Fourthly, that the coefficient of any term is formed by multiplying the coefficient of the preceding term by the exponent of x in that term, and dividing by the number of terms preceding the one in question. This rule is of much practical utility, as it enables us to form at once the expansion of any power, without recurring to the general formula. The reader may satisfy himself of this by forming the coefficients in $(x+a)^6$ by means of this rule. This law of the coefficients was discovered by Newton.

Fifthly, that the coefficient of $x^{m-p}\,a^p$, or the coefficient of the $(p+1)^{th}$ term is

$$\frac{m\,(m-1)\ldots\ldots(m-p+1)}{1\,.\,2\quad\ldots\quad p}.$$

The term itself is

$$\frac{m\,(m-1)\ldots\ldots(m-p+1)}{1\,.\,2\quad\ldots\quad p}\,x^{m-p}\,a^p.$$

This is called the general term, because, by giving any value to p between 0 and m, it represents each particular term.

And sixthly, that when m is an even number, the coefficient of the middle term is

$$\frac{m}{1}\,.\,\frac{m-1}{2}\,\ldots\,\frac{\frac{m}{2}+1}{\frac{m}{2}},$$

see art. [266], and when m is an odd number, the coefficients of the two middle terms are the same, and are [art. 266]

$$\frac{m}{1}\,.\,\frac{m-1}{2}\,\ldots\,\frac{\frac{m+1}{2}+1}{\frac{m-1}{2}}.$$

277. In the expression in art. [275] if for x we write unity, and for a we write y, it becomes

$$(1+y)^m = 1 + m\,y + m\,.\,\frac{m-1}{2}\,y^2$$

$$+\,m\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,y^3 + \&c.$$

Thus if for m we put y, this becomes

$$(1+y)^3 = 1 + 3\,y + 3\,.\,\frac{3-1}{2}\,y^2$$

$$+\,3\,.\,\frac{3-1}{2}\,.\,\frac{3-2}{3}\,y^3 + \&c.$$

All the terms after the fourth would contain $3-3$ for a factor in the numerator, and are therefore all equal to zero, so that we have,

$$(1+y)^3 = 1 + 3\,y + 3\,y^2 + y^3.$$

278. In what has gone before, m being in the first instance the number of binomial factors in a product, is necessarily a whole positive number; the binomial theorem, therefore, as we have proved it, is applicable to those powers only of a binomial that are indicated by a whole positive number. We are now to show that the same theorem is true, whether m be whole or fractional, positive or negative.

When m and m' are whole positivo numbers, we have seen that

$$(1+z)^m = 1 + m\,z + m\,.\,\frac{m-1}{2}\,z^2 + \&c.;$$

and

$$(1+z)^{m'} = 1 + m'z + m.\frac{m'-1}{2}z^2 + \&c.$$

Now if we multiply the first members of these equations together, we find for the product $(1+z)^{m+m'}$, which, by the theorem just proved, is equal to

$$1 + (m + m')z + m + m'.$$
$$\frac{m+m'-1}{2}z^2 + \&c.$$

Hence we have

$$\left(1 + mz + m\frac{m-1}{2}z^2 + \&c.\right) \times (A)$$
$$\left(1 + m'z + m'\frac{m'-1}{2}z^2 + \&c.\right) =$$
$$1 + (m+m')z + m + m'\frac{m+m'-1}{2}z^2 + \&c.$$

The latter expression, then, is the product arising from the multiplication of the two former, and the way, therefore, in which $m+m'$ enters into the product results from the way in which m and m' enter into its two factors. It is, therefore, equally the product of those two former expressions, whether m and m' be whole numbers, or fractions, positive, or negative.

Let us suppose then that we have

$$x = 1 + mz + m.\frac{m-1}{2}z^2 + \&c. \ldots [X]$$

and

$$y = 1 + m'z + m'.\frac{m'-1}{2}.z^2 + \&c. \ldots [Y];$$

where m and m' are any numbers whatever, and where we do not know any thing of x and y but that they are respectively equal to the sums of these two series. As before, putting p for $m + m'$, we have

$$xy = 1 + pz + p\frac{p-1}{2}z^2 + \&c. \ldots [Z].$$

279. Now let m be a whole positive number; in that case the expression [X] becomes the developement of $(1+z)^m$; so that we have

$$x = (1+z)^m.$$

Also let m' become equal to $-m$; then p becomes $m - m$ or zero, and the expression [Z] reduces itself to

$$xy = 1.$$

Hence we have

$$y = \frac{1}{x};$$

and, putting for x its value as above,

$$y = \frac{1}{(1+z)^m}, = (1+z)^{-m}.$$

art. [217]. Substituting for y its value in [Y] this becomes

$$1 + m'z + m'\frac{m'-1}{2}z^2 + \&c. = (1+z)^{-m}.$$

Finally, substituting for m' its value $-m$, this again becomes

$$(1+z)^{-m} = 1 + (-m)z + (-m)$$
$$\frac{(-m)+1}{2}z^2 + \&c.$$

Showing that the expression in art. [277] is true when m is a negative number.

280. Again, in the expressions [X] and [Y], let us suppose that m' is equal to m; this makes $p = 2m$, and $x = y$. Consequently xy becomes x^2, and we have, by [Z]

$$x^2 = 1 + 2mz + 2m\frac{2m-1}{2}z^2 + \&c.$$

Similarly, x^2y becomes x^3, and, as before, we have

$$x^3 = 1 + 3mz + 3m\frac{3m-1}{2}z^2 + \&c.;$$

and

$$x^4 = 1 + 4mz + 4m\frac{4m-1}{2}z^2 + \&c.;$$

so that in general, if n be any whole number, we have

$$x^n = 1 + nmz + nm.\frac{nm-1}{2}z^2 + \&c.$$

Now suppose that m is any fraction $\frac{k}{n}$; the last expression then becomes

$$x = 1 + n\frac{k}{n}z + n\frac{k}{n}.\frac{n\frac{k}{n}-1}{2}z^2 + \&c.$$

or

$$x^n = 1 + kz + k\frac{k-1}{2}z^2 + \&c.;$$

that is, [art. 277]

$$x^n = (1+z)^k,$$

since k is a whole number. Taking the n^{th} root of both members of this equation, we find

$$x = (1+z)^{\frac{k}{n}},$$

For x substitute its value in [X], and this becomes

$$1 + mz + m\frac{m-1}{2}z^2 + \&c. = (1+z)^{\frac{k}{n}};$$

and finally for m substitute its value $\frac{k}{n}$

and we have

$$(1+z)^{\frac{k}{n}} = 1 + \frac{k}{n}z + \frac{k}{n}\cdot\frac{\frac{k}{n}-1}{2}z^2 + \&c.$$

where $\frac{k}{n}$ is any fraction. This shows that the expression in art. [276] is true when m is a fractional number, and equally so whether it be positive or negative.

281. Having now shown generally [arts. 276, 279, 280], that whether m be a whole or a fractional, a positive or a negative number,* we always have

$$(1+z)^m = 1 + mz + m.\frac{m-1}{2}z^2 + \&c.$$

we can deduce at once the general expression for $(x+a)^m$.

We have

$$x + a = x\left(1 + \frac{a}{x}\right);$$

and, therefore,

$$(x+a)^m = x^m\left(1 + \frac{a}{x}\right)^m.$$

Now, as we have first seen that

$$\left(1+\frac{a}{x}\right)^m = 1 + m\frac{a}{x} + m\frac{m-1}{2}\frac{a^2}{x^2} + \&c.,$$

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* The student may perhaps be desirous of knowing whether he will be justified in applying the same formula to such indexes as $\sqrt{2}$, $\sqrt[3]{5}$, &c., which cannot be called positive or negative integers or fractions. It was shown (arts 186, 187) that we can find fractions whose value shall be as near as we please to any surd quantities. And we will now show generally that the binomial theorem is true *wherever* the index is (as in the case of surds) a quantity forming a fixed value, or *limit*, which can be expressed as nearly as we please, though never quite accurately, by rational arithmetical quantities; assuming only that the rule is, as proved in the text, true for these latter.

Suppose m to be an index which is such a limit, and suppose l and n to be rational arithmetical quantities which express the value of m to any degree of approximation, l being greater than m, and n less; so that

$$l = m + \alpha$$
$$n = m - \beta,$$

α and β being differences which can be made as small as we please.

Now
$$(1+z)^l = 1 + l.z + \frac{l.(l-1)}{1.2}.z^2 + \frac{l.(l-1).(l-2)}{1.2.3}.z^3 + \&c.;$$

or

$$(1+z)^{m+\alpha} = 1 + \overline{m+\alpha}.z + \frac{\overline{m+\alpha}.(\overline{m+\alpha}-1)}{1.2.}.z^2 + \frac{\overline{m+\alpha}.(\overline{m+\alpha}-1).(\overline{m+\alpha}-2)}{1.2.3}.z^3 + \&c. = (A).$$

Let
$$z = 1 + m.z + \frac{m.(m-1)}{1.2}.z^2 + \frac{m.(m-1).(m-2)}{1.2.3}.z^3 + \&c. = (B).$$

Again
$$(1+z)^n = 1 + n.z + \frac{n.(n-1)}{1.2}.z^2 + \frac{n.(n-1).(n-2)}{1.2.3}.z^3 + \&c.$$

or
$$(1+z)^{m-\beta} = 1 + \overline{m-\beta}.z + \frac{\overline{m-\beta}.(\overline{m-\beta}-1)}{1.2}.z^2 + \frac{\overline{m-\beta}.(\overline{m-\beta}-1).(\overline{m-\beta}-2)}{1.2.3}.z^3 + \&c. = (C).$$

Of the three series (A), (B), (C), the first terms are the same.

The coefficient of the second term of (A) is $\overline{m+\alpha}$
" " " (B) m
" " " (C) $\overline{m-\beta}$.

Now these three, the smaller α and β are, will become more nearly equal, and they would be exactly equal if we could get rid of the differences α and β altogether.

The same remark applies to the three coefficients of the third terms,

$$\frac{\overline{m+\alpha}.(\overline{m+\alpha}-1)}{1.2}$$
$$\frac{m.(m-1)}{1.2}$$
$$\frac{\overline{m-\beta}.(\overline{m-\beta}-1}{1.2},$$

and so on for all the rest of the coefficients. They will become more and more nearly equal, the less α and β become. Now let γ and δ represent the whole differences respectively between the series (A) and (B) and (B) and (C); so that

$$(A) = (B) + \gamma$$
$$(C) = (B) - \delta.$$

By the diminution of α and β, it is evident that γ and δ are diminished; so that the difference between (A) and (C) may be rendered as small as we please. Hence, the smaller α and β are, the more nearly are both $(1+z)^{m+\alpha}$ and $(1+z)^{m-\beta}$ expressed by the series (B), and the approximation may be made as close as we please. But (B) will always be of intermediate value to $(1+z)^{m+\alpha}$ and $(1+z)^{m-\beta}$.

Hence it follows that $(1+z)^m = (B)$ exactly.

For, if $(1+z)^m$ were greater than (B) by any quantity d, so that

$$(1+z)^m = (B) + d,$$

we should have $(1+z)^{m+\alpha}$ differing from (B), and still more from $(1+z)^{m-\beta}$ by a quantity necessarily greater than d. But this is absurd, since we may make $(1+z)^{m+\alpha}$, and $(1+z)^{m-\beta}$ approach each other as nearly as we please, by the continual diminution of α and β. And in the same way may be shown the absurdity of supposing $(1+z)^m$ less than (B) by any quantity.

Hence
$$(1+z)^m = (B) = 1 + m.z + \frac{m.(m-1)}{1.2.}z^2 + \frac{m.(m-1)(m-2).z^3}{1.2.3} + \&c.$$

it follows that

$$(x+a)^m = x^m\left(1+m\frac{a}{x}+m\frac{m-1}{2}\frac{a^2}{x^2}+\&c.\right),$$

or, multiplying each term of the series by x^m,

$$(x+a)^m = x^m + m\frac{a\,x^m}{x}$$
$$+ m\,.\,\frac{m-1}{2}\,.\,\frac{a^2 x^m}{x^2} + \&c.,$$

or, finally,

$$(x+a)^m = x^m + m\,a\,x^{m-1}$$
$$+ m\frac{m-1}{2}\,.\,a^2 x^{m-2} + \&c.$$

This is the most general form of the binomial theorem.

This theorem is generally (though not quite accurately) attributed to Sir Isaac Newton. He discovered the law of the coefficients already mentioned in art. [276]; and we refer the reader to our translation of Biot's *Life of Newton*, (p. 3.) where he will find some account of the manner in which Newton was led to extend the theorem to negative and positive indexes.

282. It is very important to observe that the mode of proof used in articles [279] and [280] differs materially from any that we had used before. On former occasions, in obtaining any result, we endeavoured to explain how it necessarily followed from the nature of the question proposed, as in articles [257], [261], [275], &c. In the last articles, however, we have proved that the result is true, without directly showing why it should be true; we have not sought for any reason why the developement of the negative and fractional powers of a binomial should be such as we have found it to be. The first of these sorts of proof is no doubt the more satisfactory, but very often it cannot be attained; and it is by extensively using the last, which, when rightly understood, is equally conclusive, that the mathematical sciences have been so much improved in modern times.

283. If in the expression at the end of art. [281] we substitute $-a$ for a, and observe that all the odd powers of $-a$ are negative, while its even powers are positive, art. [30], we shall find that the alternate terms in the developement beginning with the second, become negative. So that

$$(x-a)^m = x^m - m\,a\,x^{m-1}$$
$$+ m\frac{m-1}{2}a^2 x^{m-2} - \&c.$$

284. We shall now apply the binomial theorem, as given in art. [281], to a few examples. To find the expansion of $\frac{1}{1+x}$, that is, of $(1+x)^{-1}$. To find the coefficients, we must substitute -1 for m in the theorem. Now if this be done

$$m = -1$$

$$m\,.\,\frac{m-1}{2} = (-1)\,.\,\frac{(-1)-1}{2} = +1$$

$$m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3} = (-1)\,.\,\frac{(-1)-1}{2}\,.\,\frac{(-1)-2}{3}$$
$$= -1$$

&c.

So that

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \&c.$$

which is the result in art. [46].

285. To find the square root of $1+x$; that is, to find the expansion of $(1+x)^{\frac{1}{2}}$.

If we substitute $\frac{1}{2}$ for m in the theorem, we find

$$m = \frac{1}{2}$$

$$m\,.\,\frac{m-1}{2} = \frac{1}{2}\,.\,\frac{\frac{1}{2}-1}{2} = -\frac{1}{2^2}\,.\,\frac{1}{2}$$

$$m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3} = \frac{1}{2}\,.\,\frac{\frac{1}{2}-1}{2}\,.\,\frac{\frac{1}{2}-2}{3}$$
$$= \frac{1}{2^3}\,.\,\frac{1\,.\,3}{2\,.\,3}$$

$$m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,\frac{m-3}{4} = \frac{1}{2}\,.\,\frac{\frac{1}{2}-1}{2}\,.\,\frac{\frac{1}{2}-2}{3}\,.\,\frac{\frac{1}{2}-3}{4}$$
$$= -\frac{1}{2^4}\,.\,\frac{1\,.\,3\,.\,5}{2\,.\,3\,.\,4}$$

&c.

So that

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2^2}\;\frac{1}{2}x^2 + \frac{1}{2^3}\,.$$
$$\frac{1\,.\,3}{2\,.\,3}\;x^3 - \frac{1}{2^4}\;\frac{1\,.\,3\,.\,5}{2\,.\,3\,.\,4}\;x^4 + \&c.,$$
$$= 1 + \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1\,.\,3}{2\,.\,3}\left(\frac{x}{2}\right)$$
$$- \frac{1\,.\,3\,.\,5}{2\,.\,3\,.\,4}\left(\frac{x}{2}\right)^4 + \&c.$$

And, similarly,

$$(1+x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{1\,.\,3}{2\,.\,3}\left(\frac{x}{2}\right)^3$$

$$- \frac{1\,.\,3\,.\,5}{2\,.\,3\,.\,4}\left(\frac{x}{2}\right)^4 - \&\text{c.},$$

2 series which proceed very regularly. In like manner we should find

$$(1+x)^{\frac{1}{2}} = 1 - \frac{1}{1}\,.\,\frac{x}{2} + \frac{1\,.\,3}{1\,.\,2}\left(\frac{x}{2}\right)^2$$

$$- \frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3}\left(\frac{x}{2}\right)^3 + \frac{1\,.\,3\,.\,5\,.\,7}{1\,.\,2\,.\,3\,.\,4}\left(\frac{x}{2}\right)^4 - \&\text{c.}$$

And again,

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{1}\,\frac{x}{2} + \frac{1\,.\,3}{1\,.\,2}\left(\frac{x}{2}\right)$$

$$+ \frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3}\left(\frac{x}{2}\right)^3 + \frac{1\,.\,3\,.\,5\,.\,7}{1\,.\,2\,.\,3\,.\,4}\left(\frac{x}{2}\right)^4 + \&\text{c.},$$

and in the same way any other root direct or inverse of any binomial quantity may be found.

286. The general expression for $(x+a)^{-m}$ is

$$x^{-m} + (-m)\,a\,x^{-m-1} + (-m).$$

$$\frac{(-m)-1}{2}\,a\,x^{2-m-2} + \&\text{c.},$$

and this reduces itself to

$$\frac{1}{x^m} - m\frac{a}{x^{m+1}} + m\,\frac{m+1}{2}\,\frac{a^2}{x^{m+2}} - m.$$

$$\frac{m+1}{2}\,.\,\frac{m+2}{3}\,\frac{a^3}{x^{m+3}} + \&\text{c.}$$

This series goes on for ever, without coming to an end, the coefficient of every term being a whole number [art. 262]. If a be considerably less than x, the terms decrease very fast; in that case the series is said to converge, and the sum of a few of its first terms differs so little from the sum of the whole, tha the other terms may be neglected.

287. The general expression for $(x+a)^{\frac{m}{n}}$ is

$$x^{\frac{m}{n}} + \frac{m}{n}\,.\,a\,x^{\frac{m}{n}-1} + \frac{m}{n}\,.\,\frac{\frac{m}{n}-1}{2}\,.\,a^2\,x^{\frac{m}{n}-2}$$

$$+ \frac{m}{n}\,.\,\frac{\frac{m}{n}-1}{2}\,.\,\frac{\frac{m}{n}-2}{3}\,a^3\,x^{\frac{m}{n}-3} + \&\text{c.},$$

or

$$x^{\frac{m}{n}} + \frac{1}{n}\,.\,m\,a\,x^{\frac{m-n}{n}} + \frac{1}{n^2}\,.\,m\,.\,\frac{m-n}{2}\,a^2\,x^{\frac{m-2n}{n}}$$

$$+ \frac{1}{n^3}\,.\,m\,.\,\frac{m-n}{2}\,.\,\frac{m-2n}{3}\,a^3\,x^{\frac{m-3n}{n}} + \&\text{c.}$$

This series also goes on to an infinite number of terms, unless m be a multiple of n, and positive, so that $m = kn$. In that case $m - kn$, or zero, becomes, after a time, a factor in the numerators of the coefficients, and from thence all the terms of the series become zero. The series then becomes of the form in art. [275], where the exponent is a whole number.

Observe that we usually consider the developement of the m^{th} power of a binomial to go on to infinity, without stopping to consider whether m be a positive integer or not; and it is clear, from the last remark, that we introduce no error in doing so, since, in the case where the number of terms is finite, we only add a series of terms each of which is equal to zero. We shall presently find this remark of importance.

288. If we make x and a each $=1$ in the expansion of $(x+a)^m$, m being a whole number, we obtain

$$(1+1)^m, \text{ or } 2^m = 1 + m + m\,\frac{m-1}{2} + \&\text{c.} + m\,.\,\frac{(m-1)\,..\,(m-p+1)}{2\quad ..\quad p}$$

$$+ \&\text{c.} + m\,\frac{m-1}{2} + m + 1.$$

And therefore

$$2^m - 1 = m + m\,\frac{m-1}{2} + \&\text{c.} + \frac{m\,(m-1)\,..\,m-p+1}{1\,.\,2\quad ..\quad p}$$

$$+ \&\text{c.} + m\,\frac{m-1}{2} + m + 1.$$

Now, examining the several terms of this series, and referring to art. [264], we see that they are respectively the number of combinations of m things taken 1, 2, &c. p, &c. $m-1$, and m at a time. We, therefore, have $2^m - 1$ equal to the whole number of combinations of m things taken 1, 2, &c. up to m at a time; and if we consider the rejection of all the things as furnishing

the one combination of m things, taken none at a time, as in art. [265], then 2^m = number of combinations of m things taken 0, 1, 2, ... &c. up to m at a time. This result, as well as the other, is occasionally useful in the computation of chances.

289. The above developement enables us to extract with rapidity the square and other roots of numbers to any degree of approximation. Suppose N to represent the number, and that our object is to extract the square root of it. Take out of it the highest number which is a perfect square; call this a^2 and the remainder y, so that

$$N = a^2 + y.$$

Extracting the square root on both sides we have

$$\sqrt{N} = \sqrt{a^2 + y}$$

$$= a\sqrt{1 + \frac{y}{a^2}}.$$

Now, art. [283],

$$\sqrt{1 + x}, \text{ or } (1 + x)^{\frac{1}{2}}$$

$$= 1 + \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2$$

$$+ \frac{1 \cdot 3}{2 \cdot 3}\left(\frac{x}{2}\right)^3 - \&c.$$

And putting $\frac{y^2}{a^2}$ for x we get

$$\sqrt{1 + \frac{y^2}{a^2}} = 1 + \frac{y}{2a^2} - \frac{1}{2}\left(\frac{y}{2a^2}\right)^2$$

$$+ \frac{1 \cdot 3}{2 \cdot 3}\left(\frac{y}{2a^2}\right)^3 - \&c.$$

$$\text{And} \therefore \sqrt{N} = a\sqrt{1 + \frac{y}{a^2}}$$

$$= a\left\{1 + \frac{y}{2a^2} - \frac{1}{2}\left(\frac{y}{2a^2}\right)^2 + \frac{1 \cdot 3}{2 \cdot 3}\left(\frac{y}{2a^2}\right)^3 - \&c.\right\}$$

If y be much smaller than a^2, after taking a few terms of this series, it is evident that the rest will be very small. If that be not the case, but the perfect square next below N is at some distance from it, there will be a square number a little above N. Let us suppose b^2 to be such a number, and y to be the difference between it and N. We have $\quad N = b^2 - y$

$$\text{And} \therefore \sqrt{N} = \sqrt{b^2 - y} = b\sqrt{1 - \frac{y}{b^2}}.$$

Now, art. [287],

$$(1 - x)^{\frac{1}{2}} \text{ or } \sqrt{1 - x} = 1 - \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2$$

$$- \frac{1 \cdot 3}{2 \cdot 3}\left(\frac{x}{2}\right)^3 - \&c.$$

Hence putting $\frac{y}{b^2}$ for x

$$\sqrt{1 - \frac{y}{b^2}} = 1 - \frac{y}{2b^2} - \frac{1}{2}\left(\frac{y}{2b^2}\right)^2$$

$$- \frac{1 \cdot 3}{2 \cdot 3}\left(\frac{y}{2b^2}\right)^3 - \&c.$$

$$\text{And} \therefore \sqrt{N} = b\sqrt{1 - \frac{y}{b^2}}$$

$$= b\left\{1 - \frac{y}{2b^2} - \frac{1}{2}\left(\frac{y}{2b^2}\right)^2 - \frac{1 \cdot 3}{2 \cdot 3}\left(\frac{y}{2b^2}\right)^3 - \&c.\right\}(B).$$

Suppose N = 8. Here 9, the number next above 8, is a perfect square; we, therefore, make use of the latter series. We have

$$b^2 = 9 \text{ and} \therefore b = 3 \text{ and } y = 1.$$

Hence, substituting in series B, we obtain

$$\sqrt{8} = 3\left\{1 - \frac{1}{2 \times 9} - \frac{1}{2} \frac{1}{2^2 \times 9^2} - \frac{1 \cdot 3}{2 \cdot 3} \frac{1}{2^3 \cdot 9^3} - \&c.\right\}$$

$$= 3\left\{1 - \frac{1}{18} - \frac{1}{648} - \frac{1}{11664} - \&c.\right\}.$$

Now, considering all the terms in the series to be multiplied by 3, we observe that the third term, being reduced to a decimal, has no significant figure in the 2 first decimal places, and the fourth has none in the 3 first. The number of terms we take note of is determined by the degree of accuracy required. See art. [166]. If we are satisfied with a result accurate to 2 places of decimals, we only take the 3 first terms, and have

$$\sqrt{8.} = \frac{611}{216} = 2.828 \ldots\ldots$$

290. Again, suppose it were required to extract the n^{th} root of the number N. Proceeding in a similar way we will suppose a^n to be the perfect n^{th} power which is next less than N, and that $N = a^n + y$. Extracting the n^{th} root, we have

$$\sqrt[n]{N} = \sqrt[n]{a^n + y}$$

$$= a\left(1+\frac{y}{a^n}\right)^{\frac{1}{n}}.$$

Now putting $\frac{1}{n}$ for m in the expansion of $(1+x)^m$ we have

$$(1+x)^{\frac{1}{n}} = 1+\frac{1}{n}.x+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2}x^2 + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{2\cdot 3}x^3+\&c.$$

Putting $\frac{y}{a^n}$ for x, and substituting in the value of $\sqrt[n]{N}$, we obtain

$$\sqrt[n]{N}=a\left\{1+\frac{1}{n}\cdot\frac{y}{a^n}+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2}\frac{y^2}{a^{2n}} + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{2\cdot 3}\frac{y^3}{a^{3n}}+\&c.\right\}.$$

And reducing and attending to the signs of the terms

$$\sqrt[n]{N}=a\left\{1+\frac{1}{n}\frac{y}{a^n}-\frac{1}{n}\frac{n-1}{2n}\frac{y^2}{a^{2n}} + \frac{1}{n}\frac{n-1}{2n}\frac{2n-1}{3n}\frac{y^3}{a^{3n}}-\&c.\right\}.$$

Let us apply this expression to find the cube root of 31. Now, 27 being the greatest cube number contained in 31, we have $a^n = 27$, and, therefore, $a = 3$, and $y = 4$. Hence, n being equal to 3, we have, by substituting in the value of $\sqrt[n]{N}$,

$$\sqrt[3]{31}=3\left\{1+\frac{1}{3}\frac{4}{27}-\frac{1\,.\,2}{3\,.\,6}\cdot\frac{4^2}{27^2} + \frac{1\,.\,2\,.\,5}{3\,.\,6\,.\,9}\frac{4^3}{27^3}-\&c.\right\}$$

or, performing the multiplications indicated, and raising the numbers to the requisite powers,

$$\sqrt[3]{31}=3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}-\&c. (A).$$

291. In extracting the square root of 8 in the last article we only added 3 terms of our series, and because the fourth, when reduced to a decimal, contained no significant figure in the 3 first places and the terms decreased very rapidly, we concluded that the sum of all the terms after the third would not contain any significant figure in the 3 first places, so that our result was accurate so far.—The rapidity of diminution in this particular case rendered our conclusion unobjectionable. Where, however, the terms of a series which goes on to infinity are convergent, that is, where each term is less than the preceding one, and their signs are alternately positive and negative, we may estimate by the following general method the error we introduce by representing the sum of the whole series by the sum of a limited number of its terms.

Suppose the series to be

$$a-b+c-d+e-f+g-h+\&c.$$

Let S represent the sum of the series. Now when we have taken the 4 first terms, the quantity remaining is

$$e-f+g-h+k-l+\&c.,$$

and each term being greater than that which follows it, writing this in pairs as follows

$$(e-f)+(g-h)+(k-l)+\&c$$

we see that each of the pairs will be positive, and therefore their sum will be positive. This sum being the difference between S and $a-b+c-d$ it is evident that S is greater than $a-b+c-d$. Again, taking the 5 first terms, the remaining ones are

$$-f+g-h+k-\&c.,$$

which may be written

$$-(f-g)-(h-k)-\&c.,$$

when, since f is greater than g, h greater than k, and so on, it is evident that the whole is negative. We see then that S is equal to $a-b+c-d+e$ with something subtracted from it, or it is less than $a-b+c-d+e$, but it was proved to be greater than $a-b+c-d$; it is therefore comprised between the 4 and 5 first terms; e, therefore, is greater than the difference between f and the 4 first terms.—The same reasoning applies at whatever term we stop; and we, therefore, conclude that, in a series of the above nature, the numerical error introduced by representing the series by the sum of a limited number of its terms is less than the term which follows the last included in that sum.

292. Returning to series (A), in art. [290], and reducing the several terms to decimals, we have

$$
\begin{array}{rl}
3 & = 3.00000 \\
\frac{4}{27} & = \underline{.14815} \\
 & 3.14815 \\
\text{Subtracting } \frac{16}{2187} & = \underline{.00731} \\
 & 3.14084
\end{array}
$$

Now this series, after the two first terms, is of the nature indicated in the last. Stopping then at the 3 first terms the error is less than the fourth, or $\frac{320}{531441}$. But this evidently contains no significant figure in the 3 first decimal places. It is clear then that .001 is greater than the error introduced by taking only 3 terms. We indicate this by saying that 3.14084 represents $\sqrt[3]{31}$ accurately, as far as .001. The reader will find, by pursuing the same method, that $\sqrt[5]{39} = \sqrt[5]{32+7} = 2.0807$, which results from taking 4 terms of the series, and is accurate as far as .0001.

Should the perfect n^{th} power next below N be at some distance from it, so that $\frac{b}{a^n}$ is not a proper fraction, or a very large one, we must take the one which is next above it, as we did in finding the square root. In that case we shall have

$$N = a^n - y$$

$$\therefore \sqrt[n]{N} = \sqrt[n]{a^n - y}$$

$$= a\left(1 - \frac{y}{a^n}\right)^{\frac{1}{n}},$$

and expanding we shall be able to approximate as before.*

293. The same theorem enables us to find any power of a polynomial expression. Thus having

$$(x+y)^m = x^m + m\,x^{m-1}\,y$$
$$+ m\frac{m-1}{2}\,x^{m-2}\,y^2 + \text{\&c.},$$

for y write $a + z$, and this becomes

$$(x+a+z)^m = x^m + m\,x^{m-1}(a+z)$$
$$+ m\frac{m-1}{2}\,x^{m-2}(a+z)^2 + \text{\&c.},$$

expanding the terms $(a+z)^2$, $(a+z)^3$, &c. by the binomial theorem, this may be converted into a series of single terms. And in like manner, if for z we write $b + v$, we obtain the expansion of $(x + a + b + v)^m$, and so on.

In art. [276] we found the general term of the expansion of the binomial $(x+a)^m$, we may thus find the same for the polynomial $(a + b + c + \text{\&c.} + h)^m$, m being considered a whole number. We have seen, art. [276], that in this case the coefficient of $a^{m-n}\,x^n$ in $(a+x)^m$ was $\frac{m\,(m-1)\ldots(m-n+1)}{1\ .\ 2\ \ \ldots\ \ n}$, and that this was also the coefficient of $a^n\,x^{m-n}$. Call this coefficient M. In the above polynomial expression suppose $b + c +$ &c. $+ g + h = x$. It becomes $(a+x)^m$, and the general term of this expansion is $\mathrm{M}\,a^n\,x^{m-n}$, that is, giving n all successive values between 0 and n we obtain each particular term.

Call $m - n$, m_1; the general term becomes $\mathrm{M}\ a^n\,x^{m}{}_1$, and we see that n and m_1 may have any integral and positive values subject to the condition that $n + m_1 = m$.

Again, since $b + c + \text{\&c.} + h = x$.

Suppose $\quad c + \text{\&c.} + h = y,$

we have $\quad x = b + y,$

and $\quad x^{m}{}_1 = (b + y)^{m}{}_1.$

$$\text{Let } \mathrm{M}_1 = \frac{m_1\,(m_1-1)\ldots(m_1-p+1)}{1\ .\ 2\ \ \ldots\ \ p}.$$

---

* The series for the binomial may be represented in other forms which may sometimes be used with advantage for the purpose of approximation. We subjoin two instances.

$$(a+x)^n = a^n \times \left(\frac{a+x}{a}\right)^n = a^n\left(\frac{a}{a+x}\right)^{-n}$$

$$= a^n\left(1 - \frac{x}{a+x}\right)^{-n}$$

$$= a^n\left(1 + n\,.\,\frac{x}{a+x} + \frac{n\,.\,\overline{n+1}}{1\,.\,2}\,.\,\frac{x^2}{\overline{a+x}^2} + \text{\&c.}\right).$$

Again, since $\quad 1 - \frac{a-x}{a+x} = \frac{2x}{a+x},$

we have $\quad \frac{a+x}{2x} = \frac{1}{1 - \frac{a-x}{a+x}}$

$$\therefore a + x = \frac{2x}{1 - \frac{a-x}{a+x}}$$

$$\therefore (a+x)^n = 2^n\,.\,x^n\,.\left(\frac{1}{1 - \frac{a-x}{a+x}}\right)^n$$

$$= 2^n\,.\,x^n\,.\left(1 - \frac{a-x}{a+x}\right)^{-n}$$

$$= 2^n\,.\,x^n\,.\left(1 + n\,.\,\frac{a-x}{a+x} + \frac{n\,.\,\overline{n+1}}{1\,.\,2}\,.\left(\frac{a-x}{a+x}\right)^2 + \text{\&c.}\right)$$

Then the general term of the last expansion is $M_1 b^p y^{m_1-p}$, or $M_1 b^p y^{m_2}$, m_2, being equal to $m_1 - p$, that is to $m - n - p$, or $m + n + p$ being $= m$. Hence the general term of $(a + n)^m$, or $(a + b + y)^m$ is $M M_1 a^n b^p y^{m_2}$, the only condition of the 3 exponents, which must, of course, be all positive, being that their sum $= m$. Then making $y = c + z$, and proceeding as before, and so on successively, it is clear that we shall at last arrive at the following term as the general term of the expansion of $(a + b +$ &c. $+ g + h)^m$ $M M_1 M_2 \ldots M_i \,.\, a^n b^p c^q \ldots g^v h^{m_i-v}$, the little letter (i) not representing an exponent, but the subscript number to the letter m to which it is affixed. As before, the law of the exponents is, that, being all positive, their sum $= m$.

$$\text{Now } M = \frac{m(m-1)\ldots(m-n+1)}{1\,.\,2 \quad \ldots \quad n}$$

$$M_1 = \frac{m_1(m_1-1)\ldots(m_1-p+1)}{1\,.\,2 \quad \ldots \quad p},$$

$$M_2 = \frac{m_2(m_2-1)\ldots(m_2-q+1)}{1\,.\,2 \quad \ldots \quad q},$$

&c. = &c.

And the last factor will evidently be

$$M_i = \frac{m_i(m_i-1)\ldots(m_i-v+1)}{1\,.\,2 \quad \ldots \quad v}.$$

Again

$$m_1 = m - n$$
$$m_2 = m_1 - p,$$

&c. = &c.

So that the last factor of the numerator of M is greater by unity than the first of the numerator of M_1, and so on for all the quantities $M_1, M_2, \ldots M_i$; the product then of the numerator will be the product of all the whole numbers from m down to $m_i - v + 1$. We have then for tha general term calling $m_i - v$, w

$$\frac{m(m-1)\ldots\ldots(w+1)}{1\,.\,2\ldots n \times 1\,.\,2\ldots p \times \ldots 1\,.\,2\ldots v} \times a^n \times b^p \times c^q \times \text{\&c.}\, g^v h^w.$$

Multiplying, for the sake of symmetry, the numerator and denominator by $w(w-1)\ldots 1$, so that we shall have in the numerator all the numbers from m down to 1, and reversing the order of its factors, we obtain

$$\frac{1 \quad 2 \quad \ldots \quad \ldots \quad \ldots \quad m}{1.2\ldots n \times 1.2\ldots p \text{ \&c.} \times 1.2\ldots v \times 1.2\ldots w} \times a^n \times b^p \times c^q \times \ldots g^v \times h^w.$$

This quantity will represent every term of the developement by giving to $n\, p \ldots w$ all the values of which they are capable, subject to the above condition. Observe that the condition being fulfilled any of them may be equal to zero, in which case the sets of factors $1\,.\,2\ldots n$, $1\,.\,2\ldots p$, &c. belonging to those which are zero must be omitted altogether.

Of Interest.

294. Interest is the value of tha use of money. This value depends upon the plenty or scarcity of unemployed capital in the country, the rate of profits, and various considerations of the sama nature, all, however, following from, and comprised in, the first. Estimated in this way, the value of the use of a given sum of money, though it would be a difficult problem to assign it, is evidently, the circumstances of the country remaining the same, a fixed quantity.

But the consideration paid for the use of money, which is what most writers have defined interest to be, is always a matter of previous agreement between the parties to the transaction, and though necessarily dependent upon its value as above estimated is yet influenced hy other reasons;—the lender considering the probability that the money will be returned, and the interest regularly paid, and the borrower regarding his own circumstances, and the improvement of his expectations and opportunities by the command of a larger capital.

295. The method of settling the consideration to be paid for the use of any sum of money during any time is, by declaring that which is to be paid at the end of a eertain time for the use of a certain sum during that time. This section will be employed in showing how the first is derived from the second. The time made use of in the present day is ordinarily a year, and the sum £100.; £5., or any sum less than £5., (the laws at present forbid a larger,) may be agreed upon and enforced as the consideration, or interest. In the case of £5. being agreed upon, the scale of rémuneration, or as it is usually called the rate of interest, is 5 per centum per annum, or more shortly 5 per cent. Similarly had £4. been agreed upon the rate would have been 4 per cent. We might evi-

dently have referred the rate of interest to any other period besides a year. The Greeks and Romans referred it to a month, and made interest payable (and wisely, for the reader will collect from what follows, that a short interval is desirable) monthly; and, indeed, with us, though the rate of interest is fixed by the sum paid for the use during a year, that sum is usually made payable in 2 equal parts half-yearly, and sometimes in 4 equal parts quarterly.

We may here remark, that the sum upon which interest is charged is called the principal sum, or more shortly the principal.

296. Interest is either simple or compound, according to the manner in which it is calculated. Any sum being due or lent, at the end of a certain time, a year for instance, the interest upon it becomes payable, so that the sum then due, instead of being the sum originally lent, is that sum increased by the interest for a year. If the whole be still unpaid, and interest be still charged upon that sum only which was originally lent, and so on continually after any number of years, then the money is said to be charged with simple interest. In this case it is clear, that the amount of interest is in proportion to the time. But if after the first year, when interest is payable and unpaid, the principal sum and interest due upon it be considered as a new principal sum, and charged with interest accordingly, and so on continually, then the money is said to be charged with compound interest. To illustrate this briefly; £100. is due from B to A, and until payment is to be charged with 5 per cent. simple interest. At the end of the first year £5. is due for interest, at the end of the second year £5. more, and so on £5. for each year, so that the whole interest for any number of years is equal to £5. multiplied by that number. But if the £100. be charged with compound interest, at the end of the first year £5. is due for interest, but at the end of the second year not £5. more, which is the interest of £100. for a year, but a larger sum, namely the interest of £105. at the same rate, and so on.

297. The calculation of simple interest is sufficiently easy. From the interest of a hundred pounds for a year we can by a simple proportion find the interest of any other sum for the same time, and knowing the interest for a year we can find it for any number of years by multiplication. Thus to find the interest of a given sum for any number of years we have the following rule: *Multiply the given sum by the rate of interest, divide by a hundred, and multiply the result by the number of years.*

It is required to find the interest of £512. 10*s.* for 5 years at 4 per cent. per annum.

	£.	*s.*	
	512	10	
		4	rate per cent.
100)	20.50	0	
	20		
	10.00		
	£.	*s.*	
	20	10	interest for one year.
		5	number of years.
	£102	10	answer.

We have considered interest payable at the end of each year, we can, of course, find it for any fraction of a year by a simple proportion. Thus to find the amount of £73. 15*s.* for 2 years and 3 months at $3\frac{1}{2}$ per cent.

	£.	*s.*	
	73	15	
		$3\frac{1}{2}$	
	221	5	
	36	17	6
100)	2.58	2	6
	20		
	11.62		
	12		
	7.50		
	4		
	2.00		

£.	*s.*	*d.*	
2	11	$7\frac{1}{2}$	interest for 1 year.
		2	
5	3	3	interest for 2 years.
	12	$10\frac{7}{8}$	for 3 months or $\frac{1}{4}$ of a year.
5	16	$1\frac{7}{8}$	answer.

298. When a sum of money is not due for some time, but the person who owes it is willing to discharge the debt immediately, he is entitled to some compensation for giving up the use of the money during that time. This compensation is called *discount.*

The sum paid is called the present value of the debt, and is such a sum as

would, at the given rate of interest, amount to the debt at the time when it is due. The discount is the difference between the debt and the present value. Now we know immediately, from the rate per cent., what £100. would amount to in the given time. And the present value would in the same time amount to the given debt. Hence the present value of a sum of money due in a given time is the fourth term of a proportion, the three first terms of which are the numbers representing in pounds the amount of £100. in the given time, 100, and the given sum. Hence we have the following rule: *Multiply the given sum by* 100, *and divide by the number representing the amount of* £100. *in the given time.*

What is the discount on £36. 10*s*. due in 3 months, the rate of interest being 4 per cent.?

£100. in 3 months amounts to £101.

```
          £.   s.
          36   10
              100
101)3650     0(36
     303
     620
     606
      14
       20
  101)280(2
      202
       78
        12
   101)936(9
       909
        27
         4
    101)108(1
        101
          7
```

Hence the present value is £36 2*s*. 9¼*d*.

```
£.  s.  d.
36  10  0
36   2  9¼
 0   7  2¾   answer.
```

Examples of this kind are those which most frequently occur, and as they are worked out in the above manner it was thought advisable to give them so, apart from the algebraical formulæ, which we shall next proceed to investigate. We may here observe, that tables are in practice made use of in questions of interest and discount, by means of which they may be calculated with great rapidity.

299. Let r represent the interest of £1. for 1 year. (This will evidently be the rate per cent. divided by 100.) And let n represent the number of years, and P the principal sum.

Then the interest of £1. for 1 year being r

. . . . 2 is $2r$

. . . . n is nr

and the interest of P for n years is $\mathrm{P}nr$.

Hence if I and M represent respectively the interest, and amount of £P in the given time, we shall have the two following equations:

$$\mathrm{I} = \mathrm{P}nr \quad \ldots\ldots\ldots\ldots \quad (1.)$$

$$\mathrm{M} = \mathrm{P} + \mathrm{P}nr = \mathrm{P}(1 + nr) \, . \quad (2.)$$

The equation (1) is, under a slightly different form, the rule given in art. [292]. There being four different quantities in each of the above equations, by knowing any three of them we can obtain the other. We shall find it necessary in applying the equations to reduce shillings, &c. to decimals of a pound, and months, &c. to decimals of a year. Required the interest on £14. 5*s*. for a year and a half, at 5 per cent.

By equation (1) $\mathrm{I} = \mathrm{P}nr$,

Now $\mathrm{P} = £14.\ 5s. = £14.25$,

$n = 1\tfrac{1}{2}$ years $= 1.5$,

$r = .05$.

Multiplying in the manner adopted in art. [167].

```
14.25
  .05
0.713
  1.5
  743
  357
  1.1
```

which becomes by reduction

£1. 2*s*.

The interest of £73. 15*s*. at five per cent. amounted to £8. 5*s*. 11¼*d*., required the time.

By equation (1) $\quad n = \dfrac{\mathrm{I}}{r\,\mathrm{P}}$,

reducing I to the decimal of a pound,

$$\begin{array}{r} 4)\ 1. \\ 12)\overline{11.25} \\ 20)\ 5.9375 \\ \hline 8.296875 \end{array}$$

$P = £73.\ 15s. = 73.75$

$r = .05,$

$\therefore\ Pr = 3.6875,$

$$\begin{array}{r} 3.6875)8.296875(2.25 \\ 7\ 3750 \\ \hline 92187 \\ 73750 \\ \hline 184375 \\ 184375 \end{array}$$

And the answer therefore is 2.25 years, or 2 years and a quarter.

300. M being the amount of £P in n years, it is evident that P is the present value of £M due in n years. And equation (2) of last article gives us

$$P = \frac{M}{1 + nr}$$

which is, under a different form, the rule given in art. [298] for finding the present value. Similarly if D be the discount.

$$D = M - P = M - \frac{M}{1 + nr}$$

$$= \frac{Mnr}{1 + nr}.$$

It is unnecessary to give examples of these expressions, as they are of the same nature as those of the last article.

301. In almost all money transactions, it is usual, when a deduction is made by way of discount in consequence of immediate payment, to calculate the interest of the sum to be paid, instead of the discount as above given. This gives an advantage to the person so paying, inasmuch as he deducts the interest of the sum to be paid instead of the interest of its present value. But the person receiving is willing to forfeit the difference for being freed from all doubts and uncertainty.

In the same way interest is substituted for discount in the general method of calculating equations of payments.

A owes B £P_1 due at the end of n_1 years, and £P_2, due at the end of n_2 years from the present *time;* at what time must he pay B the sum of £P_1 and P_2, that neither party may gain or lose?

Let n be the number of years required. Then $(n - n_1)$ years is the extra time during which A has the use of P_1, and he is therefore benefited by the interest of £P_1 for that time, or by $P_1(n - n_1)r$. But he pays £P_2 $(n_2 - n)$ years before it is due, and is a loser, therefore, by the discount of P_2 for that time, or by

$$\frac{P_2(n_2 - n)r}{1 + (n_2 - n)r}.$$

Now, in order that he may neither gain nor lose, he must be as much a loser by paying P_2 before it is due, as he is a gainer by paying P_1 after it is due. Equating therefore his gain and loss, and dividing by r we have

$$P_1(n - n_1) = \frac{P_2(n_2 - n)}{1 + (n_2 - n)r},$$

which by reduction becomes a quadratic equation. But in the ordinary method of treating this subject A is considered a loser not by the discount, but by the interest on P_2 for $(n_2 - n)$ years, or by $P_2(n_2 - n)r$. In that case, we have, proceeding as before,

$$P_1(n - n_1) = P_2(n_2 - n);$$

the solution of which gives

$$n = \frac{P_1 n_1 + P_2 n_2}{P_1 + P_2}.$$

Similarly, if n be the equated time for the payment of any number of debts, P_1, P_2, P_3, &c., due at the several times n_1, n_2, n_3, &c., we should, by the same process, arrive at the equation

$$n = \frac{P_1 n_1 + P_2 n_2 + P_3 n_3 + \&c.}{P_1 + P_2 + P_3 + \&c.};$$

which expression is tantamount to the rule usually given: *Add together the products of each debt multiplied by the time when it is due, and divide by the sum of the debts.* Here, as before, the substitution of interest for discount is to the advantage of the debtor. The rule is so simple that it is unnecessary to illustrate it by examples.

302. As soon as a sum of money is payable, it matters little whether it be due under the name of principal or interest; the use of it would be of equal value to its owner. It would, therefore, appear to be equitable that it should be charged with interest in one case as well as the other; in other words, that a debt forborne should be charged with compound interest. It is, however, a singular fact that the laws of

this country never consider it so charged. Thus, supposing a person to have unjustly withheld for any number of years, the annual payment of a certain sum of money by way of interest; he would be simply compelled to pay the annual sum multiplied by the number of years. Considering interest payable yearly, the following rule for finding the amount of a sum of money at compound interest requires no explanation. *Multiply the principal sum by the rate per cent., divide by a hundred, and add the principal sum.* This gives the amount due at the end of the first year. *Multiply again by the rate per cent., divide by a hundred, and add the amount due at the end of the first year.* The result is the amount due at the end of the second year. *The same operation is to be performed as many times as the number of years for which the interest is to be calculated.*

The difference between the amount so found and the original sum is the compound interest. When there is any fraction of a year, it is usual to add the simple interest for this portion of the amount due at the end of the preceding year.

What is the amount of £76. 15*s.* in 2¾ years at 3 per cent. compound interest? We shall reduce the quantities to their respective decimals, and perform the operation as recommended in art.[167], neglecting all decimals beyond the third.

```
      £.
     76.75
         3
100)230.25
     2.303  interest for the first year.
    76.75
    79.053  amount due at end of do.
         3
100)237.159
     2.372  interest for the second year.
    79.053
    81.425  amount due at end of do.
```

We must find the simple interest of this for three quarters of a year.

Multiplying by the rate per cent. we have

244.275, and dividing by 100
2.443

This is the interest for a whole year. To find that for 3 quarters of a year, multiply by ¾, or by .75.

```
             2.443
              .75
             1710
              122
            1.832
adding     81.425   amount due at end
                    of second year.
           83.257   which becomes by
                    reduction.
```

	£.	s.	d.
	83	5	1½
Deducting original sum	76	15	
Compound interest is	6	10	1½.

303. The observations made in the beginning of art. [302] apply to the calculations of discount as well as those of interest. Correctly, then, the present value of a sum of money due in a given time, is such a sum as would at a given rate of compound interest amount to the given sum in that time. The same rule applies here as to the finding the present value at simple interest; with this difference, that the amount of £100. must be calculated at compound interest.

What is the discount on £173. 5*s.* due in 2 years at 5 per cent. compound interest?

	£.	s.
The amount of £100. in 1 year is	105	0
. . . . 2 years .	110	5

or, reducing to the decimal of a pound, 110.25. Reducing also £173. 5*s.* to the decimal of a pound, and proceeding as in art. [167],

```
        173.25
          100
110.25)17325(157.142
        11025
         63000
         55125
          78750
          77175
           1575
           1103
            472
            441
             31
             22
              9
```

The present value is £157.142, which becomes by reduction £157. 2*s.* 5¼*d.*, which subtracted from £173. 5*s.* gives for discount £16. 2*s.* 6¾*d.*

304. We shall retain for the algebraical formulæ for compound interest the notation adopted in art. [292], supposing, moreover, R to represent the amount of £1. in one year, which is evidently the same as $1 + r$.

Now, since £1. amounts to R in the first year, by a simple proportion R must amount to R^2 in the second year, and therefore £1. in two years amounts to R^2. It is thence clear, that in n years £1. amounts to R^n. Hence we have

$$M = PR^n \quad \ldots\ldots\ldots \quad (1.)$$

$$I = PR^n - P = P(R^n - 1) \quad (2.)$$

It is hardly necessary to observe, that these expressions present, under a slight variety of forms, the rules given in art. [302]. The following example will be sufficient.

What is the amount of £5. in 3 years at 5 per cent. compound interest?

Here $R = 1.05$, and $n = 3$.

Multiplying as in art. [167], retaining, however, 4 decimal places, as we shall multiply by 5, art. [167],

$$\begin{array}{r}
1.05 \\
1.05 \\
\hline
105 \\
525 \\
\hline
1.1025 \\
1.05 \\
\hline
1,1025 \\
551 \\
\hline
1.1576 = R \\
5 = P \\
\hline
5.788 \\
20 \\
\hline
15.76 \\
12 \\
\hline
9.12
\end{array}$$

The amount is therefore £5. 15*s*. 9*d*.

This method of calculating compound interest for any number of years is exceedingly tedious, and in practice we must have recourse to logarithms.

305. The interest having been supposed payable at the end of each year, it would seem impossible to calculate *compound* interest for a less period than a year, or to assign any but integral and positive values for n in the equation (1) of the last article. And, indeed, the manner in which the equation was obtained would appear to confine us to such values. But a little further consideration will convince us that, a proper signification being attached to the quantity which M represents, the expression is also true for fractional and negative values. We have before observed one or two instances of the extension which algebraical notation gives to the terms of a question, and the continuity which it implies in the quantities it is employed on. The following remarks will still further illustrate this fact, while the applicability of equation (1) to all values of n, presents another instance of the law of continuity.

By the method of calculating compound interest adopted in art. [295], after finding the interest for one year, we added it to the principal, considered the two together as a new principal, found the interest for a year, added again, and so on. But the algebraical mode of treating the subject, considering R as the amount of £1. in one year, mentions no particular time at which the interest is to be added to the principal, to be from that time itself considered as principal, and charged with interest; on the contrary, the generality of this language forbids that any particular time should be selected in preference to another. The conversion, therefore, of interest into principal must be considered to proceed continuously. The amount, therefore, in a year only fixes the rate of increase, and we may calculate compound interest for a less period than a year with as much propriety as for a greater.

What is the amount of £P at compound interest in 6 months?

R being the amount of £1. in one year, let x be the amount of £1. in 6 months. It is evident, then, that 1, x, and R, are continued proportionals, and, therefore,

$$x^2 = R,$$

$$\therefore x = R^{\frac{1}{2}}.$$

And the amount of P in 6 months is $PR^{\frac{1}{2}}$, the expression we should have derived from equation (1), by making n equal to $\frac{1}{2}$.

Again, what is the amount of £P at

compound interest in the $\frac{1}{m^{th}}$ part of a year.

Let x be the amount of £1. in this time. Then it is clear, from the last example, that x^2 is the amount in $\left(\frac{2}{m}\right)$th parts of a year, and so on; so that $1, x, x^2 \ldots \ldots R$ is a series of continued proportionals of $m + 1$ terms, and

$$\therefore R = 1 \times x^m,$$

$$\therefore x = R^{\frac{1}{m}}.$$

And the amount of P is $R^{\frac{1}{m}}$, the expression we should have obtained by putting $n = \frac{1}{m}$ in equation (1).

Similarly, to find the amount in 2 years, and the $\frac{1}{m^{th}}$ part of a year, we have the amount at the end of 2 years equal to $P R^2$. Let this equal P_1. The amount of this in the $\frac{1}{m^{th}}$ part of a year $= P_1 R^{\frac{1}{m}} = P R^2 R^{\frac{1}{m}} = P R^{\frac{2m+1}{m}}$.

Now M is the value in n years of £P due at the present time, and in the same way as if n refers to a succeeding period M is the amount of £P, so if it refers to a preceding one P is the amount of £M in n years.

or in the latter case $P = M . R^n$,

and $\therefore M = \frac{P}{R^n} = P . R^{-n}$.

From all this we conclude that the equation (1) of the last article originally obtained for integral is also true for fractional and negative values of n. It will have occurred to the reader that the method formerly adopted of calculating the compound interest for $2\frac{3}{4}$ years, where, after finding the amount at the end of 2 years, we took the simple interest of this amount for $2\frac{3}{4}$, was incorrect, according to the principles last laid down. We shall apply these to two examples.

Required the amount of £153. 10*s.* in 1 year and a half at 21 per cent. compound interest.

R here = 1.21.

And reducing to decimals, the amount is $(153.5) \times (1.21)^{\frac{3}{2}}$.

$$\text{But } (1.21)^{\frac{3}{2}} = (1.21) \times (1.21)^{\frac{1}{2}}$$
$$= (1.21) \times (1.1)$$
$$= 1.331$$

153.5
1.331
153.5
46 05
4 605
154
204.309

which becomes, by reduction, £204.6*s.*2*d.*, which is the answer.

What is the amount of £6. in $2\frac{1}{2}$ years at 3 per cent. compound interest?

Here R = 1.03

$$n = 2\frac{1}{2} = \frac{5}{2}$$

$$\therefore R^n = (1.03)^{\frac{5}{2}} = (1 + .03)^{\frac{5}{2}}.$$

Now the following equation, art. [280], is true as far as it goes, that is, there is no term included in the &c. which is not multiplied by a higher power of x than the second,

$$(1+x)^m = 1 + m\,x + m\frac{m-1}{2}x^2 + \&c.$$

We shall suppose $x = .03$ and $m = \frac{5}{2}$, and since we only require a result accurate to 4 places of decimals, art. [167], we may neglect every power beyond the second of .03,

for $(.03)^3 = .000027$.

We have, then,

$$1 = 1$$

$$m\,x = \frac{5}{2} \times .03$$
$$= \frac{.15}{2}$$
$$= .075$$

$$m\frac{m-1}{2}x^2 = \frac{5}{2} \times \frac{3}{4} \times .0009$$
$$= \frac{15}{8} \times .0009$$
$$= \frac{1}{8} \times .0135$$
$$= .0017,$$

and, adding, $R^n = 1.0767$

$$P = 6$$

$$\therefore P R^n = 6.4602,$$

which, by reduction, becomes £6.9s.2½d.

306. The equation $M = P R^n$ contains 4 different quantities, from knowing 3 of which we can find the other. It was observed in art. [299], that in calculating the compound interest for any number of years we were obliged to have recourse to logarithms. We are also unable to find the value of n, when that is the unknown quantity, without similar assistance. Thus taking the logarithms of both sides of the above equation, and referring to our section on logarithms,

$$\text{Log. } M = \log. P . R^n$$
$$= \log. P + \log. R^n$$
$$= \log. P + n \log. R \ldots (1)$$

$$\text{And } \therefore n = \frac{\log. M - \log P}{\log. R} \ldots (2)$$

Thus to find the amount of £15. 10s. in 10 years at 5 per cent. compound interest, we should have to multiply 1.05 by itself 10 times by the ordinary process. Referring to equation (1), and observing that £15. 10s. = £15. 5, we have

$$\text{Log. } P = 1.1903317$$
$$n \text{ Log. } R = .211893$$
$$\therefore \text{Log. } M = 1.4022247$$

And $\therefore$ M = 25.248, which, by reduction, becomes £25. 4s. 11¼d.

Of Annuities.

307. From the greater complication of the subject we shall dispense with giving arithmetical rules for calculating annuities, and proceed at once to the algebraical method. We thus find the amount of an annuity forborne any number of years, supposing it charged with simple interest.

Let the annuity be represented by A, and supposed to be payable at the end of every year. Retaining in other respects the same notation as in art. [299], and observing that the interest *for* each year is charged on the sum of the annual payments due at the end of the preceding year, we have

Due *for* the 1st year . A

„ 2nd . . . $A + A r$

„ 3rd . . . $A + 2 A r$

„ n^{th} . . . $A + (n-1) A r$,

and the sum of these or $n A + (1 + 2 + \&c. + (n - 1)) A r$ gives the whole amount. The coefficient of $A r$ is an arithmetic series of $\overline{n-1}$ terms, whose common difference is 1, and, therefore, the sum of it, art. [143], is

$$\left(2 + (n-2)\right)\frac{n-1}{2}, \text{ or } n . \frac{n-}{2},$$

$$\therefore M = n A + n . \frac{n-1}{2} r A.$$

A promised to pay B £10. at the end of every year, but neglected to do so. What was due to B at the end of the 20th year, simple interest being charged at 5 per cent.?

Here $A = 10$, $r = .05$, $n = 20$, and

$$\therefore n\frac{n-1}{2} = 190,$$

and $$r A = .5.$$

Hence $$n . \frac{n-1}{2} . r A = 95,$$

$\therefore M = 200 + 95$, and the amount due is £295.

Estimated at simple interest, the present value of an annuity is found as follows.

The present value of A, art. [298], to be paid at the end of 1 year, is equal to $\frac{A}{1+r}$, to be paid at the end of 2 years is equal to $\frac{A}{1+2r}$, and so on, and to be paid at the end of n years $= \frac{1}{1+nr}$.

Now the sum of these present values is the present value of A to be paid at the end of 1, 2, and n years, or of an annuity of £A to continue n years. Thus we have P representing the present value, P =

$$A\left(\frac{1}{1+r} + \frac{1}{1+2r} + \&c. + \frac{1}{1+nr}\right\} (1).$$

If we had supposed the first annual payment to be made at the end of the m^{th} instead of the 1st year, and then to continue n years, we should, by a like process, obtain for the present value

$$A\left\{\frac{1}{1+mr} + \frac{1}{1+(m+1)r} + \&c. + \frac{1}{1+(m+n)r}\right\}$$

What is the present value of an annuity of £5., to continue 3 years, at $1\frac{1}{2}$ per cent. simple interest?

Here $n = 3$,

$$\therefore P = 5 \times \left\{\frac{1}{1+r} + \frac{1}{1+2r} + \frac{1}{1+3r}\right\}.$$

Now $r = .015$ and r^3 and the higher powers of r are so small as not to have any influence in the result, and may, therefore, be neglected. We have then, art. [284],

$$\frac{1}{1+r} = 1 - r + r^2$$

$$\frac{1}{1+2r} = 1 - 2r + 4r^2$$

$$\frac{1}{1+3r} = 1 - 3r + 9r^2,$$

and their sum $= 3 - 6r + 14r^2$, which, putting for r its value, and performing the operations indicated, is equal to 2.9132, and multiplying by 5, $P = 14.566$, which, by reduction, becomes £14. 11*s*. $3\frac{3}{4}$*d*.

308. Some writers have defined the present value, estimated at simple interest, of an annuity to continue any number of years, to be that sum the amount of which would, in the given number of years, be equal to the amount of the annuity. But the sum thus obtained is not the present value of the annuity, but of the amount of the annuity after the given number of years. This amount, by art. [307], is $nA + n \cdot \frac{n-1}{2} rA$ and P' being the present value,

$$P'(1 + nr) = nA + n\frac{n-1}{2} rA,$$

or

$$P' = \frac{nA + n \cdot \frac{n-1}{2} rA}{1 + nr},$$

which differs from P the present value of the *annuity*, found in art. [307], as would be shown by substituting any number greater than unity for n in the values of P and P′. The meaning we give to the expression present value would naturally lead us to expect the two quantities, P and P′, to be equal. Their inequality is the strongest proof of the inadequacy of a mode of calculation, like that of simple interest, which, as it were, sets a mark upon any sums of money that may have accrued by way of interest, and forbids their future accumulation. The reason of their inequality is easily explained. Suppose p to be the present value of £m due in one year. Then $p = \frac{m}{1+r}$, and let us suppose m to be unpaid for a second year and charged with interest; it amounts to $m(1 + r)$. But p in two years amounts to $p(1 + 2r)$, or to $\frac{m(1+2r)}{1+r}$, which is different from the amount of m, and the reason is, because pr, the interest on p for the first year, is not charged with interest for the second year; and, therefore, in one case m was charged with interest and in the other only p. Therefore p, which is the present value of m, is not the present value of the amount of m after any number of years. The application of this to each payment of the annuity is manifest.

The present value of an annuity to continue for ever is found by making n infinite in the expressions for P and P′. The first becomes equal to

$$A\left(\frac{1}{1+r} + \frac{1}{1+2r} + \text{\&c.}\right),$$

the series being continued *ad infinitum*; and the latter becomes itself infinite, which is an additional proof of the inapplicability to practice of the principles upon which it rests.

309. R being, as before, the amount of £1. in one year, we thus find the amount of an annuity, forborne any number of years, at compound interest. Observing that the *whole* sum due at the end of each successive year is one of the annual payments, together with the amount in one year of the sum due at the end of the preceding year, we have due at the end of

1st year, A

2nd . $A + AR$

3rd . . $A + AR + AR^2$

n^{th} . . $A + AR + \text{\&c.} + AR^{n-1}$,

or the amount in n years is $A(1 + R + \text{\&c.} + R^{n-1})$. Now the quantity within the brackets is a geometric series of n terms, commencing with unity and having R for a common ratio, and therefore the sum of it [art. 151] is $\frac{R^n - 1}{R - 1}$,

and $\therefore M = A\frac{R^n - 1}{R - 1} \ldots\ldots (1).$

To find the present value estimated at compound interest we have, art. [299], the present value of A to be paid at the end of 1 year is equal to $\frac{A}{R}$, to be paid at the end of 2 years is equal to $\frac{A}{R^2}$, and to be paid at the end of n years is equal to $\frac{A}{R^n}$, so that the sum of these present values, which is the present value of the annuity to continue n years, is

$$\frac{A}{R} + \frac{A}{R^2} + \&c. + \frac{A}{R^n},$$

or, P being the present value,

$$P = \frac{A}{R}\left(1 + \frac{1}{R} + \&c. + \frac{1}{R^{n-1}}\right) \ldots (2).$$

The quantity within the brackets is a geometric series of n terms, whose common ratio is $\frac{1}{R}$ and first term unity, and its sum, art. [151], is $\frac{1 - \frac{1}{R^n}}{1 - \frac{1}{R}}$,

and $\therefore P = A . \frac{\left(1 - \frac{1}{R^n}\right)}{R - 1}$.

The present value of the amount M of the annuity in n years is $\frac{M}{R^n}$, or (putting for M its value in the present article) it becomes $A \times \frac{1 - \frac{1}{R^n}}{R - 1}$, which is the same as the present value of the annuity. The reader remembers that the two were different when calculated by simple interest.

Had the first payment been made at the end of the m^{th} instead of the first year, and continued n years afterwards, we should have had by the same process the present value represented by

$$\frac{A}{R^m} + \frac{A}{R^{m+1}} + \&c. + \frac{A}{R^{m+n-1}},$$

or by

$$\frac{A}{R^m}\left\{1 + \frac{1}{R} + \&c. + \frac{1}{R^{n-1}}\right\},$$

that is summing the geometric series, art. [151], by

$$\frac{A}{R^m} \times \frac{1 - \frac{1}{R^n}}{1 - \frac{1}{R}},$$

or $$\frac{A}{R^{m-1}} \times \frac{1 - \frac{1}{R^n}}{R - 1} \ldots\ldots (3).$$

If the annuity be supposed to continue for ever, the series in equation (2) will go on *ad infinitum*, and P being the present value, we shall have

$$P = \frac{A}{R}\left(1 + \frac{1}{R} + \frac{1}{R^2} + \&c.\ ad\ infinitum\right),$$

or, art. [153],

$$P = \frac{A}{R} \times \frac{1}{1 - \frac{1}{R}} = \frac{A}{R - 1} \ldots (4);$$

and similarly if the first payment be made after the m^{th} year, and then continue for ever, the present value is

$$\frac{A}{R^{m-1}} \times \frac{1}{R - 1}.$$

The rate of interest is $3\frac{2}{5}$ per cent., what sum of money is equivalent to an income of £3. a year?

$3\frac{2}{5}$ is equal to 3.4 per cent., and, therefore, $R = 1.034 \therefore R - 1 = .034$.

By equation (4) of this article

$$P = \frac{A}{R - 1} = \frac{3}{.034}, \text{ or } \frac{3000}{34},$$

which, by division, is equal to $88\frac{4}{17}$, or $88\frac{1}{4}$ nearly. The answer then is £$88\frac{1}{4}$.

Indeterminate Equations.

310. In our sections on simple and quadratic equations, art. [108], &c. and art. [204], &c., we considered those cases only where there was the same number of independent equations as of unknown quantities. The method of solution adopted in this case was, when there were several unknown quantities, by combining the equations in any manner to obtain an equation involving only one unknown quantity, see art. [119], and

art. [213]. We now propose to examine those cases where we have a less number of equations than of unknown quantities. Imperfect as our general means of solution were in the former case, except where our equations were all of the first degree, we shall find our powers in this case confined within still narrower limits.

We begin, as before, with equations of the first degree, and first with the most simple case of one equation involving two unknown quantities, which may be represented generally by

$$a x + b y = c.$$

We observe, in the first place, that if the values of x and y which we are seeking for may be of any kind, positive or negative, whole numbers or fractions, there are an infinite number of such values which satisfy this equation, see art. [119]. For we have only to substitute *any* value for one of the unknown quantities, y for instance, and then solve the equation with respect to the other, x, that is, considering x as the only unknown quantity, and the assumed value for y, and the value found for x, present us with a solution of the equation. It is for this reason, that equations of this kind are called indeterminate, because they do not fix, or determine, the values of the unknown quantities. All systems of equations, where there are more unknown quantities than independent equations, are indeterminate in the same sense of the word; for, retaining as many unknown quantities as there are equations, we may give the rest any values we please, and thus arrive at an infinite number of solutions. But returning to the equation $a x + b y = c$, suppose that we were seeking only such values of x and y as being whole numbers, or being whole numbers and positive, satisfied this equation, the above method would not be of any service, for although we might assume a whole number for y, yet the value of x obtained in the above manner, and which with the assumed value for y would form a solution of the equation, would, in all probability, be a fraction.

In the second place, the equation $a x + b y = c$, being cleared of fractions, and in its lowest terms, in order that any integral values of x and y may be capable of satisfying this equation, it is necessary that a and b be prime to each other. For, supposing for a moment this not to be the case, so that a and b having a common divisor, suppose r, may be written under the form $e r$ and $f r$, where e, f, and r are whole numbers, substituting their values of a and b in the equation $a x + b y = c$, it becomes $e r x + f r y = c$, or, dividing by r, $e x + f y = \frac{c}{r}$. But the equation having been previously in its lowest terms, $\frac{c}{r}$ must be a fraction, and it is in that case evident, that the equation $e x + f y = \frac{c}{r}$, cannot be satisfied by integral values of x and y.

We shall now show how we may find the whole numbers which satisfy the equation $a x + b y = c$, observing that questions which give rise to equations of this kind, usually require integral values of the unknown quantities. It will be best to commence by taking a numerical example of this equation, and to consider the more general case afterwards.

311. It is required to find the integral values of x and y in the equation $5 x + 7 y = 81$. We observe here, that the conditions above alluded to are satisfied, the equation being in its lowest terms, and 5 and 7 being prime to each other.

Transforming to the right-hand side of the equation, the term having the largest coefficient, we have

$$5 x = 81 - 7 y.$$

Dividing by 5,

$$x = \frac{81 - 7 y}{5}.$$

Dividing out as much as possible,

$$x = 16 + \frac{1}{5} - y - \frac{2 y}{5}$$

$$= 16 - y - \frac{2 y - 1}{5} \dots (1)$$

Our having transformed the term with the larger coefficient, has, we see, enabled us to simplify the expression by dividing out by the smaller coefficient. Now any value of y being substituted in this equation, that value of y, together with the value of x derived from the same equation, form, in the general sense of the word, a solution of the equation. But we are only seeking integral values of x and y; we shall, therefore, now find such a value of y as, being itself a whole number, will, when substituted in equation (1), make the

value of x derived therefrom a whole number.

Now, in order that x may be a whole number, it is necessary that $\frac{2y-1}{5}$ be a whole number.

Let $$\frac{2y-1}{5}=v,$$

v being any whole number,

$$\therefore 2y-1=5v.$$

This is an equation of the same kind as the original one, but its terms are simpler, and necessarily so, from the operation of dividing out, before made use of. In order to find the values of y and v we proceed as before. We thus obtain

$$y=\frac{5v+1}{2}\ldots\ldots(2)$$

$$=2v+\frac{v+1}{2}.$$

Now y being a whole number, $\frac{v+1}{2}$ must be a whole number.

Let $$\frac{v+1}{2}=w,$$

w being any whole number.

We obtain from this,

$$v=2w-1\ldots\ldots(3).$$

From all these operations we see that, w being any whole number, the value of v derived from equation (3) is integral, as also the value of y from equation (2), and that of x from equation (1). The equation (3) is of the same nature as equation (1), but the coefficient of v being unity we may assume any value for w, and are certain of an integral value for v. Although the coefficient of v might have been some whole number greater than unity, yet by pursuing the same process we are sure of arriving ultimately at an equation of this form, the operation of dividing out constantly diminishing the coefficients of the quantities y, v, &c.

Substituting the value of v derived from equation (3), in equation (2) we have

$$y=4w-2+w,$$

or $$y=5w-2\ldots\ldots\ldots(4).$$

And again, substituting this value of y in equation (1),

$$x=16-5w+2-2w+1,$$

or

$$x=19-7w\ldots\ldots\ldots\ldots\ldots(5).$$

The corresponding values of x and y obtained by giving all integral values to w in the equations (4) and (5), present so many solutions of the equation. The quantity w is called an indeterminate quantity, or, more shortly, an indeterminate.

Supposing then w equal to 1, we have

$$x=12,$$
$$y=3.$$

Supposing w equal to 2,

$$x=5,$$
$$y=8.$$

And giving to w all succeeding integral values from 1 upwards, we obtain the following corresponding values of x and y.

12 and 3,
5 8,
− 2 13,
− 9 18,
&c. &c.

We shall presently return to the law which the several values of x and y follow.

312. In solving equations of this kind we always endeavour to arrive at values of x and y expressed in terms of some indeterminate w, which is susceptible of all integral values, as in equations (4) and (5). The reader will find by trial, that unless the coefficients of x and y are prime to each other, the attainment of this result is impracticable, art. [308]. It is frequently much shortened by the employment of various artifices for which no general rule can be given. They can be only learnt by observation and practice.

Thus taking the equation

$$11x-17y=5.$$

Proceeding as before,

$$11x=17y+5,$$

and $$x=\frac{17y+5}{11}$$

$$=y+\frac{6y+5}{11}.$$

If we continued to proceed as before, we should put $\frac{6y+5}{11}=v$, but observing, that the difference between 6, the coefficient of y, and the denominator 11, is equal to the other term 5, we put the above equation under another form, namely,

$$x = y + \frac{11y - 5y + 5}{11},$$

$$x = 2y - \frac{5y - 5}{11},$$

$$= 2y - \frac{5(y-1)}{11}.$$

Now 5 being prime to 11, $\frac{y-1}{11}$ must be a whole number.

Let $\frac{y-1}{11} = w$,

$\therefore y - 1 = 11w$,

and $y = 11w + 1$.

Also $x = 2y - 5w$,

$= 22w + 2 - 5w$,

or $x = 17w + 2$.

So that the several integral values of x and y, obtained by giving to w all integral values from 0 upwards, are

2 and 1,
19 ... 12,
36 ... 23,
&c. &c.

To find a number which, when divided by 6, gives a remainder 5, and when divided by 7, a remainder 3. Let us suppose 6 to be contained x times in the number in question, with a remainder 5, then the number is $6x + 5$.

And similarly, supposing 7 to be contained y times, with a remainder 3, the number is $7y + 3$. Equating the two values of the number, we have the following equation:

$$6x + 5 = 7y + 3,$$

or $6x = 7y - 2$,

$$\therefore x = \frac{7y-2}{6},$$

$$= y + \frac{y-2}{6}.$$

Let $\frac{y-2}{6} = w$,

then $y = 6w + 2$,

and $x = 6w + 2 + w$,

$= 7w + 2$.

The several pairs of values of x and y are

2 and 2,
9 8,
16 14,
25 20,
&c. &c.

And the several values of $6x + 5$, or $7y + 3$, or of the number required, are

17, 59, 101, &c.

numbers which will be found upon trial to satisfy the proposed condition.

A person has only crowns and 3 shilling pieces in his pocket, and wishes to pay a bill of £2. 16s. How many must he give of each?

Let x = the number of 3 shilling pieces,
and y = ... crowns,
then the sum expressed in shillings is $3x + 5y$, and therefore we have by the question,

$$3x + 5y = 56,$$

$$\therefore 3x = 56 - 5y,$$

$$x = \frac{56 - 5y}{3},$$

or $x = 18 - y - \frac{2y-2}{3}$,

$$= 18 - y - \frac{2(y-1)}{3}.$$

Let $\frac{y-1}{3} = w$,

then $y - 1 = 3w$,

and $y = 3w + 1$;

also $x = 18 - (3w + 1) - 2w$,

$= 17 - 5w$.

And the several pairs of values for x and y are

17 and 1,
12 4,
7 7,
&c. &c.
− 3 and 13,
&c. &c.

So that 17 pieces of 3 shillings and 1 crown, 12 pieces of 3 shillings and 4 crowns, &c., make up the sum required. The negative values for x signify that so many pieces of 3 shillings must be given back. Thus the sum may be made up by giving 13 crowns, 3 pieces of 3 shillings being returned. Suppose it had been required to pay the same sum in crowns and half sovereigns. A moment's consideration shows this to be impossible, and forming the equation we shall find that the coefficients of x and y admit a divisor 5, to which 56, the number on the other side of the equation, is prime, art. [310].

313. If we observe the several values of either of the unknown quantities in any of the above examples, we shall find that they form an arithmetical progres-

sion, the common difference of which is the coefficient of the other quantity. For instance, in the first example, $5x + 7y = 81$, the several values of x were

$$12, \quad 5, \quad -2, \quad -9, \quad \&c.$$

and those of y

$$3, \quad 8, \quad 13, \quad 18, \quad \&c.$$

In the former we observe each term is less than the preceding by 7, the coefficient of y, and in the latter, each term is greater by 5, the coefficient of x.

In order to show that this is necessarily the case, we will consider the general equation $ax + by = c$. This equation is in its lowest terms, and a and b are prime to each other. Suppose we have found x' and y', two integral values of x and y, which satisfy this equation, we have

$$ax' + by' = c \dots . (1).$$

Now let $x' + m$, and $y' + n$, be two other integral values of x and y, so that

$$a(x' + m) + b(y' + n) = c \dots (2).$$

We propose to find what relations must subsist between m and n.

Substracting equation (1), term by term, from equation (2), we have

$$am + bn = 0,$$

or $$am = -bn,$$

therefore $$m = -\frac{bn}{a} \dots . (3).$$

Now observing that b and a are prime to each other, in order that m may be a whole number it is necessary that n should be a multiple of a; and n must also be a whole number. Let then $n = aw$, w being capable of all integral values, positive as well as negative. Equation (3) gives us $m = bw$. Hence x' and y' being any two values of n and y, we have all others represented by

$$x' - bw,$$

$$y' + aw,$$

w being capable of all integral values.

From this result we learn, *first*, that the several integral values of x and y, which satisfy the equation $ax + by = c$, must necessarily be of the form above indicated.

Secondly. That a and b being both positive, while the value of y is increased by the coefficient of x, that of x is diminished by the coefficient of y, and *vice versâ*; but if a or b be negative, then their corresponding values are both increased or both diminished by the coefficient of the other.

Thirdly. That the indeterminate w being capable of negative as well as positive values, the arithmetic series at the beginning of the present article giving the several values of x and y, may be continued indefinitely to the left as well as the right.

It is clear from what has preceded, that having obtained any two integral values of x and y, which satisfy the equation, we can immediately (from the equations $x = x' - bw$, $y = y' + aw$) find an infinite number of such values. The main object then is the finding with rapidity two such values. A property of a *continued fraction* has been made use of for this purpose, and we shall briefly explain the manner in treating of that subject.

314. When we have one equation of the first degree involving more than two unknown quantities, the method of proceeding is very similar. We shall simply go through the operations, their explanation being the same as that given in the case of two unknown quantities.

$$4x + 9y + 10z = 103,$$

transforming

$$4x = 103 - 9y - 10z,$$

and $$x = \frac{103 - 9y - 10z}{4},$$

$$= 25 - 2y - 2z - \frac{y + 2z - 3}{4} \dots (1).$$

Let $$\frac{y + 2z - 3}{4} = w,$$

$$\therefore y + 2z - 3 = 4w,$$

and $$y = 4w + 3 - 2z \dots . (2).$$

Substituting this value of y in equation (1),

$$x = 25 - 8w - 6 + 4z - 2z - w,$$

or $$x = 19 - 9w + 2z \dots . (3).$$

We have thus the values of x and y expressed in terms of z, one of the unknown quantities, and an indeterminate w, but the quantity z is so involved that giving to it any integral value, the corresponding values of x and y are also integral. Suppose z equal to 0, and giving to w sucessively the values 0, 1, 2, &c., we find the corresponding values of x and y to be

19 and 3,
10 ... 7,
1 ... 11,
&c. &c.

Next suppose $z = 1$, the corresponding values of x and y are

21 and 1,
12 5,
3 9,

and so on.

If it be necessary that the values of all the unknown quantities be positive, we must not give to z a greater value than 9. In that case x and y are each equal to unity.

A similar method may be adopted whatever be the number of unknown quantities, and we shall ultimately arrive at two values of x and y, expressed in terms of the other unknown quantities, and one indeterminate, or we shall have the values of several of the unknown quantities expressed in terms of the others, and of several indeterminates.

315. Still considering only simple equations, suppose that we have several equations involving, however, a greater number of unknown quantities, as, for example, two equations involving three.

$$14x + 11y + 9z = 360 \ldots\ldots (1),$$

$$x + y + z = 30 \ldots\ldots (2).$$

Our object is to find such *integral* values of x, y, and z as satisfy at the same time both these equations. The process adopted is analogous to that in art. [119], where we had two equations between two unknown quantities, and to the reasoning of that article the reader is referred for an explanation of what follows.

Multiplying equation (2) by 14, the coefficient of x in equation (1), we have

$$14x + 14y + 14z = 420.$$

Subtracting equation (1), term by term, from this, $14x$ disappears, and we have

$$3y + 5z = 60 \ldots\ldots (3).$$

From this equation we find the values of y and z in terms of an indeterminate w, which values are as follows,

$$y = 5 + 5w \ldots\ldots (4),$$

$$z = 9 - 3w \ldots\ldots (5).$$

Subtracting these values of y and z in equation (2), and reducing, we obtain

$$x = 16 - 2w \ldots\ldots (6).$$

Making w then successively equal to 0, 1, 2, &c., we obtain the following corresponding values of x, y, and z:

16, 5 and 9,
14, 10 . . . 6,
12, 15 . . . 3,
10, 20 . . . 0,
&c. &c.

In the above equation the coefficients of the unknown quantities in one equation being equal to unity, the process was very simple. The following example will show more fully the method of solving equations of this kind.

$$2x + 5y + 3z = 108 \ldots\ldots (1),$$

$$3x - 2y + 7z = 95 \ldots\ldots (2).$$

Multiplying equation (1) by 3, the coefficient of x in equation (2), and equation (2) by 2, the coefficient of x in equation (1), we obtain

$$6x + 15y + 9z = 324,$$

$$6x - 4y + 14z = 190,$$

and, subtracting,

$$19y - 5z = 134 \ldots\ldots (3).$$

The values of y and z obtained from this equation in terms of an indeterminate t, are

$$y = 11 + 5t \ldots\ldots (4),$$

$$z = 15 + 19t \ldots\ldots (5).$$

Substituting these values of y and z in equation 2, we have

$$3x - 22 - 10t + 105 + 133t = 95,$$

and, transposing,

$$3x + 123t = 12.$$

Here we have again an equation between two unknown quantities, to which we should apply the method for solution of equations of this nature, and thus have values of x and t in terms of another indeterminate w. The substitution of the value of t in the equations (4) and (5), would give us the values of y and z in terms of w. We should thus have the values of x, y, and z in terms of the same indeterminate w. But observing the above equation, we see that all its terms are divisible by 3, and, dividing, we obtain

$$x + 41t = 4,$$

which gives us at once

$$x = 4 - 41t \ldots\ldots (6).$$

Thus equations (4), (5), and (6) give us at once the values of the three unknown quantities in terms of a quantity t, to which we may give any integral value. It will frequently happen that equations of this kind do not admit a solution in whole numbers, and that will be indicated by one of the equations which we arrive at between two unknown quantities, having its coefficients divisible by a number, of which the other sum is not a multiple. See art. [310].

316. By combining any two simple equations, whatever be the number of

unknown quantities comprised in them, in the way in which equations (1) and (2), in the last example, were combined, we may always arrive at an equation involving one unknown quantity less than those two equations. See art. [119]. When we do this, we are said to *eliminate* that unknown quantity. Thus equation (3), in the two last examples, arose from the elimination of x from the equations (1) and (2). Suppose then that we have p equations involving any greater number of unknown quantities. Combining the first of these equations with the second, third, and every other successively, and eliminating by each combination the same unknown quantity, we arrive at $p - 1$ equations, involving one unknown quantity less than the p original equations. Thus combining the first of the $p - 1$ equations with every other successively, we get rid of another unknown quantity, and arrive at $p - 2$ equations. Proceeding similarly, we shall at last arrive at *one* equation, involving a certain number of the unknown quantities, which we may solve by one of the methods which we have given, and find values of the unknown quantities comprised in it in terms of one or more indeterminates. This will lead us, by the continuation of a similar process, to the values of all the unknown quantities. It might at first appear, that, after combining the first of our p equations with each of the other, we might also combine the second with the third, and so on, and thus, eliminating the same unknown quantity as before, obtain more independent equations. But that, this is not the case may be proved as follows. Having transposed all the terms of each of the equations to the left-hand side, art. [109], we may write them $A = 0$, $B = 0$, $C = 0$, &c. Suppose a, b, c, &c. to be the coefficients of x, the quantity which we propose to eliminate, in the equations $A = 0$, $B = 0$, $C = 0$, &c. Then in order to get rid of x from A and B, we multiply the first by b, and the second by a, and subtract. We thus obtain

$$A b - B a = 0 \quad \ldots\ldots\ldots \quad (1).$$

Similarly, combining $A = 0$ with $C = 0$, we obtain

$$A c - C a = 0 \quad \ldots\ldots\ldots \quad (2).$$

Now combining $B = 0$ with $C = 0$, the equation we should arrive at, independent of x, is

$$B c - C b = 0 \quad \ldots\ldots\ldots\ldots \quad (3);$$

and this equation is not independent of the equations (1) and (2), but derivable from them, as follows. Multiply equation (1) by c, and equation (2) by b, and subtract, we obtain

$$B a c - C a b = 0,$$

or, dividing by a,

$$B c - C b = 0,$$

which is the same as equation (3). The same may be proved of any combinations of the equations. We may hence conclude, that, when we have any number of simple equations less than the number of unknown quantities, we cannot, by combining them in any manner, arrive at the same number of independent equations as unknown quantities.

317. When indeterminate equations are above the first degree, their solution is one of considerable difficulty. If in one equation with two unknown quantities even the square of either of them occurs, its discussion would require a separate treatise. We will take one example, where the only term above the first degree is the product of the two unknown quantities, the solution of which will give some notion of the method to be generally adopted.

Required two numbers whose product, added to three times the first, and five times the second, is equal to 48.

x and y being the two numbers, the question gives us the following equation:

$$x y + 3 x + 5 y = 48.$$

Transposing the term not involving x, and collecting the coefficients of x,

$$x (y + 3) = 48 - 5 y,$$

and dividing by the coefficient of x,

$$x = \frac{48 - 5 y}{y + 3}.$$

We may get rid of y from the numerator by writing this

$$x = \frac{63 - 5 (y + 3)}{y + 3},$$

or

$$x = \frac{63}{y + 3} - 5.$$

Now, in order that x may be a whole number, 63 must be a multiple of $y + 3$, and the several numbers of which 63 is a multiple are 3, 7, 9, 21. Hence $y + 3$ must be equal to one of these numbers. In the first case y would be equal to 0, but if we take 7, we have $y = 4$, and

$$x = \frac{63}{y+3} - 5$$
$$= 9 - 5 = 4,$$

or the two numbers are 4 and 4, as will be found correct by trial. Again, taking 9, we have $y = 6$, and

$$x = \frac{63}{9} - 5 = 2,$$

and we shall find the numbers 2 and 6 answer the proposed condition. The other value of $y + 3$, viz. 21, would make x negative.

On Continued Fractions.

318. When we have any fraction, as for instance $\frac{61}{13}$, dividing out as far as we can, the fractional part will become a proper fraction, and we have a mixed fraction $4\frac{9}{13}$, or, representing it algebraically,

$$4 + \frac{9}{13}.$$

Now this may be written

$$4 + \frac{1}{\frac{13}{9}}.$$

We may now proceed further with the division, for

$$\frac{13}{9} = 1 + \frac{4}{9}.$$

Writing this in the denominator we have

$$4 + \frac{1}{1 + \frac{4}{9}}.$$

Again, $$\frac{4}{9} = \frac{1}{\frac{9}{4}} = \frac{1}{2 + \frac{1}{4}}.$$

And putting this value of $\frac{4}{9}$ in the previous expression, we have the fraction $\frac{61}{13}$ represented by the following expression:

$$4 + \frac{1}{1 + \frac{1}{2 + \frac{1}{4}}}.$$

A fraction represented in this way, where we observe the numerators of the several fractions which enter into it are all equal to unity, is called a *continued fraction*. We shall here be able to enter into a few only, and those the most simple and obvious, of the properties which it possesses.

In the first place it enables us to express in simpler terms the value of any given fraction to any degree of accuracy which the question may require. Thus taking the above expression representing the fraction $\frac{61}{13}$, and neglecting the fractional part altogether, we have for our first approximation 4, which differs from the accurate value of the fraction by $\frac{9}{13}$. Again, taking only the first term of the denominator, and neglecting the rest, our second approximation will be $4 + \frac{1}{1}$, or 5, and this differs from the accurate value of the fraction by $\frac{4}{13}$. Observe that this approximation is greater than the fraction itself, while the former one was less. Proceeding one step further, and representing the fraction by

$$4 + \frac{1}{1 + \frac{1}{2}},$$

our third approximation is $4 + \frac{2}{3}$, or $\frac{14}{3}$, which, like the first approximation, is less than the accurate value of the fraction, and differs from it by $\frac{1}{39}$, approaching nearer to it than either of the former values. We shall presently treat the subject more generally, and shall then see that it necessarily results from the form of the expression that the successive approximations, derived in the above manner, err alternately by excess and deficiency, and approach nearer and nearer to the accurate value of the fraction.

As an example of this, we will examine the length of the tropical year, or of the interval between the sun's leaving

and returning to the same equinox.* The most accurate calculations have proved this interval (upon which the seasons mainly depend) to be equal to 365,242264 days. Representing this fractionally, we have

$$365 + \frac{242264}{1000000},$$

or, reducing the fractional part to its lowest terms,

$$365 + \frac{30283}{125000},$$

or $$365 + \frac{1}{\frac{125000}{30283}}.$$

And, dividing as before, we obtain

$$365 + \cfrac{1}{4 + \cfrac{3868}{30283}}.$$

This, by a further reduction, becomes

$$365 + \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{3207}{3868}}}.$$

Not proceeding any further with the division, we see that the three first approximations to the given fraction are 365, $365\frac{1}{4}$, $365\frac{7}{29}$. The first number gives us a very rude approximation to the length of the year, and one which, in the course of a few centuries, would completely invert the order of the seasons. The second answers to the Julian Calendar, by which one day was intercalated every four years; and the third differs, by a very small quantity, from the length of the day which is the basis of the Gregorian Calendar, and which is now acted upon in almost all the countries of Europe. According to this we intercalate a single day every four years, but omit three of these intercalations in four centuries. This makes us intercalate ninety-seven days in four centuries, and gives therefore for the length of each year $365\frac{97}{400}$ days.

319. But the utility of continued fractions in giving us approximations to the value of any fraction in simpler terms than the fraction itself, is not to be estimated very highly. Their principal utility consists in our frequently being able to express the value of an unknown quantity under this form only, or under no other so easily. On this account every thing connected with them becomes important. For instance, when we have the quantity sought for, *only* under this form, it is convenient to know, in stopping at any term of the continued fraction, how far our approximation differs from the value of the whole of it, or to know some limit to the error, so as to estimate the degree of accuracy. See art. [292]. We shall presently show how this limit may be assigned.

In art. [254] we showed how, from an equation of the form $a^x = b$, we might find the value of x by means of a table of logarithms. We may also express the value of x derived from such an equation in the form of a continued fraction. Then take the equation

$$2^x = 3.$$

Now, observing that

$$2^1 = 2,$$

and $$2^2 = 4,$$

it is evident that in the above equation x must be greater than 1, and less than 2.

Suppose, then,

$$x = 1 + \frac{1}{x'},$$

where x' is greater than 1, (since x is less than 2.)

We have from the original equation

$$2^{1+\frac{1}{x'}} = 3,$$

or $$2 \times 2^{\frac{1}{x'}} = 3,$$

or $$2^{\frac{1}{x'}} = \frac{3}{2};$$

or, raising both sides of the equation to the x'^{th} power,

$$2 = \left(\frac{3}{2}\right)^{x'} \ldots\ldots (1).$$

Now $\left(\frac{3}{2}\right)^1 = \frac{3}{2}$, which is less than 2; but $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$, which is greater than 2, and therefore x' must be greater than 1, and less than 2.

* In reality, the earth moves round the sun, but it aids conception sometimes to consider the earth at rest, and the sun moving round it. The sun is said to be in the *equinox* in those two positions where the plane of the earth's equator being produced passes through him. At these times the duration of day is equal to that of night all over the earth.

Suppose, then,

$$x' = 1 + \frac{1}{x''}.$$

Substituting this in equation (2),

$$2 = \left(\frac{3}{2}\right)^{1+\frac{1}{x''}},$$

or

$$2 = \frac{3}{2} \times \left(\frac{3}{2}\right)^{\frac{1}{x''}},$$

or

$$\frac{4}{3} = \left(\frac{3}{2}\right)^{\frac{1}{x''}};$$

which gives us

$$\left(\frac{4}{3}\right)^{x''} = \frac{3}{2} \ldots . (2).$$

Now

$$\left(\frac{4}{3}\right)^1 \text{ is less than } \frac{3}{2}$$

and

$$\left(\frac{4}{3}\right)^2, \text{ or } \frac{16}{9}, \text{ is greater than } \frac{3}{2};$$

whence we infer, as before, that x'' is greater than 1, and less than 2. Let, then,

$$x'' = 1 + \frac{1}{x'''}.$$

Equation (2) gives us

$$\left(\frac{4}{3}\right)^{1+\frac{1}{x'''}} = \frac{3}{2},$$

or

$$\frac{4}{3} \times \left(\frac{4}{3}\right)^{\frac{1}{x'''}} = \frac{3}{2},$$

or

$$\left(\frac{4}{3}\right)^{\frac{1}{x'''}} = \frac{9}{8},$$

which gives us

$$\frac{4}{3} = \left(\frac{9}{8}\right)^{x'''}.$$

We should here find that x''' was greater than 2, and less than 3, and might again proceed in the same manner as before.

The result of all this is, that

$$x = 1 + \frac{1}{x'},$$

or substituting for x'

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{x''}},$$

or substituting for x''

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x'''}}},$$

or

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \&c.}}},$$

since x''' is greater than 2.

The several approximations to the value of x derived from this expression are

$$1, 2, \frac{3}{2}, \frac{8}{5}, \&c.$$

320. In order to consider this subject generally, and exhibit the relations which the successive approximations bear to each other, and their respective degrees of accuracy, we will take the general form of a continued fraction, or

$$y + \cfrac{1}{y_1 + \cfrac{1}{y_2 + \cfrac{1}{y_3 + \&c.}}},$$

where y, y_1, y_2, y_3, &c. are whole numbers. The first approximation to the value of this fraction is y,

the second is $\quad y + \frac{1}{y_1}$,

the third is $\quad y + \cfrac{1}{y_1 + \cfrac{1}{y_2}}$,

and so on. With regard to their several approximations, we immediately observe that the *first* y is too *small*, there being some quantity, $\frac{1}{y_1 + \&c.}$, to be added to it; that the *second* $y + \frac{1}{y_1}$ is too *great*, the denominator of the fractional part not being y_1 but some quantity, $y_1 + \frac{1}{y_2 + \&c.}$, greater than y_1; the *third*, $y + \cfrac{1}{y_1 + \cfrac{1}{y_2}}$, is too *small* for $y_1 + \frac{1}{y_2}$, the denominator of the fractional part is too great for the

same reason that $y + \frac{1}{y_1}$ was too great in the second approximation, and so on. It is unnecessary to proceed further to prove the following principle. *The successive approximations to the value of a continued fraction are alternately too great and too small, the even ones too great, and the odd ones too small.*

321. In order to exhibit the connections of the several approximations with each other, let the *first* be represented by $\frac{p}{q}$, the *second* by $\frac{p_1}{q_1}$, the *third* by $\frac{p_2}{q_2}$, and so on. The several fractions $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, &c. are called converging fractions, or more shortly convergents; $\frac{p_1}{q_1}$ the first, $\frac{p_2}{q_2}$ the second, &c. and their several numerators and denominators are intended to represent the numerators and denominators of the several approximations, when reduced to the form of simple fractions by the process made use of in art. [318]. The n^{th} approximation then is the $(n-1)^{th}$ converging fraction; $\frac{p}{q}$ is not called a converging fraction, because, being equal to y, it is in fact not a fraction; q is of necessity equal to unity, but it is put under this form for the sake of symmetry.

The following equations need no explanation,

$$\frac{p}{q} = \frac{y}{1},$$

$$\frac{p_1}{q_1} = y + \frac{1}{y_1},$$

reducing this becomes

$$\frac{p_1}{q_1} = \frac{y\,y_1 + 1}{y_1},$$

so that

$$\left.\begin{aligned} p_1 &= y\,y_1 + 1 \\ q_1 &= y_1 \end{aligned}\right\} \ldots\ldots\ldots (1),$$

$\frac{p_2}{q_2}$ is found by writing $y_1 + \frac{1}{y_2}$ for y_1 i the expression for $\frac{p_1}{q_1}$, so that it is equal to

$$\frac{y\left(y_1 + \frac{1}{y_2}\right) + 1}{y_1 + \frac{1}{y_2}},$$

or

$$\frac{y\,y_1\,y_2 + y + y_2}{y_1\,y_2 + 1},$$

which may be written

$$\frac{(y\,y_1 + 1)\,y_2 + y}{y_1\,y_2 + 1}.$$

Now observing equations (1), and also that $y = p$ and $q = 1$, we obtain by substitution in the above value of $\frac{p_2}{q_2}$

$$\frac{p_2}{q_2} = \frac{p_1\,y_2 + p}{q_1\,y_2 + q},$$

so that

$$\left.\begin{aligned} p_2 &= p_1\,y_2 + p \\ q_2 &= q_1\,y_2 + q \end{aligned}\right\} \ldots\ldots\ldots\ldots (2).$$

Again $\frac{p_3}{q_3}$ is found by substituting $y_2 + \frac{1}{y_3}$ for y_2 in the value of $\frac{p_2}{q_2}$, and is therefore equal to

$$\frac{p_1\left(y_2 + \frac{1}{y_3}\right) + p}{q_1\left(y_2 + \frac{1}{y_3}\right) + q},$$

which may be written

$$\frac{(p_1\,y_2 + p)\,y_3 + p_1}{(q_1\,y_2 + q)\,y_3 + q_1};$$

and observing equations (2), and substituting for $p_1\,y_2 + p$, and $q_1\,y_2 + q$, we obtain

$$\frac{p_3}{q_3} = \frac{p_2\,y_3 + p_1}{q_2\,y_3 + q_1},$$

so that

$$\left.\begin{aligned} p_3 &= p_2\,y_3 + p_1 \\ q_3 &= q_2\,y_3 + q_1 \end{aligned}\right\} \ldots\ldots\ldots (3).$$

Proceeding similarly, it is evident from the way in which the quantities $y_1\,y_2$, &c. enter into the several converging fractions, that we shall arrive at a similar result, and we have the following equation connecting each of the converging fractions with the two which precede it:

$$\frac{p_n}{q_n} = \frac{p_{n-1}\,y_n + p_{n-2}}{q_{n-1}\,y_n + q_{n-2}};$$

whence

$$\left.\begin{aligned} p_n &= p_{n-1}\,y_n + p_{n-2} \\ q_n &= q_{n-1}\,y_n + q_{n-2} \end{aligned}\right\} \ldots\ldots (n).$$

322. Multiplying the first of these equations by q_{n-1}, and the second by p_{n-1}, and subtracting, we obtain

$$p_n \times q_{n-1} - q_n \times p_{n-1} = p_{n-2}\,q_{n-1} - q_{n-2}\,p_{n-1},$$

or

$$p_n \times q_{n-1} - q_n \times p_{n-1} = -\left\{p_{n-1}\,q_{n-2} - q_{n-1}\,p_{n-2}\right\} \ldots\ldots (A).$$

This expression shows us *that the difference between the product of the numerator of any converging fraction, and the denominator of that which precedes it, and the product of the denominator of the same converging fraction, and the numerator of that which precedes it, is alternately positive and negative, and differs only in sign.*

Now going back to equations (1), and observing that $y = p$ and $q = 1$, we have

$$p_1 \times q - q_1 \times p = y y_1 + 1 - y y_1,$$

or $\quad p_1 \times q - q_1 \times p = 1.$

Again recurring to equations (2), we find

$$\begin{aligned} p_2 \times q_1 - q_2 \times p_1 &= p \times q_1 - q \times p_1 \\ &= -(p_1 \times q - q_1 \times p) \\ &= -1, \end{aligned}$$

a result which might have been at once inferred from equation (A). The following equation immediately results from equation (A) and the remark which follows it:

$$p_n \times q_{n-1} - q_n \times p_{n-1} = \pm 1 \; \ldots . \text{(B)}.$$

And it is also evident from the values of $p_1 \times q - q_1 \times p$, and $p_2 \times q_1 - q_2 \times p_1$, that the positive sign must be taken when n is odd, and the negative one when it is even. We may hence derive the following property. *The difference between the product of the numerator of the n^{th} converging fraction, and the denominator of the $(n-1)^{th}$, and the product of the denominator of the n^{th}, and the numerator of the $(n-1)^{th}$, is equal to $+1$ when n is odd, and -1 when n is even.*

323. In treating of indeterminate equations, we observed, art. [311], that by means of a property of continued fractions we might quickly arrive at a solution in whole numbers of the equation $a x + b y = c$. We may thus find it. The fraction $\frac{a}{b}$ being reduced into the form of a continued fraction, let $\frac{P}{Q}$ represent the converging fraction previous to the last, which will be the complete fraction $\frac{a}{b}$. We have then by the last rule

$$a \times Q - b \times P = \pm 1.$$

In any particular case we should know whether $\frac{a}{b}$ was an even or an odd converging fraction, and therefore know which sign to take. We will suppose it to be an odd one, and according to the rule adopting the positive sign, and then multiplying by c, we have

$$a \times P \times c - b \times Q \times c = c \ldots (1.$$

Now observing the equation

$$a x + b y = c,$$

and comparing it with equation (1), we see that it is satisfied by the substitution of $P \times c$ for x, and $Q \times c$ for y. Hence $P \times c$, and $-Q \times c$, afford us a solution of the equation. By applying this to the equation we before examined, $5x + 7y = 81$, we should obtain for the values of x and y, 243, and -162.

Returning to equation (B),

$$p_n \times q_{n-1} - q_n \times p_{n-1} = \pm 1,$$

it is clear from this equation that every divisor of p_n and q_n must also divide 1, so that they admit no divisor greater than unity. This furnishes us with another property. *The numerator and denominator of each converging fraction are prime to each other.*

324. By writing $y_1 + \frac{1}{y_2}$ for y_1 in the expression for $\frac{p_1}{q_1}$, art. [319], we obtained the expression for $\frac{p_2}{q_2}$. But supposing that we had written

$$y_1 + \cfrac{1}{y_2 + \cfrac{1}{y_3 + \text{\&c.}}}$$

as far as the expression went, for y_1, in the same expression, we should evidently have obtained, upon reduction, the fraction which the continued fraction represents. Similarly, if in the expression for $\frac{p_n}{q_n}$, we put for y_n

$$y_n + \frac{1}{y_{n+1} + \text{\&c.}},$$

continuing the expression as far as it goes, we shall again obtain the original fraction. Representing then

$$y_n + \frac{1}{y_{n+1} + \text{\&c.}}$$

by Y, and the original fraction by F, and writing Y for y_n in the expression for $\frac{p_n}{q_n}$, in art. [319], we have

$$F = \frac{p_{n-1} Y + p_{n-2}}{q_{n-1} Y + q_{n-2}}.$$

This expression will lead us to the several errors we commit by representing the original fraction by its several converging fractions. For

$$F - \frac{p_{n-2}}{q_{n-2}} = \frac{p_{n-1}\,Y + p_{n-2}}{q_{n-1}\,Y + q_{n-2}} - \frac{p_{n-2}}{q_{n-2}},$$

$$= \frac{Y\,\{p_{n-1}\,q_{n-2} - q_{n-1}\,p^{n-2}\}}{(q_{n-1}\,Y + q_{n-2})\,q_{n-2}} \;\ldots\ldots\; (1).$$

And by a similar reduction we should obtain

$$F - \frac{p_{n-1}}{q_{n-1}} = \frac{-\{p_{n-1}\,q_{n-2} - q_{n-1}\,p_{n-2}\}}{q_{n-1}\,\{q_{n-1}\,Y + q_{n-2}\}} \;\ldots\ldots\ldots\; (2).$$

From these expressions we learn that the several converging fractions are alternately too great and too small; for if

$F - \frac{p_{n-2}}{q_{n-2}}$ is positive, $F - \frac{p_{n-1}}{q_{n-1}}$ is negative,

and *vice versâ*. This result we formerly arrived at from the way in which the successive approximations were derived from each other.

325. Now, in order to estimate the degree of accuracy of the several converging fractions, we observe in the first place that art. [320],

$$p_{n-1}\,q_{n-2} - q_{n-1}\,p_{n-2} = \pm 1.$$

Hence, from equation (2) in the last article,

$$F - \frac{p_{n-1}}{q_{n-1}} = \frac{\pm 1}{q_{n-1}\,(q_{n-1}\,Y + q_{n-2})};$$

now Y is greater than unity, and, therefore,

$F - \frac{p_{n-1}}{q_{n-1}}$ is less than $\frac{1}{q_{n-1}\,(q_{n-1} + q_{n-2})}$,

neglecting the sign, and consequently, *à fortiori*,

$F - \frac{p_{n-1}}{q_{n-1}}$ is less than $\frac{1}{(q_{n-1})^2}$.

The following property then is manifest. *The error committed by representing a fraction by any of its convergents is less than unity divided by the square of the denominator of that convergent.*

Thus the error committed by representing the length of the year, art. [316], by $365\,\frac{7}{29}$ days, was less than $\frac{1}{(29)^2}$ days, or less than the $\frac{1}{841}$*th* part of a day. And similarly the error introduced by representing the value of x in the equation $2^x = 3$ by $\frac{8}{5}$ was less than $\frac{1}{25}$.

Once more, observing equations (1) and (2), and considering that Y is greater than unity, and q_{n-1} greater than q_{n-2}, (for $q_{n-1} = q_{n-2}\,y_{n-1} + q_{n-3}$, art. [319]) it will be manifest that $F - \frac{p_{n-1}}{q_{n-1}}$ is less than $F - \frac{p_{n-2}}{q_{n-2}}$, from which we derive the following property. *Each converging fraction approaches nearer to the value of the continued fraction than any of those which precede it.* It is for this reason that they are called *converging* fractions, or *convergents.*

Of the expansion of a^x and the formation of Logarithmic Tables.

326. We have already explained the purposes of logarithmic tables, art. [234], &c., and also the method of using them. We now proceed to show how they may be formed. It appeared from the nature of logarithms, art. [237], that comparatively few of them could be expressed in whole numbers, or terminating decimals, and we observed that this was not necessary, since they were sufficiently exact for all common purposes when carried to seven decimal digits. This naturally leads us to endeavour to express them in the shape of a series, and if we can arrive at one which is rapidly *convergent*, neglecting those terms which have no influence on the first seven decimal places, we shall form a table accurate to the degree required, art. [292]. A preliminary step in the arrival at this series is the expansion of a^x.

327. Before, however, we proceed to the direct consideration of the subjects placed at the head of this section, we must prove the following theorem.

If the equation

$$A + B\,x + C\,x^2 + \&c. = a + b\,x + c\,x^2 + \&c.$$

be true whatever value be given to x and A, B, C, &c. a, b, c, &c. being independent of x, we propose to prove that it is a necessary consequence of the above equation, that the coefficients of like powers of x in both sides of the equation are equal, that is, that $A = a$, $B = b$, $C = c$, &c. For since this equation is true, whatever value be given to x, it is true when $x = 0$, and the two expressions are reduced to their first terms, so that we have

$$A = a.$$

But A and a are independent of the value of x, and therefore being equal for one value of x, they are equal for all. We have then, striking out A and a from the two sides of the original equation, the following equation subsisting for all values of x,

$$Bx + Cx^2 + \&c. = bx + cx^2 + \&c.$$

Divide by x and the following equation is true for all values of x.

$$B + Cx + \&c. = b + Cx + \&c.$$

Hence for the same reason that A was equal to a, we have

$$B = b,$$

and so on for all the other coefficients.

This theorem is one of the strongest instruments in analytical reasoning, and we shall find it in the subsequent part of Algebra of frequent and extensive application.

328. We shall now consider the expansion of a^x, and endeavour to express it by a series ascending by integral powers of x. We have already seen, art. [281], that any power of a binomial for instance $(1 + y)^n$ may be represented by a series ascending by integral powers of y. We have in fact proved that

$$\left.\begin{aligned}(1+y)^n = 1 + ny + n\frac{n-1}{2}y^2 + n\frac{n-1}{2}\frac{n-2}{3}y^3 + \&c.\\ + n.\frac{n-1}{2}\;\frac{n-2}{3}\cdots\frac{n-r+1}{r}y^r + \&c.\end{aligned}\right\} \text{(A)}.$$

This series goes on for ever when n is fractional or negative, but when it is a positive integer, contains only $n + 1$ terms. Now here the remark made in art. [287] is of importance to show that even if n be a positive integer, we introduce no error in supposing the series to go on to infinity. The method in which we shall alter the arrangement of the factors of the several terms in series (A) will show the necessity of retaining all the terms if we wish to arrive at a general result.

Returning to equation (A), and representing the exponent by x, and writing a for $1 + y$, so that $y = a - 1$, we have

$$\left.\begin{aligned}a^x = 1 + x(a-1) + x\frac{x-1}{2}(a-1)^2 + x\frac{x-1}{2}\frac{x-2}{3}(a-1)^3 + \&c.\\ + x\;\frac{x-1}{2}\;\frac{x-2}{3}\cdots\frac{x-(r-1)}{r}(a-1)^r + \&c.\end{aligned}\right\} \text{(B)}.$$

We here observe, in the first place, that only integral powers of x enter into the expression, and in the next, that all the terms after the first have x for a factor, so that 1 is the first term of the result arranged according to the ascending powers of x. In order to find the *second* term, or, which comes to the same thing, the coefficient of x, we proceed as follows. Observing the several terms of the series, we see that in the second the coefficient of x is $(a-1)$, in the third, $-\frac{1}{2}(a-1)^2$, in the fourth, $-\frac{1}{2} \times -\frac{2}{3}(a-1)^3$ or $+\frac{1}{3}(a-1)^3$, and so on for the remainder of the coefficients; or that in each term it is the product of the second terms of the numerators of all the factors (except x, which has no second term) which compose the coefficient, divided by the product of the denominators of all the factors. Now the second term of the numerator of each factor is always the same as the denominator of the preceding factor. A little consideration then will show, attention being paid to the sign, which is positive when there are an even number of factors besides x, and negative when there are an odd number, that the coefficient of x is

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \&c.,$$

this series going on to infinity. It is doubtless possible by a similar method to examine the coefficients of x^2, x^3, &c., but the operation would be tedious and almost impracticable, and would be different for each coefficient. The theorem demonstrated at the beginning of this section will enable us to obtain these several coefficients with great facility, and has the advantage of furnishing them all by the same process.

329. Resuming equation (B), and writing A for

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{\&c.}$$

and calling the several coefficients of x^2, x^3, &c. (the expressions for which we are about to investigate) M, N, &c. we have

$$\left.\begin{aligned} a^x = 1 + \mathrm{A}\,x + \mathrm{M}\,x^2 \\ + \mathrm{N}\,x^3 + \text{\&c.} \end{aligned}\right\} \ldots (1).$$

This being true for all values of x, and A, M, N, &c. being independent of x, we may write z for x, and have the following equation, in which the values of A, M, N, &c. are the same as before,

$$\left.\begin{aligned} a^z = 1 + \mathrm{A}\,z + \mathrm{M}\,z^2 \\ + \mathrm{N}\,z^3 + \text{\&c.} \end{aligned}\right\} \ldots (2).$$

Subtracting equation (2) term by term from equation (1), and combining the terms with like coefficients, we have

$$a^x - a^z = \mathrm{A}\,(x - z) + \mathrm{M}\,(x^2 - z^2) + \mathrm{N}\,(x^3 - z^3) + \text{\&c.}$$

Now every term of the right-hand side of this equation is divisible by $x - z$, for every term has a multiplier of the form $x^r - z^r$, and $x^r - z^r$ being divided by $x - z$, gives

$$x^{r-1} . + x^{r-2} . z + x^{r-3} . z^2 + \text{\&c.} + z^{r-1} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

a quotient in which there are r terms. The above equation then may be written, making $x - z$ a common factor of the several terms,

$$a^x - a^z = (x - z)\,\{\mathrm{A} + \mathrm{M}\,(x + z) + \mathrm{N}\,(x^2 + x\,z + z^2) + \text{\&c.}\} \ldots . (4).$$

Again, $a^x - a^z = a^z\,(a^{x-z} - 1)$.

But by equation (1), putting $x - z$ for x,

$$a^{x-z} = 1 + \mathrm{A}\,(x - z) + \mathrm{M}\,(x - z)^2 + \mathrm{N}\,(x - z)^3 + \text{\&c.}$$

Whence it appears, making $x - z$ a factor of the sum of all the terms which it multiplies, that

$$a^{x-z} - 1 = (x - z)\left\{\begin{aligned} \mathrm{A} + \mathrm{M}\,(x - z) \\ + \mathrm{N}\,(x - z)^2 + \text{\&c.} \end{aligned}\right\},$$

so that

$$\begin{aligned} a^x - a^z &= a^z\,(a^{x-z} - 1) \\ &= a^z\,(x - z)\left\{\begin{aligned} \mathrm{A} + \mathrm{M}\,(x - z) \\ + \mathrm{N}\,(x - z)^2 + \text{\&c.} \end{aligned}\right\}. \end{aligned}$$

Whence substituting this expression for $a^x - a^z$ in equation (4), and dividing by $x - z$, which is a factor on both sides of the resulting equation, we obtain

$$\mathrm{A} + \mathrm{M}\,(x + z) + \mathrm{N}\,(x^2 + x\,z + z^2) + \text{\&c.} =$$
$$a^z\,(\mathrm{A} + \mathrm{M}\,(x - z) + \mathrm{N}\,(x - z)^2 + \text{\&c.})$$

Now this equation subsisting for all values of x and z, we may suppose z equal to x, and we obtain, since all the terms after A on the right-hand side of the equation vanish,

$$\mathrm{A} + \mathrm{M} . 2\,x + \mathrm{N} . 3\,x^2 + \text{\&c.} = a^x \times \mathrm{A};$$

and it is evident from equation (3), and the remark which follows it, that the coefficients on the left-hand side of this equation would go on following the same law. Putting then for a^x its value from equation (1), and multiplying each term by A, we have

$$\mathrm{A} + 2\,\mathrm{M}\,x + 3\,\mathrm{N}\,x^2 + \text{\&c.} =$$
$$\mathrm{A} + \mathrm{A}^2 x + \mathrm{A}\,\mathrm{M}\,x^2 + \text{\&c.}$$

Hence by the theorem proved at the beginning of this section, the coefficients of like powers of x must be equal; so that

$$2\,\mathrm{M} = \mathrm{A}^2,$$
$$3\,\mathrm{N} = \mathrm{A} \times \mathrm{M},$$
$$\text{\&c.} \quad \text{\&c}$$

From the first we obtain

$$\mathrm{M} = \frac{\mathrm{A}^2}{2},$$

from the second

$$\mathrm{N} = \frac{\mathrm{A}\,\mathrm{M}}{3},$$

$$= \frac{A^3}{2\ \ 3}$$

and the form of the two expressions shows that we should have for the coefficient of x^4, $\frac{A^4}{2 \cdot 3 \cdot 4}$, and so on for the other coefficients. Hence substituting the values of the several coefficients in equation (1),

$$a^x = 1 + Ax + \frac{A^2 . x^2}{1 . 2} + \frac{A^3 . x^3}{1 . 2 . 3} + \&c. \ldots\ldots\ldots\ldots\ (C),$$

the series continuing to follow the same law. This is the expansion required.

330. We now propose to deduce from this expression a series for the logarithm of a number to any base. See art. [234]. The process, though at first view circuitous and indirect, requires only to be understood to appear simple.

In equation (C) any value may be given to a; and A, therefore, which, art. [326], is equal to

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.,$$

admits a corresponding variety of values. Let us suppose A equal to *unity*, and represent by e the value of a corresponding to this value of A. We have, by equation (C), A being equal to 1, and a to e,

$$e^x = 1 + x + \frac{x^2}{1 . 2} + \frac{x^3}{1 . 2 . 3} + \&c.\ (D),$$

and making x equal to 1,

$$e = 1 + 1 + \frac{1}{1 . 2} + \frac{1}{1 . 2 . 3} + \&c.$$

Of this series we may take any number of terms. The summation of them is sufficiently easy, since each term, after the second, is derived from the preceding one by dividing successively by 2, 3, &c. The student can perform the operation, and neglecting the terms after the eleventh, which have no influence on the seven first places of decimals, he will find

$$e = 2{,}7182818.$$

This quantity then is known. The discovery of it does not at present appear to have brought us nearer our object, but we shall find it a necessary instrument in arriving at it. It is the base of a system of logarithms called the Napierean, from Napier, a celebrated mathematician of the seventeenth century, who invented logarithms and calculated them to this base. Logarithms calculated to this base are also sometimes, though without much propriety, called hyperbolic logarithms, from their entering into some of the properties of the hyperbola.

Resuming equation (C), suppose x equal to unity, the equation becomes

$$a = 1 + A + \frac{A^2}{1 . 2} + \&c.$$

But making x equal to A in equation (D) the series for e^A is the same as the above series for a. We thus have

$$a = e^A \ldots\ldots\ldots\ldots\ldots\ (1).$$

As we shall consider a as the base of the system for which we are forming our tables, we will put equation (1) under another form, and write n for a, and represent

$$(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.,$$

which is what A becomes, by p. This being done, equation (1) becomes

$$n = e^p \ldots\ldots\ldots\ldots\ (2).$$

From the definition of logarithms, this equation shows us that p is the logarithm of n to base e, or Nap. log. $n = p$. But

$$p = n - 1 - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.$$

Substituting, then,

$$\text{Nap. log. } n = n - 1 - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c. \ldots\ldots\ldots\ (3).$$

We have thus obtained a series for the logarithm of a number in the Napierean system. We thus proceed to find it in any other.

331. Taking the logarithms of both sides of equation (2) in the system required, for instance, in that whose base is a, we have

$$\log. n = \log. e^p$$
$$= p \log. e,$$

or $$\log. n = \log. e \left\{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.\right\} \ldots\ldots(4),$$

since $$p = (n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.$$

We here appear to be no further advanced, since log. e being taken to base a, we have at present no means of finding it. Equation (1) will help us out of this difficulty.

The equation (1) of the last article $a = e^A$ shows, taking the logarithm of both sides to base a, that $1 = \log. e^A = A \log. e$, therefore

$$\frac{1}{A} = \log. e,$$

the logarithm being taken in the system whose base is a, and $\frac{1}{A}$ is known, since A is equal to

$$a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$$

But this, it is evident, from equation (3), is the logarithm of a in the Napierean system. Representing, then, for the sake of distinction, logarithms taken in the Napierean system by log.', and those taken to base a by log., we have, putting for A its value, in the above equation

$$\frac{1}{\log.' a} = \log. e,$$

and substituting for log. e in equation (4),

$$\log. n = \frac{1}{\log.' a}\left\{n - 1 - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.\right\}.$$

The series within brackets in the above equation is the Napierean logarithm of n given in equation (3). The quantity $\frac{1}{\log.' a}$, by which the Napierean logarithm is multiplied, so as to produce the logarithm of the same number in the system calculated to base a, is called the modulus of that system, and is evidently the same for all values of n, depending only on the base of the system. It is usually represented by M; representing it so, the equation becomes

$$\log. N = M\left\{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.\right\} \ldots\ldots(5).$$

(332.) In order that this equation, which affords the complete algebraical solution of the question, may be practically adequate to the computation of logarithms, we must alter its form, for if n be any number greater than 2, it is evident that the terms of the series instead of *converging*, become continually greater and greater. In fact, although we stated in the last article, that A could be found, being equal to

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.,$$

yet we should find this expression of little use in computing its numerical value. We proceed then to alter the form of the series in the above equation, and to the numerical computation of M.

Putting $n = 1 + n'$ in equation (5), and therefore $n - 1 = n'$, it becomes

$$\log. (1 + n') = M\left\{n' - \frac{1}{2}n'^2 + \frac{1}{3}n'^3 - \&c.\right\}.$$

Again putting $n = 1 - n'$ in equation (5), and therefore $n - 1 = -n'$, it becomes

$$\log. (1 - n') = M\left\{-n' - \frac{1}{2}n'^2 - \frac{1}{3}n'^3 - \&c.\right\}.$$

Subtracting this equation from the last, and observing that

$$\log. (1 + n') - \log. (1 - n') = \log. \frac{1 + n'}{1 - n'},$$

and that the series within the brackets go on following the same law, so that all the even powers of n' destroy each other, and all the terms involving the odd powers become doubled, we have

$$\log. \frac{1+n'}{1-n'} = 2\,\mathrm{M}\left\{n' + \frac{n'^3}{3} + \frac{n'^5}{5} + \&c.\right\}.$$

To reduce this to a more favourable form, let

$$\frac{1+n'}{1-n'} = \frac{b}{c},$$

the solution of which simple equation gives

$$n' = \frac{b-c}{b+c}.$$

Substituting for $\frac{1+n'}{1-n'}$ and n' in the above equation

$$\log. \frac{b}{c} = 2\,\mathrm{M}\left\{\frac{b-c}{b+c} + \frac{1}{3}\left(\frac{b-c}{b+c}\right)^3 + \frac{1}{5}\left(\frac{b-c}{b+c}\right)^5 + \&c.\right\} \ldots . (6).$$

(333.) The series here is convergent, and rapidly so where the difference between b and c is very small. It is convenient to make b and c consecutive numbers, so that $b - c$ is equal to unity, and we can by means of the above series, observing that

$$\log. b = \log. \frac{b}{c} + \log. c,$$

find the logarithms of all numbers successively. Supposing, for a moment, that we have found the value of M for Brigg's system when the base is 10. It being equal to

$$.43429448,$$

so that 2 M is equal to

$$.86858896,$$

we have the following rule which is taken substantially from Dr. Hutton's mathematical tables. *Call* s *the sum of any number* (b) *whose logarithm is sought, and the number* (c) *next less by unity. Divide* .86858896 *by* s, *and reserve the quotient; divide the reserved quotient by the square of* s, *and reserve this quotient; divide this last quotient by the square of* s, *and again reserve this quotient, and thus proceed continually dividing the last quotient by the square of* s, *as long as division can be made. Then write these quotients under one another, the first uppermost, and divide them respectively by the uneven numbers* 1, 3, 5, *&c. Add all these last quotients together, then the sum will be the logarithm of* $\frac{b}{c}$, *as given by equation* (6).

And therefore to this logarithm, adding also the logarithm of c, *the next less number, the sum will be the required logarithm of* b, *the number required.*

The operations for finding the logarithms of 2 are indicated below. The next less number is 1, so that s = 3, and the square of s = 9.

```
3).86858896
9).28952965
9).03216996
9).00357444
9).00039716
9).00004412
9).00000490
9).00000054
  .00000006
```

Proceeding with the rule

```
 1).28952965(.28952965
 3).03216996(.01072332
 5).00357444(.00071488
 7).00039716(.00005673
 9).00004412(.00000490
11).00000490(.00000044
13).00000054(.00000004
```

Adding the quotients

$$\log. \frac{2}{1} = .30102996$$

$$\text{Add } \log. 1 = .00000000$$

$$\log. 2 = .30102996$$

By a similar process we may find the logarithm of 3 and of all the higher numbers. But it is only necessary to find the prime numbers by this direct method, the others being easily found by composition and division.

Thus $\log. 4 = \log. 2^2$
$= 2 \log. 2,$
$\log. 6 = \log. (2 \times 3)$
$= \log. 2 + \log. 3,$

and so on.

$$\text{Log. } 5 = \log. \frac{10}{2}$$
$$= \log. 10 - \log. 2$$
$$= 1 - \log. 2, \quad .$$

&c. &c.

The series in equation (6) admits other transformations which are of service to the practical computer of logarithms by affording *verifications* of his result; it being evident that when we have arrived at the same result by several different processes, we may feel a more perfect assurance of its accuracy. But the discussion of them here would be misplaced, our object being only to *explain* the construction of tables.

(334.) The multiplier M may be found immediately from equation (6). It is equal to $\frac{1}{\log.' a}$, the accent indicating as before that the logarithms are taken in the Napierean system. This, as was before remarked, is the quantity by which we multiply the Napierean logarithm of a number in order to arrive at the logarithm to base a. Equation (6) then gives us, withdrawing M,

$$\log.' \frac{b}{c} = 2\left(\frac{b-c}{b+c} + \frac{1}{3}\left(\frac{b-c}{b+c}\right)^3 + \frac{1}{5}\left(\frac{b-c}{b+c}\right)^5 + \&c.\right) \;\ldots\ldots (7).$$

Now for Brigg's system $a = 10$ In order, then, to find M, or $\frac{1}{\log.' a}$, we must first find $\log.' 10$.

But
$$\log.' 10 = \log.' (2 \times 5)$$
$$= \log.' 2 + \log.' 5.$$

By equation (7) b being equal to 2, and c to 1,

$$\log.' 2 = 2\left(\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \&c.\right),$$

By a process similar to that used in the last article we shall find

$$\log.' 2 = .69314718.$$

From this we obtain $\log.' 4$, or $2 \log.' 2$.

Again, putting 5 for b, and 4 for c in equation (7), we have

$$\log.' \frac{5}{4} = 2\left(\frac{1}{9} + \frac{1}{3} \cdot \frac{1}{9^3} + \frac{1}{5} \cdot \frac{1}{9^5} + \&c.\right).$$

which may be easily computed, being very convergent. Adding $\log.' 4$, we obtain $\log.' 5$. These operations being performed, in which the reader will find no difficulty, and $\log.' 2$ being added to $\log.' 5$, we shall obtain

$$\log.' 10 = 2.30258509,$$

and therefore

$$M = \frac{1}{\log.' 10}$$
$$= .43429448.$$

(335.) In art. [233] we promised to inquire into the real value of such quantities as $a^{\cdot 00000029}$, and to show that it differed from unity by a quantity which, within a certain degree of accuracy, might be neglected.

By art. [327] we have

$$a^x = 1 + \text{A}x + \frac{\text{A}^2 x^2}{1.2} + \&c.,$$

where

$$\text{A} = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.,$$

which is the Napierean logarithm of a. It is in most cases a small number, and is not greater than 10 unless a be greater than 10000. Now x being equal to .00000029, the third term will not influence the twelve first places of decimals, and may, therefore, with those that follow it, be put out of the question. Supposing A to be less than 10, Ax will have no significant figure in the five first decimal places, and may therefore be neglected if the degree of accuracy required does not extend so far. In this case we may therefore consider $a^{\cdot 00000029}$ as equal to 1.

In art. [240] we remarked that the *common* logarithms of consecutive numbers of five or more digits were nearly in arithmetical progression. We may thus prove this to be the case:

$$\log.(c+1) - \log. c = \log. \frac{c+1}{c},$$

$$\log. (c+2) - \log. (c+1) = \log. \frac{c+2}{c+1}.$$

Calling the first D, and the second D′, and subtracting

$$D - D' = \log. \frac{c+1}{c} - \log. \frac{c+2}{c+1} = \log. \frac{\frac{c+1}{c}}{\frac{c+2}{c+1}},$$

$$= \log. \frac{(c+1)^2}{c(c+2)} = \log. \frac{c^2+2c+1}{c(c+2)}$$

$$= \log. \left\{1 + \frac{1}{c(c+2)}\right\},$$

or, expressing the logarithm in a series,

$$D - D' = M\left\{\frac{1}{c(c+2)} - \frac{1}{2}\,\frac{1}{c^2(c+2)^2} + \&c.\right\}.$$

Now c containing five digits, the denominator of the first term contains at least nine digits, and M being less than $\frac{1}{2}$, (as was shown in the last article,) $D - D'$ can have no significant figure in the eight first decimal places. We have therefore in the tables calculated to seven places of decimals,

$$D - D' = 0,$$

or $$D = D',$$

so that the differences between the logarithms of consecutive numbers are equal. From this property the method of finding the logarithms of numbers of six and seven digits in art. [242] directly flowed.

(336.) We have subjoined a small table of common logarithms, which will, for many purposes, supply the place of a whole volume of them. By means of this table the logarithms of all numbers, from *one* up to *ten thousand*, or of all numbers consisting of *four* digits, wherever the decimal point be placed, may be found to a certain number of decimal places. It will be seen, by reference to the table, that the logarithms *directly* given are those of all numbers only from 10 up to 999. But since the logarithms of 20 and 2, 30 and 3, &c., as well as those of all numbers consisting of the same digits, but varying in the position of the decimal point, differ only in their *characteristics*, art. [238], the manner of deducing from this table the logarithms of all numbers consisting of *one*, *two*, or *three* digits, is evident. These logarithms, it will be observed, are given as far as *four* decimal places only, while those tables, the construction of which we more fully explained in arts. [236], &c., were calculated as far as *seven*. For an explanation of the use of the *proportional parts* arranged to the right of the double line in each page, we must refer to arts. [240]...[243]. We are enabled, by means of them, to find from our table the logarithm of any number consisting of *four* digits, as in the articles last referred to, we derived the logarithms of numbers, consisting of 6 digits, from tables where the logarithms of numbers up to 99999 only, were tabulated. Thus, to find from our table the logarithm of 9863:—From the table we see that the logarithm of 986 is 2.9939 (2 being the characteristic), and, consequently, the logarithm of 9860 is 3.9939. Now the *proportional part* corresponding to 3, the last digit of the number whose logarithm we proposed to find, is 1, and, therefore, adding this, we have

$$\log. 9863 = 3.9940.$$

See arts. [240]...[243].

(337.) The table of antilogarithms contains the natural numbers corresponding to all logarithms of four places, arranged in the order of the logarithms. This table is an extract from Dodson's Antilogarithmic Canon. The first two figures of the given logarithm being found in the first column, and the third figure on the top of the page, the corresponding natural number is opposite to the former, and under the latter. The quantity to be added for the fourth logarithmic figure must be taken from

the columns headed proportional parts, in the same line with the number already found, and under the given fourth figure. The natural numbers thus obtained will generally be exact to a unit in the last place, except towards the end of the table; the number of decimal places will depend upon the *characteristic* of the logarithm. This table is chiefly intended to save time when many logarithms in succession are to be looked out.

Nat. Nums.	0	1	2	3	4	5	6	7	8	9	Proportional Parts. 1	2	3	4	5	6	7	8	9
10	·0000	·0043	·0086	·0128	·0170	0212	·0253	·0294	·0334	·0374	4	8	12	17	21	25	29	33	37
11	·0414	·0453	·0492	·0531	·0569	0607	·0645	·0682	·0719	·0755	4	8	11	15	19	23	26	30	34
12	·0792	·0828	·0864	·0839	·0934	·0969	·1004	·1038	·1072	·1106	3	7	10	14	17	21	24	28	31
13	·1139	·1173	·1206	·1239	·1271	·1303	·1335	·1367	·1399	·1430	3	6	10	13	16	19	23	26	29
14	·1461	·1492	·1523	·1553	·1584	·1614	·1644	·1073	·1703	·1732	3	6	9	12	15	18	21	24	27
15	·1761	·1790	·1818	·1847	·1875	·1903	·1931	·1959	·1987	·2014	3	6	8	11	14	17	20	22	25
16	·2041	·2068	·2095	·2122	·2148	·2175	·2201	·2227	·2253	·2279	3	5	8	11	13	16	18	21	24
17	·2304	·2330	·2355	·2380	·2 05	·2430	·2455	·2480	·2504	·2529	2	5	7	10	12	15	17	20	22
18	·2553	·2577	·2601	·2625	2648	·2672	·2695	·2718	2742	·2765	2	5	7	9	12	14	16	19	21
19	·2788	·2810	·2833	·2856	·2878	·2900	·2923	·2945	·2967	2989	2	4	7	9	11	13	16	18	20
20	·3010	·3032	·3054	·3075	·3096	3118	·3139	·3160	·3181	·3201	2	4	6	8	11	13	15	17	19
21	·3222	·3243	·3263	·3284	·3304	·3324	·3345	·3365	·3385	·3404	2	4	6	8	10	12	14	16	18
22	·3424	·3444	·3464	·3483	·3502	3522	·3541	·3560	·3579	·3598	2	4	6	8	10	12	14	15	17
23	·3617	·3636	·3655	·3674	3692	·3711	·3729	·3747	·3766	·3784	2	4	6	7	9	11	13	15	17
24	·3802	·3820	·3838	·3856	3874	·3892	·3909	·3927	·3945	·3962	2	4	5	7	9	11	12	14	16
25	·3979	·3997	·4014	·4031	·4048	·4065	·4082	·4099	·4116	·4133	2	3	5	7	9	10	12	14	15
26	·4150	·4166	·4183	·4200	·4216	·4232	·4249	·4265	·4281	·4298	2	3	5	7	8	10	11	13	15
27	·4314	·4330	·4346	·4362	·4378	·4393	·4409	·4425	·4440	·4456	2	3	5	6	8	9	11	13	14
28	·4472	·4487	·4502	·4518	·4533	4548	·4564	·4579	·4594	·4609	2	3	5	6	8	9	11	12	14
29	·4624	·4639	·4654	·4669	·4683	·4698	·4713	4728	4742	·4757	1	3	4	6	7	9	10	12	13
30	·4771	·4786	·4800	·4814	·4829	·4843	·4857	·4871	·4886	·4900	1	3	4	6	7	9	10	11	13
31	·4914	·4928	·4942	·4955	·4969	·4983	·4997	·5011	5024	·5038	1	3	4	6	7	8	10	11	12
32	·5051	·5065	·5079	·5092	·5105	·5119	·5132	·5145	5159	·5172	1	3	4	5	7	8	9	11	12
33	·5185	·5198	·5211	·5224	·5237	·5250	·5263	5276	5289	·5302	1	3	4	5	6	8	9	10	12
34	·5315	·5328	·5340	·5353	5366	·5378	·5391	·5403	·5416	·5428	1	3	4	5	6	8	9	10	11
35	·5441	·5453	·5465	5478	·5490	·5502	·5514	·5527	5539	·5551	1	2	4	5	6	7	9	10	11
36	·5563	·5575	·5587	·5599	5611	·5623	·5635	·5647	5658	5670	1	2	4	5	6	7	8	10	11
37	·5682	·5694	·5705	·5717	·5729	·5740	·5752	·5763	5775	·5786	1	2	3	5	6	7	8	9	10
38	·5798	·5809	·5821	·5832	·5843	·5855	·5866	·5877	5888	·5899	1	2	3	5	6	7	8	9	10
39	·5911	·5922	·5933	5944	·5955	·5966	5977	·5988	·5999	·6010	1	2	3	4	5	7	8	9	10
40	·6021	·6031	·6042	6053	·6064	·6075	·6085	·6096	·6107	·6117	1	2	3	4	5	6	8	9	10
41	·6128	·6138	·6149	·6160	·6170	·6180	6191	·6201	·6212	·6222	1	2	3	4	5	6	7	8	9
42	·6232	·6243	·6253	·6263	·6274	·6284	·6294	·6304	·6314	·6325	1	2	3	4	5	6	7	8	9
43	·6335	·6345	·6355	·6365	·6375	·6385	·6395	·6405	·6415	·6425	1	2	3	4	5	6	7	8	9
44	·6435	·0444	6454	·6464	·6474	·6484	·6493	·6503	·6513	·6522	1	2	3	4	5	0	7	8	9
45	·6532	·6542	·6551	·6561	·6571	·6580	·6590	·6599	·6609	·6618	1	2	3	4	5	6	7	8	9
46	·6628	·6637	6646	·6656	·6665	·6675	·6684	·6693	·6702	6712	1	2	3	4	5	6	7	7	8
47	·6721	·6730	·6739	·6749	·6758	·6767	·6776	·6785	·6794	·6803	1	2	3	4	5	5	6	7	8
48	6812	·6821	6830	·6839	·6848	·6857	·6866	·6875	·6884	·6893	1	2	3	4	4	5	6	7	8
49	·6902	·6911	·6920	·6928	·6937	·6946	·6955	6964	·6972	·6981	1	2	3	4	4	5	6	7	8
50	·6990	·6998	·7007	·7016	·7024	7033	·7042	·7050	·7059	·7067	1	2	3	3	4	5	6	7	8
51	·7076	·7084	·7093	·7101	·7110	·7118	·7126	7135	·7143	·7152	1	2	3	3	4	5	6	7	8
52	·7160	·7168	·7177	·7185	7193	·7202	·7210	·7218	·7226	·7235	1	2	2	3	4	5	6	7	7
53	·7243	·7251	·7259	·7267	·7275	7284	7292	·7300	7308	·7316	1	2	2	3	4	5	6	6	7
54	·7324	·7332	·7340	·7348	·7356	·7364	·7372	·7380	·7388	·7396	1	2	2	3	4	5	6	6	7

TABLE OF LOGARITHMS.

PROPORTIONAL PARTS.

Nat. Nums.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	·7404	·7412	·7419	·7427	·7435	7443	7451	·7459	·7466	·7474	1	2	2	3	4	5	5	6	7
56	·7482	·7490	·7497	·7505	7513	7520	7528	·7536	·7543	7551	1	2	2	3	4	5	5	6	7
57	·7559	·7566	·7574	·7582	7589	7597	7604	·7612	·7619	·7627	1	2	2	3	4	5	5	6	7
58	·7634	·7642	·7649	·7657	7664	7672	·7679	·7686	·7694	·7701	1	1	2	3	4	4	5	6	7
59	·7709	·7716	·7723	·7731	·7738	7745	·7752	·7760	·7767	·7774	1	1	2	3	4	4	5	6	7
60	·7782	·7789	·7796	·7803	·7810	7818	·7825	·7832	·7839	·7846	1	1	2	3	4	4	5	6	6
61	·7853	·7860	·7868	·7875	·7882	7889	·7896	·7903	·7910	·7917	1	1	2	3	4	4	5	6	6
62	·7924	·7931	·7938	·7945	·7952	7959	·7966	·7973	·7980	·7987	1	1	2	3	3	4	5	6	6
63	·7993	·8000	·8007	·8014	·8021	8028	·8035	·8041	·8048	·8055	1	1	2	3	3	4	5	5	6
64	·8062	·8069	·8075	·8082	·8089	9096	·8102	·8109	·8116	·8122	1	1	2	3	3	4	5	5	6
65	·8129	·8136	·8142	·8149	·8156	8162	·8169	·8176	·8182	·8189	1	1	2	3	3	4	5	5	6
66	·9195	·8202	·8209	·8215	·8222	·8228	·8235	·8241	·8248	·8254	1	1	2	3	3	4	5	5	6
67	·8261	·8267	·8274	·8280	·8287	8293	·8299	·8306	·8312	·8319	1	1	2	3	3	4	5	5	6
68	·8325	·8331	·8338	·8344	·8351	·8357	·8363	·8370	·8376	·8382	1	1	2	3	3	4	4	5	6
69	·8388	·8395	·8401	·8407	·8414	·8420	·8426	·8432	·8439	·8445	1	1	2	2	3	4	4	5	6
70	·8451	·8457	·8463	·8470	·8476	·8482	·8488	·8494	·8500	·8506	1	1	2	2	3	4	4	5	6
71	·8513	·8519	·8525	·8531	·8537	8543	·8549	·8555	·8561	·8567	1	1	2	2	3	4	4	5	5
72	·8573	·8579	·8585	·8591	·8597	·8603	·8609	·8615	·8621	·8627	1	1	2	2	3	4	4	5	5
73	·8633	·8639	·8645	·8651	·8657	·8663	·8669	·8675	·8681	·8686	1	1	2	2	3	4	4	5	5
74	·8692	·8698	·8704	·8710	·8716	·8722	·8727	·8733	·8739	·8745	1	1	2	2	3	4	4	5	5
75	·8751	·8756	·8762	·8768	·8774	·8779	·8785	·8791	·8797	·8802	1	1	2	2	3	3	4	5	5
76	·8808	·8814	·8820	·8825	·8831	8837	·8842	·8848	·8854	·8859	1	1	2	2	3	3	4	5	5
77	·8865	·8871	·8876	·8882	·8887	·8893	·8899	·8904	·8910	·8915	1	1	2	2	3	3	4	4	5
78	·8921	·8927	·8932	·8938	·8943	·8949	·8954	·8960	·8965	·8971	1	1	2	2	3	3	4	4	5
79	·8976	·8982	·8987	·8993	·8998	·9004	·9009	·9015	·9020	·9025	1	1	2	2	3	3	4	4	5
80	·9031	·9036	·9042	·9047	·9053	·9058	·9063	·9069	·9074	·9079	1	1	2	2	3	3	4	4	5
81	·9085	·9090	·9096	·9101	·9106	·9112	·9117	·9122	·9128	·9133	1	1	2	2	3	3	4	4	5
82	·9138	·9143	·9149	·9154	·9159	·9165	·9170	·9175	·9180	·9186	1	1	2	2	3	3	4	4	5
83	·9191	·9196	·9201	·9206	·9212	·9217	·9222	·9227	·9232	·9238	1	1	2	2	3	3	4	4	5
84	·9243	·9248	·9253	·9258	·9263	·9269	·9274	·9279	·9284	·9289	1	1	2	2	3	3	4	4	5
85	·9294	·9299	·9304	·9309	·9315	·9320	·9325	·9330	·9335	·9340	1	1	2	2	3	3	4	4	5
86	·9345	·9350	·9355	·9360	·9365	·9370	·9375	·9380	·9385	·9390	1	1	2	2	3	3	4	4	5
87	·9395	·9400	·9405	·9410	·9415	·9420	·9425	·9430	·9435	·9140	0	1	1	2	2	3	3	4	4
88	·9445	·9450	·9455	·9460	·9465	·9469	·9474	·9479	·9484	·9489	0	1	1	2	2	3	3	4	4
89	·9494	·9499	·9504	·9509	·9513	·9518	·9523	·9528	·9533	·9538	0	1	1	2	2	3	3	4	4
90	·9542	·9547	·9552	·9557	·9562	·9566	·9571	·9576	·9581	·9586	0	1	1	2	2	3	3	4	4
91	·9590	·9595	·9600	·9605	·9609	·9614	·9619	·9624	·9628	·9633	0	1	1	2	2	3	3	4	4
92	·9638	·9643	·9647	·9652	·9657	·9661	·9666	·9671	·9675	·9680	0	1	1	2	2	3	3	4	4
93	·9685	·9689	·9694	·9699	·9703	·9708	·9713	·9717	·9722	·9727	0	1	1	2	2	3	3	4	4
94	·9731	·9736	·9741	·9745	·9750	·9754	·9759	·9763	·9768	·9773	0	1	1	2	2	3	3	4	4
95	·9777	·9782	·9786	·9791	·9795	·9800	·9805	·9809	·9814	·9818	0	1	1	2	2	3	3	4	4
96	·9823	·9827	·9832	·9836	·9841	·9845	·9850	·9854	·9859	·9863	0	1	1	2	2	3	3	4	4
97	·9868	·9872	·9877	·9881	·9886	·9890	·9894	·9899	·9903	·9908	0	1	1	2	2	3	3	4	4
98	·9912	·9917	·9921	·9926	·9930	·9934	·9939	·9943	·9948	·9952	0	1	1	2	2	3	3	4	4
99	·9956	·9961	·9965	·9969	·9974	·9978	·9983	·9987	·9991	·9996	0	1	1	2	2	3	3	3	4

TABLE OF ANTILOGARITHMS.

Logs.	0	1	2	3	4	5	6	7	8	9	Proportional Parts. 1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	7

PROPORTIONAL PARTS.

Log.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	6192	3199	3206	3214	3221	3228	1	1	2	3	4	4	6	6	7
51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3312	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	6	4	5	6	6	7
55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	6	6	7	8
.57	3716	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3812	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	6	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	6	2
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	2
63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	2
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	6	6	7	8	2
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	6	5	7	8	9
66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	2	10
.68	4786	4797	4803	4812	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	6117	1	2	4	5	6	7	8	2	11
.71	6122	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	6248	5260	5272	6284	5297	5309	5321	5333	5346	5358	1	2	4	6	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	6	5	6	2	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	6	7	8	9	10	12
.76	5754	6768	5781	6794	5808	5821	5834	5848	5861	5875	1	3	4	6	7	5	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	6	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	5142	1	3	4	6	7	9	10	12	13
.81	6457	6471	5486	6501	6516	6531	6546	6561	6577	6592	2	3	6	6	8	2	11	12	14
82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	6	6	6	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	6	6	6	8	2	11	13	14
84	6916	6934	6950	6966	6982	5998	7015	7031	7047	7063	2	3	6	6	8	10	11	13	15
.85	7072	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	6	7	8	10	12	13	15
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
89	7762	7780	7798	7815	7834	7852	7870	7889	7907	7925	2	4	5	7	2	11	13	14	16
90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	16	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
97	9333	9354	9376	9397	2419	9441	9462	2484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9615	9638	9661	9683	9705	9727	9750	2	4	7	9	11	16	15	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

ERRATA IN SOME OF THE EDITIONS.

Page 86, column 1, line 38, for $(x+a)^b$ read $(x+a)^6$

Page 87, column 2, line 27, for x read x^n

Page 88, column 2, line 23 of note, for a read α

line 24 ditto, for $(1+z)^{m+a}$ read $(1+z)^{m+\alpha}$

line 27 ditto, for $(1+z)^{m+a}$ read $(1+z)^{m+\alpha}$

The same in lines 32 and 35

Page 90, column 1, line 2, for $(1+x)^{\frac{1}{2}}$ read $(1-x)^{\frac{1}{2}}$

line 6, for $(1+x)^{\frac{1}{2}}$ read $(1+x)^{-\frac{1}{2}}$

line 19, dele $-m$ at the end of the line

line 20, at beginning, for $\frac{m+1}{2}$ read $-m.\frac{m+1}{2}$

column 2, line 5, for $\frac{\frac{m}{n}-1}{2}$ read $\frac{\frac{m}{n}-1}{2}$

line 7 from bottom, for $\frac{m(m-1)\ldots m-p+1}{1\,.\,2\,.\,.\,.\,.\,p}$ read $\frac{m(m-1)\ldots(m-p+1)}{1\,.\,2\,.\,.\,.\,.\,p}$

Page 91, column 1, line 26, for $\frac{y^2}{a^2}$ read $\frac{y}{a^2}$

line 27, for $\sqrt{1+\frac{y^2}{a^2}}$ read $\sqrt{1+\frac{y}{a^2}}$

Page 92, column 1, lines 4 and 5 should stand thus:

$$(1+x)^{\frac{1}{n}}=1+\frac{1}{n}\cdot x+\frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}.x^2$$

$$+\frac{1}{n}.\frac{\frac{1}{n}-1}{2}\cdot\frac{\frac{1}{n}-2}{3}.x^3+\&c.$$

line 8 thus: $\sqrt[n]{N}=a\left\{1+\frac{1}{n}\cdot\frac{y}{a^n}+\frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}\cdot\frac{y^2}{a^{2n}}\right.$

line 9 thus: $\left.+\frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}\cdot\frac{\frac{1}{n}-2}{3}\cdot\frac{y^3}{a^{3n}}+\&c.\right\}$

line 12, for $+\frac{1}{}\frac{y}{a.^n}$ read $+\frac{1}{n}.\frac{y}{a^n}$

line 13 thus: $+\frac{1}{n}.\frac{n-1}{2n}.\frac{2n-1}{3n}.\frac{y^3}{a^{3n}}-\&c.\}$

column 2, line 40, for f read S

Page 93, column 2, line 5 of note, for $\frac{x^2}{a+x^2}$ read $\frac{x^2}{(a+x)^2}$

line 9 ditto, for $\left(\frac{1}{1-\frac{a-x}{a+x}}\right)^n$ read $\left(\frac{1}{1-\frac{a-x}{a+x}}\right)^n$

Page 94, column 1, line 2, dele comma at the end of the line.

line 5, for $(a+n)^m$ read $(a+x)^m$

Page 96, column 2, line 22, for [292] read [297].

(3

EXAMPLES OF THE PROCESSES

OF

ARITHMETIC AND ALGEBRA.

To prevent any misconception as to the use of this treatise, we state that it is intended only for those who study the principles of arithmetic and algebra, and the reasons of the rules laid down in those sciences. The plan we should recommend is the following:—Let the student repeat examples of each rule upon paper, choosing the most simple numbers which can be found, as well those given in this work as others, until he is capable of solving such instances mentally. Let him then proceed to the cases which contain more complicated numbers or expressions. This is by much the shortest way of proceeding, and eventually the easiest.

We presume a knowledge of the four fundamental operations of arithmetic in whole numbers, and shall therefore content ourselves with showing how examples may be formed which shall contain their own verification.

As soon as the pupil knows the processes of addition and subtraction, let him take a series of numbers, each of which contains one more figure than the preceding; say 154, 2879, 31673, 200104, and 7172618. Let him subtract each of these from the succeeding as follows:—

2879	31673	200104	7172618
154	2879	31673	200104
2725	28794	168431	6972514

Let him then add all his results, together with the least number chosen. The result ought to be the greatest number.

$$\begin{array}{r} 6972514 \\ 168431 \\ 28794 \\ 2725 \\ 154 \\ \hline 7172618 \end{array}$$

As an exercise in multiplication, let two numbers be written down for the student, each of which he is to multiply by itself. For instance, 142 and 361.

$$\begin{array}{rr} 361 \times 361 = & 130321 \\ 142 \times 142 = & 20164 \\ \text{Subtract} & \overline{110157} \end{array}$$

Let him then take the sum and difference of the two numbers first chosen, and multiply these together, which should give the same result as the preceding.

$$\begin{array}{rr} 361 & 361 \\ 142 & 142 \\ \text{add } \overline{503} & \overline{219} \text{ subtract.} \end{array}$$

$$503 \times 219 = 110157$$

For division, let the student multiply two numbers by themselves, and divide the difference of the results by the difference of the numbers; which should give their sum. But the division of any two numbers by one another may be made, and the result verified by multiplication as usual.

SECTION 1.—*Common Fractions.*

OPERATIONS containing fractions with very high numbers are of little practical use; decimal fractions being pre-

ferred. But as exercises of arithmetical accuracy we shall give, among the rest, a few cases of high numbers.

I.—*To reduce a fraction to its lowest terms.*

Definition. — A fraction is in its lowest terms, when there is no fraction equal to it which has a smaller numerator and denominator.

Principle employed.—The value of a fraction is not altered by dividing both its numerator and denominator, or multiplying both its numerator and denominator by the same number.

$$\frac{1}{2} \quad \frac{2}{4} \quad \frac{3}{6} \quad \frac{4}{8} \quad \frac{5}{10} \text{ \&c., \&c., are all equal.}$$

$$\frac{3}{7} \quad \frac{6}{14} \quad \frac{9}{21} \quad \frac{12}{28} \quad \frac{15}{35} \text{ \&c., \&c., are all equal.}$$

Rule. Divide both numerator and denominator by the greatest whole number which will divide them both without remainder.

Case 1. Where it is evident that a certain number will divide both numerator and denominator without remainder, and that the result is in its lowest terms.

$$\frac{11}{33}=\frac{1}{3} \qquad \frac{12}{48}=\frac{1}{4} \qquad \frac{21}{14}=\frac{3}{2} \qquad \frac{18}{99}=\frac{2}{11} \qquad \frac{27}{15}=\frac{9}{5}$$

Point out here by what numbers the numerators and denominators are divided.

Case 2. Where it is evident that the numerator and denominator are divisible by some number, but not evident that the result is in its lowest terms, divide by that whole number, and proceed as in *Case* 3. (Observe that *Case* 3 may be employed without this, if preferred.)

A number is divisible by

Two, when the last digit is divisible by *two*, or even; as in 66, 48, 132.

Three, when the sum of its digits is divisible by *three*, as 162, in which 1 + 6 + 2 or 9 is divisible by 3.

Four, when the two last digits are divisible by *four*, as in 16864, in which 64 is divisible by 4.

Five, when the last digit is either 0 or 5, as in 180, 965. (To divide by 5, multiply by 2, and strike off the cipher.)

Six, when it is *even* and divisible by *three*, as 486.

Seven, according to no rule sufficiently simple to be useful.

Eight, when the three last digits are divisible by *eight*, as 2794216, in which 216 is divisible by 8.

Nine, when the sum of its digits is divisible by *nine*, as 729, in which 7 + 2 + 9 or 18 is divisible by 9.

Ten, when the last digit is a cipher.

Eleven, when the two sets of sums made by taking alternate digits are either equal, or differ by a multiple* of 11, as 1034, in which 1 + 3 is the same as 0 + 4, 121 in which 1 + 1 is the same as 2, 129382 in which 1 + 9 + 8 or 18, differs from 2 + 3 + 2 or 7, by 11.

Twelve, when it is divisible by *four* and *three*.

The preceding rules may be applied to the following fractions: find out which is employed in each.

$$\frac{5665}{5720}=\frac{1133}{1144}=\frac{103}{104} \text{ in the lowest terms.}$$

$$\frac{7944}{8916}=\frac{1986}{2229}=\frac{662}{743} \quad (\textit{Case } 3.)$$

$$\frac{8904}{4494}=\frac{1272}{642}=\frac{212}{107} \quad (\textit{Case } 3.)$$

Case 3. When there is no very evident divisor of the numerator and denominator, divide the greater by the less, the divisor by the remainder, the last-mentioned remainder by the new remainder, &c., &c., (as afterwards

* A multiple of *a* is any number which can be divided by *a* without remainder.

shown,) until there is no remainder, or until it is evident that two successive remainders have no common divisor. In the first case, the last divisor used will divide both terms of the given fraction, and will reduce it to its lowest terms; in the second case, the fraction is already in its lowest terms.

*** Observe that whatever divides two numbers divides their difference: therefore 102 and 107 can have no common divisor; if they had, it would be either 5 or would divide 5. This will often be useful.

Reduce $\frac{4466}{1856}$ to its lowest terms.

```
1856)4466(2
     3712
      754)1856(2
          1508
           348)754(2
               696
                58)348(6
                   348
                     0
```

therefore 58 is the greatest divisor which 4466 and 1856 have in common.

```
58)4466(77        58)1856(32
   406               174
   ---               ---
    406               116
    406               116
      0                 0
```

Therefore $\frac{4466}{1856} = \frac{77}{32}$

Reduce $\frac{1847}{8209}$ to its lowest terms.

```
1847)8209(4
     7388
      821)1847(2
          1642
           205)821(4
               820
                 1)205(205
                   205
                     0
```

This tells us the greatest common divisor of 1847 and 8209 is 1, or that there is no divisor which will reduce the fraction to lower terms.

$$\frac{2433}{13787} = \frac{3}{17} \qquad \frac{156933}{19557} = \frac{329}{41}$$

$$\frac{314175}{100005} = \frac{355}{113} \qquad \frac{100110}{31866} = \frac{355}{113}$$

$$\frac{7992}{11544} = \frac{9}{13} \qquad \frac{54369}{73355} = \frac{63}{85}$$

instances of higher numbers,

$$\frac{7241379310344827586206896551}{9999999999999999999999999999} = \frac{63}{87}$$

$$\frac{42614574994432}{149720237927424} = \frac{16807}{59049}$$

The following is a table containing some *prime* numbers (or numbers which have no whole divisors greater than 1) by which examples may be formed.

23	367	857	1637	3299	8443	18583
29	397	883	1709	3389	8573	20611
83	433	947	1759	4591	8669	32801
149	509	953	1831	4673	9011	43717
179	541	967	1847	5189	9151	58573
181	619	971	1861	5407	9181	60013
191	647	977	2081	6329	9403	72053
257	709	983	2111	6449	9521	84229
271	761	991	2287	7237	9631	97073
311	809	997	2749	7321	9967	99991

Take any two of the preceding numbers, say 23 and 149; multiply both by any number, say 8, giving 184 and 1192, then

$$\frac{184}{1192} \text{ reduced to lowest terms, gives } \frac{23}{149}$$

Many thousands of examples may be thus formed.

II.—*To reduce Fractions to a common Denominator,*—

That is, to find fractions having the same denominator which shall be respectively equal to a set of fractions having different denominators.

Case 1. When the fractions have denominators, of which all the divisors can be easily seen. An example of this case will be better than any rule.

To reduce to a common denominator the following fractions,

$$\frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{4} \quad \frac{3}{5} \quad \frac{5}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{2}{9} \quad \frac{3}{10} \quad \frac{1}{12} \quad \frac{13}{16}$$

write down all the denominators which are not evidently divisors of some of the rest.

7, 9, 10, 12, 16

Write these down in prime factors, that is, make them by multiplication.

$7 \quad 3 \times 3 \quad 5 \times 2 \quad 2 \times 2 \times 3 \quad 2 \times 2 \times 2 \times 2$

Take each prime number as often as it occurs in that one of the preceding which has it most often.

2 2 2 2 3 3 5 7

Multiply all these together, which gives

$$2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 = 5040;$$

Divide this by all the denominators in succession.

$5040 \div 2 = 2520$	$5040 \div 7 = 720$
$5040 \div 3 = 1680$	$5040 \div 8 = 630$
$5040 \div 4 = 1260$	$5040 \div 9 = 560$
$5040 \div 5 = 1080$	$5040 \div 10 = 504$
$5040 \div 6 = 840$	$5040 \div 12 = 420$

$5040 \div 16 = 315$

These need not all be formed by actual division, for it is clear that to divide by 9, we may take the third part of 1680, in which 5040 has been already divided by 3.

Now look to the original fractions: multiply every numerator by the result of its denominator in the preceding list, and we shall thus have the numerators of the fractions required, while 5040 will be the common denominator, as follows:—

$1 \times 2520 = 2520$	$\frac{1}{2}$ is $\frac{2520}{5040}$
$2 \times 1680 = 3360$	$\frac{2}{3}$ is $\frac{3360}{5040}$
$1 \times 1260 = 1260$	$\frac{1}{4}$ is $\frac{1260}{5040}$

Similarly

$\frac{3}{5}$ is $\frac{3240}{5040}$	$\frac{2}{9}$ is $\frac{1120}{5040}$
$\frac{5}{6}$ is $\frac{4200}{5040}$	$\frac{3}{10}$ is $\frac{1512}{5040}$
$\frac{1}{7}$ is $\frac{720}{5040}$	$\frac{1}{12}$ is $\frac{420}{5040}$
$\frac{1}{8}$ is $\frac{630}{5040}$	$\frac{13}{16}$ is $\frac{4095}{5040}$

Fractions given.	The same reduced to a common denominator.
$\frac{3}{8} \quad \frac{5}{12} \quad \frac{7}{100}$	$\frac{225}{600} \quad \frac{250}{600} \quad \frac{42}{600}$
$3^* \quad \frac{1}{12} \quad \frac{5}{16} \quad \frac{7}{30}$	$\frac{720}{240} \quad \frac{20}{240} \quad \frac{75}{240} \quad \frac{56}{240}$

* Consider this as $\frac{3}{1}$.

The most convenient common denominator is the *least* number which is divisible by all the denominators or their *least* common multiple; but any common multiple will answer. The least common multiple is found in the preceding process.

Case 2. — Where the least common multiple of all the denominators is evident. This, generally speaking, is when the denominators are very low numbers, and the least common multiple is found by multiplying the denominators together, rejecting any factor out of each, which is evidently contained in a preceding one. For instance, the least common multiple of 4 and 6 is not 4 × 6 but 4 × 3, because the factor 2, which is thrown out when 6 is made 3, is already in 4. The following are instances :—

Numbers given.	Least common multiple.
2, 3, 4, 6,	2 × 3 × 2 × 1 = 12
4, 9, 10, 12, 18,	4 × 9 × 5 × 1 = 180
2, 6, 10,	2 × 3 × 5 = 30
6, 8, 10,	6 × 4 × 5 = 120
3, 7, 9,	3 × 7 × 3 = 63
5, 8, 10, 15,	5 × 8 × 3 = 120
7, 9, 12, 15,	7 × 9 × 4 × 5 = 1260
2, 4, 6, 8, 10,	2 × 2 × 3 × 2 × 5 = 120
18, 20, 24,	18 × 10 × 2* = 360
16, 18, 22	16 × 9 × 11 = 1584

When the least common multiple has been found, perception derived from practice, rather than rules, must be the guide; if that fails, go back to *Case* 1.

Fractions given.	Reduced to a common denominator.
$\frac{1}{2}$ $\frac{2}{3}$ $\frac{1}{4}$ $\frac{5}{6}$	$\frac{6}{12}$ $\frac{8}{12}$ $\frac{3}{12}$ $\frac{10}{12}$
$\frac{3}{7}$ $\frac{3}{14}$ $\frac{1}{4}$	$\frac{12}{28}$ $\frac{6}{28}$ $\frac{7}{28}$
$\frac{9}{2}$ $\frac{5}{6}$ $\frac{1}{18}$	$\frac{81}{18}$ $\frac{15}{18}$ $\frac{1}{18}$
$\frac{7}{15}$ $\frac{3}{10}$ $\frac{5}{9}$	$\frac{42}{90}$ $\frac{27}{90}$ $\frac{50}{90}$
$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$ $\frac{1}{6}$	$\frac{30}{60}$ $\frac{20}{60}$ $\frac{15}{60}$ $\frac{12}{60}$ $\frac{10}{60}$

Case 3.—When there are only two fractions with complicated denominators, either multiply numerator and denominator of each by the denominator of the other; or, if considered worth while, find the greatest common measure of the two denominators, and their quotients when divided by it; multiply each numerator and denominator by the quotient of the other denominator.

To reduce $\frac{33}{82}$ and $\frac{11}{25}$ to a common denominator.

$$\frac{33}{82} = \frac{33 \times 25}{82 \times 25} = \frac{825}{2050}$$

$$\frac{11}{25} = \frac{11 \times 82}{25 \times 82} = \frac{902}{2050}$$

To reduce $\frac{81}{7700}$ and $\frac{37}{1540}$ to a common denominator. Here the greatest common measure of 1540 and 7700 is 1540; therefore the fractions are $\frac{81}{7700}$ and $\frac{185}{7700}$

Fractions given.		Reduced to a common denominator.	
$\frac{53}{181}$	$\frac{27}{936}$	$\frac{49608}{169416}$	$\frac{4887}{169416}$
$\frac{113}{355}$	$\frac{355}{113}$	$\frac{12769}{40115}$	$\frac{126025}{40115}$

* For 24 write 2, because 6 is already a factor of 18, and of the residuary factor 4, 2 is already in 20.

III.—*Estimation of the Value of Fractions.*

Rule 1. When fractions have a common denominator, the greater has the greater numerator.

$$\frac{16}{12} > \frac{9}{12} \qquad \frac{14}{11} > \frac{13}{11}$$

Rule 2. When fractions have a common numerator, the greater has the less denominator.

$$\frac{16}{7} > \frac{16}{8} \qquad \frac{21}{11} > \frac{21}{15}$$

Rule 3. If the numerators of two fractions be added for a numerator, and the denominators for a denominator, the resulting fraction lies between the two first.

$\frac{2}{5}$ lies between $\frac{1}{2}$ and $\frac{1}{3}$

$\frac{8}{17}$ lies between $\frac{3}{10}$ and $\frac{5}{7}$

$\frac{23}{60}$ lies between $\frac{20}{40}$ and $\frac{3}{20}$

Rule 4. By adding the same number to both numerator and denominator of a fraction, the fraction is brought nearer to 1; by subtracting the same number from the numerator and denominator, the fraction is removed farther from 1; that is, addition to both decreases fractions greater than 1, and increases fractions less than 1; subtraction from both increases fractions greater than 1, and decreases fractions less than 1.

$$\frac{1}{2}\ \ \frac{2}{3}\ \ \frac{3}{4}\ \ \frac{4}{5}\ \ \frac{5}{6}, \text{ \&c.}$$

is a continually increasing series.

$$\frac{3}{2}\ \ \frac{4}{3}\ \ \frac{5}{4}\ \ \frac{6}{5}\ \ \frac{7}{6}, \text{ \&c.}$$

is a continually decreasing series.

$\frac{11}{4}$ is less than $\frac{10}{3}$,

$\frac{28}{29}$ is greater than $\frac{27}{28}$.

IV.—*To add and subtract Fractions.*

Rule. Reduce the fractions to a common denominator; do with the numerators what is directed to be done with the fractions; let the result be the numerator; let the common denominator be the denominator. Reduce the result to its lowest terms, if thought worth while.

What is $\frac{1}{2} + \frac{1}{3}$? These are $\frac{3}{6}$ and $\frac{2}{6}$; hence $\frac{1}{2} + \frac{1}{3}$ has $3 + 2$ for numerator, and 6 for denominator, or is $\frac{5}{6}$. Similarly, $\frac{1}{2} - \frac{1}{3}$ is $\frac{3-2}{6}$, or $\frac{1}{6}$.

$$\frac{7}{3} + \frac{2}{11} = \frac{77}{33} + \frac{6}{33} = \frac{83}{33}$$

$$\frac{7}{3} - \frac{2}{11} = \frac{77}{33} - \frac{6}{33} = \frac{71}{33}$$

$$2 - \frac{4}{5} = \frac{6}{5} \qquad \frac{12}{13} + \frac{2}{7} = \frac{110}{91}$$

$$\frac{7}{10} + \frac{8}{15} = \frac{37}{30} \qquad 1 + \frac{1}{2} = \frac{3}{2}$$

$$1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$$

$$\frac{1}{8} + \frac{1}{63} + \frac{1}{560} + \frac{1}{5040} = \frac{1}{7}$$

$$\frac{4}{17} + \frac{7}{50} + \frac{21}{850} - \frac{11}{90} = \frac{5}{18}$$

$$\frac{1}{3} + \frac{2}{5} + \frac{4}{21} = \frac{97}{105}$$

$$2 - \frac{1}{7} + \frac{12}{13} = \frac{253}{91}$$

$$\frac{1}{3} + \frac{1}{5} + \frac{6}{7} - \frac{4}{15} = \frac{118}{105}$$

$$\frac{53}{89} + \frac{19}{347} = \frac{20082}{30883}$$

$$\frac{6}{197} + \frac{11}{12} = \frac{2239}{2364}$$

$$\frac{44}{3} - \frac{153}{427} = \frac{18329}{1281}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$$

$$1 - \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} = \frac{163}{60}$$

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}=\frac{23}{60}$$

$$\frac{1}{2}+\frac{4}{3}+\frac{3}{4}+\frac{6}{5}+\frac{5}{6}=\frac{277}{60}$$

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}=\frac{213}{60}$$

Such reductions as the following are particular cases which often occur:

$$2\frac{1}{3}=2+\frac{1}{3}=\frac{6}{3}+\frac{1}{3}=\frac{7}{3}.$$

Rule. Multiply the whole number by the denominator, and add it to the numerator; let the denominator remain.

$$3\frac{1}{4}=\frac{13}{4} \qquad 7\frac{1}{9}=\frac{64}{9}$$

$$4\frac{4}{5}=\frac{24}{5} \qquad 8\frac{5}{12}=\frac{101}{12}$$

$$16\frac{3}{100}=\frac{1603}{100} \qquad 2\frac{93}{100}=\frac{293}{100}$$

$$14\frac{11}{25}=\frac{361}{25} \qquad 12\frac{9}{17}=\frac{213}{17}$$

V.—*To multiply or divide Fractions by a Whole Number.*

Rule. Do as directed with the numerator, or the contrary with the denominator; that is, *to multiply*, multiply the numerator, or divide the denominator; *to divide*, divide the numerator, or multiply the denominator.

Multiply $\frac{6}{35}$ by 7,

either $\frac{6\times 7}{35}$ or $\frac{6}{35\div 7}$, that is,

either $\frac{42}{35}$ or $\frac{6}{5}$;

the latter is the more simple.

Divide $\frac{6}{35}$ by 3,

either $\frac{6\div 3}{35}$ or $\frac{6}{35\times 3}$, that is,

either $\frac{2}{35}$ or $\frac{6}{105}$;

the former is the more simple.

$$\frac{7}{15}\times 5=\frac{7}{3} \qquad \frac{7}{15}\div 5=\frac{7}{75}$$

$$\frac{8}{21}\times 10=\frac{80}{21} \qquad \frac{3}{19}\div 4=\frac{3}{76}.$$

$$\frac{144}{107}\times 2=\frac{288}{107} \qquad \frac{144}{107}\div 12=\frac{12}{107}.$$

When the multiplier is composed of factors, it may happen that some factors may be most conveniently used in one way, some in the other. The student must render himself very familiar with the following:

$$\left.\begin{array}{l}\text{Multiplication of Numerator}\\ \text{Division of Denominator}\end{array}\right\}\text{is Multiplication,}$$

$$\left.\begin{array}{l}\text{Division of Numerator}\\ \text{Multiplication of Denominator}\end{array}\right\}\text{is Division.}$$

Multiply $\frac{4}{77}$ by 14, or 7×2,

$$\frac{4\times 2}{77\div 7}=\frac{8}{11}.$$

Divide $\frac{60}{77}$ by 25 or 5×5,

$$\frac{60\div 5}{77\times 5}=\frac{12}{385}.$$

$$\frac{4}{21}\times 28=\frac{16}{3} \qquad \frac{4}{21}\div 28=\frac{1}{147}$$

$$\frac{12}{39}\times 26=\frac{24}{3} \qquad \frac{12}{39}\div 8=\frac{3}{78}$$

$$\frac{108}{35}\times 20=\frac{432}{7} \qquad \frac{108}{35}\div 24=\frac{9}{70}$$

$$\frac{55}{63}\times 45=\frac{275}{7} \qquad \frac{55}{63}\div 110=\frac{1}{126}$$

$$\frac{25}{43}\times 86=50 \qquad \frac{25}{43}\div 300=\frac{1}{516}.$$

VI.— *To multiply and divide Fractions by one another.*

Definition 1. The product of $\frac{2}{3}$ and $\frac{4}{5}$. This is the answer to such questions as the following: What is two-thirds of $\frac{4}{5}$? What is four-fifths of $\frac{2}{3}$? A. gave B. $\frac{2}{3}$ of his share, and B. gave C. $\frac{4}{5}$ of what he got. How much of A.'s share did C. get? If 1 gallon cost $\frac{2}{3}$ of a shilling, how much of a shilling does $\frac{4}{5}$ of a gallon cost? What is $\frac{4}{5}$ taken two-thirds of a time? What is twice the third part of $\frac{4}{5}$? &c.

Definition 2. The quotient of $\frac{2}{3}$ divided by $\frac{4}{5}$. This is the answer to such questions as the following: What number of times, or what parts of a time, does $\frac{2}{3}$ contain $\frac{4}{5}$? How must $\frac{4}{5}$ be treated, so as to give $\frac{2}{3}$; that is, into how many parts must $\frac{4}{5}$ be divided, and how many of these parts must be taken, so that $\frac{2}{3}$ may result? If 1 gallon cost $\frac{2}{3}$ of a shilling, how many gallons, or how much of a gallon, may be bought for $\frac{4}{5}$ of a shilling?

Rule. To *multiply*, multiply numerators by numerators, and denominators by denominators. To *divide*, divide numerator by numerator, and denominator by denominator; or invert the divisor, and multiply. Or to perform either operation, invert the multiplier or divisor, and proceed as in the other.

Multiply $\frac{2}{3}$ by $\frac{4}{5}$, divide the product by $\frac{5}{7}$, and multiply the result by $\frac{7}{11}$.

$$\frac{2}{3}\times\frac{4}{5}=\frac{2\times4}{3\times5}=\frac{8}{15},\quad \frac{8}{15}\div\frac{5}{7}=\frac{8}{15}\times\frac{7}{5}=\frac{56}{75},\quad \frac{56}{75}\times\frac{7}{11}=\frac{392}{825}.$$

$$\frac{16}{3}\times\frac{20}{7}=\frac{320}{21}\qquad \frac{4}{11}\div\frac{3}{5}=\frac{20}{33}.$$

$$\frac{20}{7}\times\frac{16}{3}=\frac{320}{21}\qquad \frac{3}{5}\div\frac{4}{11}=\frac{33}{20}.$$

Before the *multiplication* is made, strike out any factors which are common to a numerator and a denominator; before the *division* is made, strike out any factors which are common to both numerators, or to both denominators.

$$\frac{16}{21}\times\frac{35}{24}=\frac{8\times2}{7\times3}\times\frac{7\times5}{8\times3}=\frac{2}{3}\times\frac{5}{3}=\frac{10}{9}$$

$$\frac{22}{35}\div\frac{33}{25}=\frac{11\times2}{7\times5}\div\frac{11\times3}{5\times5}=\frac{2}{7}\div\frac{3}{5}=\frac{10}{21}$$

$$\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}=\frac{1}{120}\qquad \frac{1}{2}\div\frac{1}{3}\times\frac{1}{4}\div\frac{1}{5}=\frac{15}{8}$$

$$\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{1}{5}\qquad \frac{1}{2}\times\frac{1}{3}\times\frac{3}{4}\div\frac{2}{7}=\frac{7}{16}$$

$$\frac{21}{16}\times\frac{10}{7}=\frac{15}{8}\qquad \frac{7}{18}\times\frac{11}{10}\times\frac{27}{14}\times\frac{5}{2}=\frac{33}{16}$$

$$\frac{51}{96}\times\frac{51}{96}\times\frac{51}{96}=\frac{132651}{884736}\qquad \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}=\frac{64}{729}$$

$$\frac{113}{355} \times \frac{221}{118} = \frac{24973}{41890} \qquad \frac{412}{167} \times \frac{397}{777} = \frac{163564}{129759}$$

$$\frac{1}{169} \times \frac{228}{963} = \frac{228}{162747} \qquad \frac{93}{4600} \times \frac{125}{662} = \frac{11625}{30452}$$

$$\frac{60}{61} \times \frac{60}{61} \times \frac{60}{61} \times \frac{60}{61} \times \frac{60}{61} \times \frac{60}{61} = \frac{46656000000}{51520374361}$$

$$\frac{6}{7} \times \frac{1}{2} \times \frac{4}{3} \times \frac{2}{11} \times \frac{7}{9} \times \frac{18}{25} = \frac{16}{275} \qquad \frac{14}{13} \times \frac{13}{14} = 1$$

$$\frac{17}{2} \div \frac{6}{5} = 7\frac{1}{12} \qquad \frac{25}{18} \div \frac{25}{18} = 1$$

$$\frac{25}{18} \div \frac{18}{25} = \frac{625}{324} \qquad \frac{119}{27} \div \frac{338}{113} = \frac{13447}{9126}$$

$$\frac{3163}{468} \div \frac{799}{25} = \frac{79075}{373932} \qquad 3\frac{1}{2} \div \frac{1}{2} = 7$$

$$\left(\frac{7}{6} \times \frac{1}{2} \times \frac{3}{10}\right) \div \left(\frac{6}{11} \times \frac{7}{8} \times \frac{9}{2}\right) = \frac{11}{135}$$

$$\left(\frac{1}{7} \times \frac{4}{9} \times 3\right) \div \left(\frac{6}{11} \times \frac{8}{9} \times 4\right) = \frac{11}{112}$$

$$\left(\frac{1}{2} + \frac{1}{3}\right) \div \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{10}{7} \qquad 12 \div 8 = \frac{3}{2}$$

$$\left(2\frac{1}{2} + \frac{1}{6}\right) \div \left(3\frac{1}{2} - \frac{1}{8}\right) = \frac{64}{81} \qquad 6\frac{1}{8} \div 2\frac{1}{3} = 2\frac{5}{8}$$

$$17 \div \frac{1}{3} = 51 \qquad 1 \div \frac{1}{10} = 10$$

$$9 \div 1\frac{1}{2} = 6 \qquad 10\frac{1}{2} \div \frac{3}{4} = 14$$

$$\frac{4}{5} \div \frac{7}{5} = \frac{4}{7} \qquad \frac{100}{3} \div \frac{101}{3} = \frac{100}{101}$$

$$6 \times \frac{2}{3} \div \frac{3}{2} = \frac{8}{3} \qquad 10 \times \frac{7}{11} \div \frac{7}{11} = 10$$

$$9 = 1 + 2 \times \frac{4}{3} + 3 \times \frac{16}{9}$$

VII.—*Fractions having Fractions in the Numerator or Denominator, or both.*

Rule. To reduce such fractions to equivalent simple fractions, multiply the numerator and denominator by the least common multiple of the denominators of the fractions contained in them.

To reduce $\dfrac{3\frac{1}{2}}{4\frac{1}{3}}$ to a simple fraction.

$$\frac{3\frac{1}{2}}{4\frac{1}{3}} = \frac{3\frac{1}{2} \times 6}{4\frac{1}{3} \times 6} = \frac{18 + 3}{24 + 2} = \frac{21}{26}$$

To reduce $\dfrac{4\frac{1}{7} - 2\frac{1}{4}}{6\frac{1}{2} - 2\frac{1}{7}}$ to a simple fraction.

The least common multiple of 7, 4, 2, and 7, is 28:

$$\frac{4\frac{1}{7}-2\frac{1}{4}}{6\frac{1}{2}-2\frac{1}{7}}=\frac{4\frac{1}{7}\times 28-2\frac{1}{4}\times 28}{6\frac{1}{2}\times 28-2\frac{1}{7}\times 28}=\frac{116-63}{182-60}=\frac{53}{122}$$

That is, $4\frac{1}{7}$, when diminished by $2\frac{1}{4}$, is the same proportion of $6\frac{1}{2}$ diminished by $2\frac{1}{7}$ which 53 is of 122.

$$\frac{3\frac{1}{4}}{4\frac{1}{4}}=\frac{13}{17}\qquad\frac{2\frac{1}{9}}{3\frac{1}{5}}=\frac{95}{144}\qquad\frac{2\frac{1}{12}}{8\frac{1}{4}}=\frac{25}{99}$$

$$\frac{8\frac{2}{3}}{9\frac{3}{4}}=\frac{104}{117}\qquad\frac{1\frac{1}{2}}{1\frac{1}{8}}=\frac{4}{3}\qquad\frac{\frac{22}{3}}{1\frac{1}{4}}=\frac{88}{15}$$

$$\frac{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}}{1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}}=\frac{12-6+4-3}{12+6-4-3}=\frac{7}{11}$$

$$\frac{2\frac{1}{4}-1\frac{1}{3}}{2\frac{1}{8}+1\frac{1}{4}}=\frac{54-32}{51+30}=\frac{22}{81}$$

VIII.—*Miscellaneous Exercises in the preceding Rules.*

$$\frac{\frac{1}{4}-\frac{1}{9}}{\frac{1}{2}+\frac{1}{3}}\times\frac{3}{5}+\frac{\frac{2}{3}+\frac{3}{2}}{1+\frac{4}{9}}\times\frac{1}{18}=\frac{11}{60}$$

$$\frac{7-3\frac{1}{4}}{7+3\frac{1}{4}}\times\frac{\frac{2}{3}}{\frac{4}{7}}-\frac{1}{2}\times\frac{1}{5}\times\frac{1+\frac{1}{10}}{1000}=\frac{1749547}{4200000}$$

$$\frac{\frac{2}{19}+\frac{1}{3}}{3-\frac{1}{3}}\left(\frac{1}{3}+\frac{1}{5}\right)=\frac{5}{57}$$

$$\frac{18}{17}\times\left(1-\frac{64}{81}\right)+\frac{.8}{11}\times\frac{1}{6}\times\left(\frac{1}{2}+\frac{5}{12}\right)=\frac{1}{3}$$

$$\frac{2}{7}\times\frac{1-\frac{2}{7}}{2}+\frac{4}{5}\times\frac{1}{10}+\frac{3}{5}\left(\frac{1}{2}+\frac{11}{14}\right)+\frac{3}{70}\left(\frac{2}{7}+\frac{4}{5}\right)=1$$

$$\frac{\frac{13}{21}\times\frac{1}{2}-\frac{11}{14}\times\frac{1}{3}}{\frac{16}{21}\times\frac{1}{2}-\frac{13}{14}\times\frac{1}{3}}=\frac{2}{3}$$

$$\frac{2\frac{1}{2} \times 2\frac{1}{2} \times 2\frac{1}{2} - 1}{2\frac{1}{2} \times 2\frac{1}{2} - 1} = 2\frac{1}{2} + \frac{2}{7}$$

$$\frac{6\frac{1}{4} \times 6\frac{1}{4} \times 6\frac{1}{4} - 8}{6\frac{1}{4} \times 6\frac{1}{4} - 4} = 6\frac{1}{4} + \frac{4}{8\frac{1}{4}}$$

$$\frac{3}{7} + \frac{16}{49} = \frac{4}{7} + \frac{9}{49} = 5\frac{2}{7} \times \frac{1}{8} + \frac{37}{392}$$

IX.—*Verification of Algebraical Processes.*

The following are some algebraical equations which are always true, whatever numbers or fractions may be placed instead of the letters; provided only, that wherever a subtraction occurs, such as $a - b$, a must be greater than b. The student must attempt to verify them; and the proof that he is correct consists in his finding the same number on each side of the equation. For instance, in the first example, let a stand for $\frac{1}{2}$, and b for $\frac{1}{3}$: then

$$\frac{b}{a+b} = \frac{\frac{1}{3}}{\frac{1}{2}+\frac{1}{3}} = \frac{2}{5} \quad \frac{a-b}{b} = \frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}} = \frac{1}{2}$$

$$\frac{b}{a+b} + \frac{a-b}{b} = \frac{2}{5} + \frac{1}{2} = \left(\frac{9}{10}\right)$$

Again,

$$\frac{aa}{ab+bb} = \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}} = \frac{\frac{1}{4}}{\frac{1}{6}+\frac{1}{9}} = \left(\frac{9}{10}\right)$$

The following list may be considered long, but it must be remembered that every one is also an example in algebra, and that the young student cannot* be more usefully employed at this stage of his progress, whether the operation be considered with reference to arithmetic or algebra. The student should first try each expression with some whole numbers, before he proceeds to use fractions, in order to be certain that he understands the meaning of the terms.

$$\frac{b}{a+b} + \frac{a-b}{b} = \frac{aa}{ab+bb}$$

$$\frac{a-b}{a+b} + \frac{a+b}{a-b} = \frac{2aa+2ab}{aa-bb}$$

$$\frac{1}{1+a} - \frac{1}{1+2a} = \frac{a}{1+3a+2aa}$$

$$(a+b) \times (a-b) = aa - bb$$

$$(a+b) \times (a+b) = aa + 2ab + bb$$

$$(a-b) \times (a-b) = aa - 2ab + bb$$

$$\frac{1+x}{1+\frac{1}{x}} = x \qquad \frac{ax}{a+x} = a - \frac{aa}{a+x}$$

$$(x+a)\times(x+b) = xx+(a+b)\times x+ab$$

$$(x-a)\times(x-b) = xx-(a+b)\times x+ab$$

$$(ax - by)(ax - by) = (aa + bb)(xx + yy) - (ay + bx)(ay + bx)$$

$$\frac{a+b}{2} + \frac{a-b}{2} = a \qquad \frac{a+b}{2} - \frac{a-b}{2} = b$$

$$\frac{xx - 2x + 1}{xx - 1} = \frac{x-1}{x+1} \qquad \frac{xx - 3x + 2}{xx - 10x + 9} = \frac{x-2}{x-9}$$

$$\frac{xxx + 9xx + 26x + 24}{xxx + 6xx + 11x + 6} = \frac{x+4}{x+1}$$

$$\frac{(n+1)\ (n+2)}{2} = \frac{n(n+1)}{2} + n + 1$$

$$\frac{(n+1)(n+2)(n+3)}{6} = \frac{n(n+1)(n+2)}{6} + \frac{n(n+1)}{2} + n + 1$$

* The very little power which even advanced students generally possess, of turning their algebraical into arithmetical results, is one of the principal features of school instruction, as it exists at present, and is a very serious impediment to higher studies.

$$\frac{c}{a-b}=\frac{c}{a}+\frac{bc}{aa}+\frac{bbc}{aaa}+\frac{bbbc}{aaa\ (a-b)}$$

$$\frac{c}{a+b}=\frac{c}{a}-\frac{bc}{aa}+\frac{bbc}{aaa}-\frac{bbbc}{aaa\ (a+b)}$$

$$\frac{1+x}{1-x}-\frac{1-x}{1+x}-\frac{3x}{1-xx}=\frac{x}{1-xx}$$

$$a+\frac{1}{a}=\frac{(a+1)\,(a+1)}{a}-2 \qquad a-\frac{1}{a}=\frac{(a+1)\,(a-1)}{a}$$

$$(a+b+c)\,(b+c-a)\,(c+a-b)\,(a+b-c)=$$
$$=2aabb+2aacc+2bbcc-aaaa-bbbb-cccc$$

X.—*Algebraical Theorems of Approximation, for Verification.*

If p be very nearly equal to 1, then the following theorems are *nearly* true :

$$pp=2p-1 \qquad 1\div p=2-p$$
$$ppp=3p-2 \qquad 1\div pp=3-2p$$
$$pppp=4p-3 \text{ \&c.} \qquad 1\div ppp=4-3p \text{ \&c.}$$

If x be very small, the following theorems are *nearly* true.

$$\frac{1}{1+x}=1-x \qquad \frac{1}{1-x}=1+x \qquad \frac{1+x}{1-x}=1+2x$$

$$\frac{1-3x}{1-2x}=1-x \qquad \frac{1+6x}{1-4x}=1+10x \qquad \frac{6-4x}{3+2x}=2-\frac{8}{3}x$$

If x be very great, the following theorems are *nearly* true :

$$\frac{x}{x+1}=1-\frac{1}{x} \qquad \frac{x-1}{x+1}=1-\frac{2}{x} \qquad \frac{1}{1+x}=\frac{1}{x}-\frac{1}{xx}$$

Section 2.—*Decimal Fractions.*

I.—*Exercises on the Meaning of the Decimal Notation.*

·1 is read *decimal, one.*
·123 is read *decimal, one, two, three.*
36·012 is read *thirty-six, decimal, nought, one, two.*

The student should now write the following and similar tables :—

·1 means $\frac{1}{10}$	·01 means $\frac{1}{100}$	·001 means $\frac{1}{1000}$
·2 . . . $\frac{2}{10}$	·02 . . . $\frac{2}{100}$	·002 . . . $\frac{2}{1000}$
·3 . . . $\frac{3}{10}$	·03 . . . $\frac{3}{100}$	·003 . . . $\frac{3}{1000}$
·4 . . . $\frac{4}{10}$	·04 . . . $\frac{4}{100}$	·004 . . . $\frac{4}{1000}$

&c., &c.

·123 is $\frac{1}{10}+\frac{2}{100}+\frac{3}{1000}$; which is $\frac{100}{1000}+\frac{20}{1000}+\frac{3}{1000}$; which is $\frac{123}{1000}$

·0104 is $\frac{1}{100}+\frac{4}{10000}$; which is $\frac{100}{10000}+\frac{4}{10000}$; which is $\frac{104}{10000}$

6·7 is $6+\frac{7}{10}$ which is $\frac{60}{10}+\frac{7}{10}$; which is $\frac{67}{10}$

$$35{\cdot}013 = 35 + \frac{1}{100} + \frac{3}{1000} = \frac{35103}{1000}$$

$$2{\cdot}008 = 2 + \frac{8}{1000} = \frac{2008}{1000}$$

$${\cdot}0174 = \frac{1}{100} + \frac{7}{1000} + \frac{4}{10000} = \frac{174}{10000}$$

$$12{\cdot}11 = 12 + \frac{1}{10} + \frac{1}{100} = \frac{1211}{100}$$

$$\begin{aligned}
12345 &= 10000 + 2000 + 300 + 40 + 5\\
1234{\cdot}5 &= 1000 + 200 + 30 + 4 + \frac{5}{10}\\
123{\cdot}45 &= 100 + 20 + 3 + \frac{4}{10} + \frac{5}{100}\\
12{\cdot}345 &= 10 + 2 + \frac{3}{10} + \frac{4}{100} + \frac{5}{1000}\\
1{\cdot}2345 &= 1 + \frac{2}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000}
\end{aligned}$$

$$\begin{aligned}
12345 &= 1234{\cdot}5 \times 10 = 123{\cdot}45 \times 100 = 12{\cdot}345 \times 1000\\
1234{\cdot}5 &= 123{\cdot}45 \times 10 = 12{\cdot}345 \times 100 = 1{\cdot}2345 \times 1000\\
123{\cdot}45 &= 12{\cdot}345 \times 10 = 1{\cdot}2345 \times 100 = {\cdot}12345 \times 1000\\
1{\cdot}2345 &= {\cdot}12345 \times 10 = {\cdot}012345 \times 100 = {\cdot}0012345 \times 1000\\
{\cdot}12345 &= {\cdot}012345 \times 10 = {\cdot}0012345 \times 100 = {\cdot}00012345 \times 1000
\end{aligned}$$

$$\begin{aligned}
{\cdot}12345 &= \frac{1{\cdot}2345}{10} = \frac{12{\cdot}345}{100} = \frac{123{\cdot}45}{1000} = \frac{1234{\cdot}5}{10000}\\
1{\cdot}2345 &= \frac{12{\cdot}345}{10} = \frac{123{\cdot}45}{100} = \frac{1234{\cdot}5}{1000} = \frac{12345}{10000}\\
12{\cdot}345 &= \frac{123{\cdot}45}{10} = \frac{1234{\cdot}5}{100} = \frac{12345}{1000} = \frac{123450}{10000}\\
123{\cdot}45 &= \frac{1234{\cdot}5}{10} = \frac{12345}{100} = \frac{123450}{1000} = \frac{1234500}{10000}
\end{aligned}$$

$$8{\cdot}2 = 8{\cdot}20 = 8{\cdot}200 = 8{\cdot}2000 = 8{\cdot}20000, \text{ \&c.}$$

Instances like the preceding should be continued until the student is so familiar with the changes of the decimal point as instantly to point out the effect produced by it, without recurring to a rule.

II.—*To find a Decimal Fraction which shall be nearly equal to a given Common Fraction.*

Principle. No common fraction has a decimal fraction exactly equal to it, unless its denominator is divisible by nothing but 2 or 5, or is composed of the product of some numbers of *twos* and *fives*. But a decimal fraction can be found, which shall be as near to a given common fraction as we please, though not exactly equal to it.

Rule. Annex ciphers to the numerator, divide by the denominator, and neglect the remainder. Cut off as many places from the quotient as there were ciphers annexed to the numerator, for decimals. If one cipher was annexed, the decimal so obtained is within $\frac{1}{10}$ of the given fraction; if two ciphers, within $\frac{1}{100}$; if three ciphers, within $\frac{1}{1000}$; and so on.

In this and all other decimal operations, when directions are given to cut off a certain number of places, and there

are not places enough to be so cut off, affix ciphers to the beginning, in sufficient number to make up the deficiency. Thus, to cut off three decimal places from 25, write ·025; to cut off ten decimal places from 118, write ·0000000118.

Find a decimal fraction which shall be within $\frac{1}{10000}$ of $\frac{18}{23}$.

Annex *four* ciphers to 18, and divide by 23.

23)180000(7826
rem. 2.

Cut off *four* places from 7826, and the answer, ·7826, is within $\frac{1}{10000}$ of $\frac{18}{23}$.

Verification. $\frac{18}{23} - \frac{7826}{10000} = \frac{2}{230000}$, and $\frac{2}{230000}$ is $\frac{1}{115000}$, which is less than $\frac{1}{10000}$.

Find a fraction which shall be within $\frac{1}{1000000}$ of $\frac{1}{913}$.

913)1000000(1095
rem. 265.

Make *six* decimal places in 1095, which gives ·001095, the fraction required.

Definition. A decimal is said to be *true to the (first, second, third, &c.) place of figures* when any alteration in the (*first, second, third, &c.*) place of figures would remove it farther from the truth than it is as it stands. For instance:

$$\frac{61}{99} = 61616 \text{ very nearly.}$$

It is also very nearly ·6161, but not quite so near to this as to ·6162. The second is a little too great, the first a little too small; but the second is not so much in excess as the first is in defect.

Rule. To make a decimal true to the last figure, find one more figure than is wanted; if the last figure be 5, or upwards, increase the preceding by 1. Thus:

·18829976.

If we wish to retain	one place only, write		·2
,,	,,	two places	·19
,,	,,	three	·188
,,	,,	four	·1883
,,	,,	five	·18830
,,	,,	six	·188300
,,	,,	seven	·1882998.

What is the nearest decimal fraction to $\frac{1}{309}$ true to *five* places of decimals.

Annex *six* ciphers to 1, and divide by 309.

309)1000000(3236

Answer: ·003236, which, made true to five places, is ·00324.

The following examples of decimal fractions are all true to the last place:

$\frac{7}{24} = ·2917$ $\qquad \frac{370}{873} = ·42383$

$\frac{11}{28} = ·3929$ $\qquad \frac{61}{827} = ·07376$

$\frac{1}{29} = ·0345$ $\qquad \frac{447}{257} = 1·73930$

$\frac{1}{940} = ·0011$ $\qquad \frac{1012}{582} = 1·73883$

$\frac{41}{741} = ·0553$ $\qquad \frac{410}{555} = ·73874$

$\frac{16}{289} = ·0554$ $\qquad \frac{355}{113} = 3·14159$

* $\frac{33}{596} = ·0554$ $\qquad \frac{1}{3090} = ·00032$

$\frac{700}{793} = ·8827$ $\qquad \frac{100}{633} = ·15798$

* The student must not be surprised at this fraction having the same decimal (to four places) as the preceding. The two fractions do not differ by so much as *one ten-thousandth.*

Instances of the above process carried to a greater number of places:

$$\frac{653}{633} = 1\cdot 031595576619273301737756714060031 5955, \text{ \&c.}$$

$$\frac{43}{953} = \cdot 045120671563483735571878279118572927597 0619$$

$$\frac{1}{561} = \cdot 0017825311942959001782531194295900, \text{ \&c.}$$

Instances of fractions which can be exactly expressed decimally; that is, in which the denominators are made by multiplying *twos* or *fives*, or both:

$$\frac{1}{2} = \cdot 5 \quad \frac{1}{5} = \cdot 2 \quad \frac{1}{4} = \cdot 25 \quad \frac{1}{25} = \cdot 04$$

$$\frac{1}{8} = \cdot 125 \quad \frac{1}{16} = \cdot 0625 \quad \frac{1}{32} = \cdot 03125$$

$$\frac{1}{64} = \cdot 015625 \quad \frac{1}{125} = \cdot 008 \quad \frac{1}{128} = \cdot 0078125$$

$$\frac{1}{512} = \cdot 001953125 \quad \frac{7}{512} = \cdot 013671875$$

$$\frac{63}{64} = \cdot 984375 \quad \frac{101}{8192} = \cdot 0123291015625.$$

III.—*Reduction of Decimal Fractions to a common Denominator.*

Rule. Annex ciphers to all which have a less number of places than are in that which has the greatest number of places, so that all shall have the same number of places. Thus ·1, ·12, ·123, reduced to a common denominator, are ·100, ·120, ·123.

Fractions given.			Reduced to a common Denominator.		
·06,	·031,	·0148	·0600,	·0310,	·0148
12·3,	2·4,	·197	12·300,	2·400,	·197.

IV.—*Addition and Subtraction of Decimal Fractions.*

Rule. Proceed in every respect as in whole numbers, but keep decimal points under one another, and place the decimal point of the result under the other points. (See page 1 for methods.)

$$\begin{array}{r} 12 \\ 12\cdot 1 \\ 1\cdot 42 \\ \cdot 0081 \\ \hline 25\cdot 5281 \end{array} \qquad \begin{array}{lr} \text{From} & 66\cdot 112 \\ \text{Take} & 2\cdot 01783 \\ \hline & 64\cdot 09417 \end{array}$$

Add 12, 12·1, 1·42, and ·0081.

$$1 + \cdot 1 + \cdot 01 + \cdot 001 + \cdot 0001 = 1\cdot 1111$$
$$1 + \cdot 2 + \cdot 03 + \cdot 004 + \cdot 0005 = 1\cdot 2345$$
$$67 + 7\cdot 8 + \cdot 89 - 1\cdot 2168 = 74\cdot 4732$$
$$6\cdot 718909 - 2\cdot 1488 = 4\cdot 570109.$$

V.—*Multiplication of Decimals.*

Throw away the decimal points, and all preliminary ciphers; multiply the results together, and take as many decimal places in the result as there are in both multiplier and multiplicand.

Multiply together the following:

$$\begin{array}{r} 1\cdot 2 \\ 1\cdot 1 \\ \\ 12 \\ 11 \\ \hline 121 \\ \\ \\ 1\cdot 21 \end{array} \qquad \begin{array}{r} 6\cdot 3 \\ \cdot 84 \\ \\ 63 \\ 84 \\ \hline 252 \\ 504 \\ \hline 5292 \\ 5\cdot 292 \end{array} \qquad \begin{array}{r} 2\cdot 99 \\ \cdot 011 \\ \\ 299 \\ 11 \\ \hline 3289 \\ \\ \\ \cdot 03289 \end{array} \qquad \begin{array}{r} \cdot 001 \\ \cdot 01 \\ \\ 1 \\ 1 \\ \hline 1 \\ \\ \\ \cdot 00001 \end{array} \qquad \begin{array}{rl} 6\cdot 0 & \text{Multiplicands.} \\ \cdot 5 & \text{Multipliers.} \\ \\ 60 & \\ 5 & \\ \hline 300 & \\ \\ \\ 3. & \text{Answers.} \end{array}$$

8 × 8 = 64   8 × ·8 = 6·4   ·8 × ·8 = ·64   ·08 × ·8 = ·064
80 × ·8 = 64   ·008 × ·08 = ·00064   800 × ·0008 = ·64
15·94 × 254·0836 = 4050·092584
·004716 × ·22240656 = ·00104886933696
·923521 × ·28629151 = ·26439622160671
·155 × 24·025 = 3·723875   14·2 × ·142 = 2·0164.

VI.—*Division of Decimals.*

Rule. Case 1. When the divisor has no decimals, or is a whole number, proceed as in common division, and let the *first decimal place* of the quotient be that figure, in the making of which the *first decimal place* of the dividend is brought down; but if more than one decimal place of the dividend is used in making the first figure of the quotient, put the decimal point first, and then a cipher for every decimal place after the first which is used in making the first quotient-figure.

9)173·43
19·27

18)·0041(·0002
36
5

23)4·61(·2
4 6
1

Case 2. When the divisor has decimal places, strike out the decimal point, and remove the point in the dividend as many places to the right as the number of places which have been thus destroyed in the divisor, previously annexing ciphers to the right of the dividend, if necessary.

In both cases, ciphers may be annexed at pleasure to the right of the dividend, and used in forming additional quotient-figures.

·09)1·68(
9)168·00...
16·22...

·4)·0192(
4) ·192
·048

·11)3
11)300·00...
27·27... Quotients.

2·5)·1793
25)1·793(7172
1 75
43
25
180
175
50
50
0
·07172

·0025)179·3
25)1793000(71720
175
43
25
180
175
50
50
0
71720 Quotient.

Case 3. If the dividend be a number followed by ciphers, as 86400, strike out the ciphers, proceed as before, and when the process is finished, remove the decimal point one place to the left for every cipher so struck out.

2500)1·793($\frac{1·793}{25}$ = ·07172

25)1·793(·07172 $\frac{1·793}{2500}$ = ·0007172

Dividend.	Divisor.	Altered Dividend.	Altered Divisor.	First Quotient-Figure.	Part of the Altered Dividend which gives it.	Column in which it must stand.
1·9628	64·19	196·28	6419	3	196·28	2nd Decimal Place.
·0019	·134	1·9	134	1	1·90	2nd Decimal Place.
674	·012	674000	12	5	67	Ten Thousands Column.
6·221	·9136	62210	9136	6	62210	Units Column.
·7021	123·65	70·21	12365	5	70·210	3rd Decimal Place.
1	·001	1000	1	1	1	Thousands Column.
118	190·5	1180	1905	6	1180·0	1st Decimal Place.
1	116·4	10	1164	8	10·000	3rd Decimal Place.

$$\frac{\cdot 6}{6} = \cdot 1 \qquad \frac{6}{\cdot 6} = 10 \qquad \frac{\cdot 06}{60} = \cdot 001 \qquad \frac{\cdot 006}{\cdot 6} = \cdot 01$$

$$\frac{600}{\cdot 6} = 1000 \qquad \frac{600}{\cdot 06} = 10000 \qquad \frac{\cdot 006}{600} = \cdot 00001$$

$$\frac{8\cdot 4}{12} = \cdot 7 \qquad \frac{8\cdot 4}{1\cdot 2} = 7 \qquad \frac{8\cdot 4}{\cdot 12} = 70 \qquad \frac{8\cdot 4}{\cdot 012} = 700$$

$$\frac{\cdot 84}{12} = \cdot 07 \qquad \frac{\cdot 084}{\cdot 12} = \cdot 7 \qquad \frac{\cdot 084}{1\cdot 2} = \cdot 07 \qquad \frac{\cdot 0084}{\cdot 0012} = 7$$

$$\frac{1}{\cdot 159} = 6\cdot 289308 \qquad \frac{\cdot 2}{23\cdot 2} = \cdot 00862069$$

$$\frac{8792}{937\cdot 6567} = 9\cdot 37657 \qquad \frac{6821691\cdot 97627}{88\cdot 03} = 77492\cdot 809$$

$$\frac{37\cdot 96416}{\cdot 156} = 243\cdot 36 \qquad \frac{\cdot 00636056}{\cdot 86} = \cdot 007396$$

$$\frac{\cdot 59}{79800} = \cdot 000007393483 \qquad \frac{61000}{\cdot 825} = 73939\cdot 393939$$

$$\frac{\cdot 59}{80000} = \cdot 000007375 \qquad \frac{23}{\cdot 000579} = 39723\cdot 66148532$$

When the student has acquired sufficient knowledge of the meaning of decimals, and expertness in using them, he will need no other rule for all the cases than the following:—Put a semicolon in the place where the decimal point ought to be, in order that the result should contain no higher or lower denomination than *units*, that is, should lie between 1 and 10; pass from the semi-colon to the decimal point as it stands, repeating *tens*, *hundreds*, *thousands*, &c., as successive figures are passed over, *if to the left*, and TENTHS, HUNDREDTHS, THOUSANDTHS, &c., as successive figures are passed over, IF TO THE RIGHT. Let the first figure of the quotient have the denomination last named. We give underneath the place of the semi-colon, and the value of the first place of the quotient.

Divisions required.	Places of the semi-colon.	Value of first places of the quotient.
$\frac{84}{\cdot 19}$	$\frac{84}{\cdot 19;}$	400
$\frac{\cdot 31}{22}$	$\frac{\cdot 31}{;22\cdot}$	$\cdot 01$
$\frac{5630}{11\cdot 9}$	$\frac{5630}{11\cdot 90;}$	400
$\frac{369\cdot 7}{49872\cdot 3}$	$\frac{369\cdot 7}{49;872\cdot 3}$	$\cdot 007$
$\frac{216\cdot 4}{193\cdot 2}$	$\frac{216\cdot 4}{193;2}$	1

VII.—*Contracted Multiplication of Decimals.*

Rule.—To multiply two decimals together, so as to retain only a certain number of places in the product, without the trouble of finding the rest—invert the order of the figures of the multiplier, and write them under those of the multiplicand in such a way that what was the units figure of the multiplier may come under the last place of decimals, which is to be retained. Multiply as usual, with this exception, that each figure of the multiplier begins with the figure of the multiplicand which comes immediately over it, the figure next to that being only used to carry from (as in the subsequent example). Put the several lines directly under one another, instead of removing each one place to the left.

*** As it is almost impossible to make this rule clear in words, we subjoin an example at length.

Ex. To multiply $147\cdot 3861$ by $\cdot 6457$, retaining only *three* places of decimals. The second factor, written so as to show a unit's place, is $0\cdot 6457$, and in reversing, the 0 must fall under the *third* decimal place of the other factor, thus:—

1473861 75460	Multiplier reversed; units place 0 falling under third decimal 6 of the upper line.
88432	Multiplier 6; figure to begin with, 8, figure to carry from, 6. Six times 6 is 36, nearest ten, *four* tens, carry *four*. Six times 8 is 48, and 4 is 52, put down 2 and carry 5. The rest as usual.
5895	Multiplier 4; figure to begin with, 3; figure to carry from 8. Four times 8 is 32; nearest ten, *three* tens, carry *three*. Four times 3 is 12 and 3 is 15, &c. The rest as usual.
737	Multiplier 5; figure to begin with, 7; figure to carry from, 3. Five times 3 is 15; nearest* ten, *two* tens, carry *two*. Five times 7 is 35 and 2 is 37, &c. The rest as usual.
103	Multiplier 7; figure to begin with, 4; figure to carry from, 7. Seven times 7 is 49, nearest ten, *five* tens, carry *five*. Seven times 4 is 28 and 5 is 33, &c. The rest as usual.
95·167	Add as usual, and mark off three places; (the number proposed) for decimals.

The full product of 147·3861 and ·6457 is 95·16720477, which in thousandths only is nearest to 95·167, our result.

The following multiplications have the proper arrangement and result given. *No* decimal places means that the whole number of the result is required, without fractions.

Multiplication required.	No. of Decimals retained.	Arrangement of Multiplier and Multiplicand.	Result.
36·3771 × 9·99339	three	36·3771 933 999	363·529
19·081137 × 523·36	two	19·081137 6 3325	9986·30
·0699268 × ·9975641	seven	·0699268 1 4657990	·0697565
13763819 × ·05877853	one	13763819·0 35877850 0	809017·0
753554·1 × 7·986355	none	753554·1 5536897	6018150
1·2799416 × ·6156615	seven	1·2799416 5 1665160	·7880108

Where the figures of the multiplier extend to the left of the multiplicand, continue as long as there is either multiplication or carriage. Thus in the first example, the first 9 of the arranged multiplier has no figure above it; but the carriage from the 3 ($9 \times 3 = 27$) is *three* tens, and three must be written under the right hand column of the preceding lines.

VIII.—*Contracted Division of Decimals.*

Rule.—Proceed as usual, until the number of quotient figures remaining to be found does not exceed the number of figures in the divisor. Then, instead of annexing a cipher, or bringing a figure down from the dividend, cut off the last figure of the divisor; that is, do not employ it except to carry from, as in the last rule. See how often this abridged divisor is contained in the remainder; multiply, carrying from the figure cut off; find a new remainder; cut off another figure from the divisor, and repeat the process until all the figures of the divisor are cut off. When the abridged divisor is not contained in the remainder, cut off a second figure from the divisor, put a cipher in the quotient, and proceed. We subjoin a detailed example.

To divide ·1299494 by ·9915206, as far as nine decimal places. The first quotient figure being a decimal, and there being seven places in the divisor, two quotient figures must be found by the usual method; after which, the process is explained.

* In this case, 15 is equally near to *one* ten and *two* tens. It is usual, and generally more correct, to take the higher of the two.

Divisor, afterwards abridged.		Dividend.	Quotient.	
9915206		) 12994940 (	·131060717	
		9915206		
		30797340		
		29745618		
991520	6	1051722	. .	Cut off the 6, reserving it to carry from; 991520 is contained in 1051722 once; once 6 is 6, nearest ten, one ten, carry one. The rest as usual.
		991521		
99152	06	60201	. .	Figure to carry from, 0; 99152 not contained in 60201, cut off another figure from divisor, and put 0 in quotient. Figure to carry from, 2; 9915 contained in 60201, six times. Six times 2 is 12; nearest ten, one ten, carry 1. Six times 5 is 30, and 1 is 31. The rest as usual.
9915	206			
		59491	.	
991	5206	710	. .	991 not contained in 710; cut off one more figure, and put 0 in quotient. Carrying figure 1; 99 contained in 710, seven times. Seven times 1 is 7, nearest ten, one ten, carry one. Seven times 9 is 63, and 1 is 64. The rest as usual.
99	15206			
		694		
9	915206	16	. .	Carrying figure 9. Divis. 9, contained once in 16. Once 9 is 9; nearest ten, one ten, carry one. Once 9 is 9, and 1 is 10.
		10		
	9915206	6	. .	No divisor, carrying figure 9. What number of times 9 will carry 6, or be most nearly 60? Seven times 9 is 63; put 7 in the quotient, and carry 6, which finishes the process.
		6		
		0		

Dividend.	Divisor.	No. of Decimals to be retained.	Quotient.
1	3·14159265	7	·3183099
1	2·7182818	7	·4342944
2992·9	51·77717	5	57·80347
171·8	414·487636	9	·414487636
·273	74·529	9	·003663004
·0008202	·67272804	8	·00121922

When the divisor itself contains more places than are required in the quotient, as many places may be cut from the right as will make the two the same; and the dividend may be cut down in the same way until no more places are left than will give one figure in the quotient, to the abridged divisor, remembering the rule for increasing the last figure. Thus ·1648267 ÷ ·7263, to two places only, may be found from ·16483 ÷ 73, by the rule exemplified above.

Both in multiplication and division, it is best to retain one more place than is absolutely required to be correct.

Section 3.—*Extraction of the Square Root. Examples of Surds and Irrational Quantities.*

I.—*Extraction of the Square Root.*

The rule for this will be better understood by a detailed example than by any verbal explanation. Though the quantities operated upon are decimal, it is to be understood that a whole number may be used in the same way. For 5, for instance, is 5·0000, &c.

The following contains the working of the rule at length for the extraction of the square root of 32·19, to four places of decimals. Annex so many ciphers that the decimal point shall be followed by twice as many places (eight) as there are to be decimals in the root (four). This gives 32·19000000. Point the unit's place, and every *other* place from it, to the right and left, which give 3219000000.

Divisors.	Given number pointed.	Root found, figure by figure, as below.
	3219000000	(5·6736
	25	First *period* 32; nearest square, 25; root 5. Put 5 in the root, and subtract 25 from 32.
106)	719	Remainder 7; bring down next period, 19. Double 5 (10), which place in divisor.
		Cut off one figure from 719,—71. This contains the divisor 10 seven times; *try* 7, as follows: annex it to divisor, 107; multiply by it, 107 × 7 is 749: this is greater than 719: 7 will not do.
	636	Try* 6. Then 106 × 6, is 636—less than 719. Put 6 in the root and in the divisor, and subtract 636 from 719; remainder, 83.
1127)	8300 7889	Bring down next period, 00: add 6 last found to 106, giving new divisor, 112. Cut one figure from 8300—830. This contains 112 seven times. Try 7, and 1127 × 7 is 7889. Put 7 in the root and in the divisor, and subtract 7889 from 8300.
11343)	41100 34029	Remainder, 411. Bring down next period, 00; add 7 last found to 1127, giving new divisor 1134. Cut one figure from 41100, —4110. This contains 1134 three times; trial * no longer necessary. Put 3 in the root, annex 3 to divisor, giving 11343. Subtract 11343 × 3, or 34029.
113466)	707100 680796	Remainder, 7071. Bring down last period, 00; add 3 last found to 11343, giving new divisor 11346. Cut one figure from 707100;—70710. This contains 11346 six times. Put six in the root; annex 6 to divisor, giving 113466. Subtract 113466 × 6, or 680796.
	26304	Remainder, 26304; less than half of 113466, which shows that there is no occasion to change the last found 6 into 7, to have the nearest decimal of four places.

The required root is therefore 5·6736; by which we mean, that though 32·19 has no exact square root, yet 5·6736, multiplied by itself, will give a result *nearer to* 32·19 than any other number with four decimal places. This we will try. Multiply the three successive fractions, 5·6735, 5·6736, 5·6737, each by itself, retaining five decimal places in the product.

5·67350	5·67360	5·67370
53765	63765	73765
2836750	2836800	2836850
340410	340416	340422
39715	39715	39716
1702	1702	1702
284	340	397
32·18861	32·18973	32·19087

Find the difference between each of these, and the quantity which we first set out with, and we have

·00139 ·00027 ·0008,

of which the second is the smallest.

When, after cutting off one figure from the altered remainder, the divisor is not contained in the result, bring down a second period, and place a cipher in the root and the divisor. The following is an instance in the extraction of the square root of 100406552374249. Where the calculator would simply annex a cipher or period to a line, we write the line again with the cipher, that the student may see the several steps.

* This trial will rarely be necessary after the second step. So that having cut one figure from the increased remainder, the number of times which the divisor is therein contained may be written down on the right, and the whole divisor, *thus altered*, multiplied by its last figure.

```
         . . . . . . . .
         100406552374249(1
         1
    2)  00                          *
- - - - - - - - - - - - - - - - -
    20)  0040            10        *
- - - - - - - - - - - - - - - - - - - -
  2002)  004065          1002
           4004
  2004)    6152
- - - - - - - - - - - - - - - - - - - -
  200403     615237      100203
             601209
  200406)   1402842
- - - - - - - - - - - - - - - - - - - -
  20040607)  140284249    10020307
             140284249
                 0
```

Wherever a dotted line occurs, the augmented remainder, with the last figure cut off, is found not to contain the dividend, a new period is brought down below the line, a cipher is annexed to the divisor and to the root (also brought down), and the figure which, after this, answers the purpose, appears at the end of the divisor and of the root. There being no remainder at last, the exact square root required is 10020307.

The student should perform the preceding operation in this form:

```
           . . . . . . . .
           100406552374249(10020307
           1
   2002)  004065
           4004
  200403)    615237
             601209
20040607)    140284249
             140284249
                 0
```

We must notice one more case in which a cipher may occur. We will first write the beginner's attempt, as it would be if he were not cautious. To extract the square root of 2034:

First Attempt.

```
     . .
     203400, &c. (45·1
     16
 85) 434
     425
901)  900
      901
```

Corrected Process.

```
      203400, &c. (45·09, &c.
      16
  85) 434
      425
9009)  90000
       81081
  9018) 892900
                 &c.
```

In the first he has gone wrong, for though 900, stripped of its last figure, contains 90 once exactly, yet 901 (the new figure being annexed) is not contained in 900. He therefore puts a cipher in the divisor and the root, and brings down another period. The decimal point of the root always precedes that root figure in forming which the first decimal period was used, annexed

* The preliminary ciphers may be omitted; 20 is not contained in 004 or 4.

ciphers being always considered as decimals. If periods of ciphers be thrown away in the beginning of the operation, the root is all decimal, and has a cipher at the beginning for every period so thrown away; but this rule does not apply to the throwing away of a single cipher (not a whole period) at the beginning, or to a cipher in the unit's place.

In the following examples, the number whose root is to be extracted is in the first column; the pointing at full length in the second; the same with the decimal point and preliminary ciphers, if any, thrown away, in the third; and the answer in the last.

No. given.	Do., pointed.	Do., simplified.	Square Root, nearly.
·1	0·1000 &c.	1000 &c.	·31622776602
·85	0·8500 &c.	8500 &c.	·92195444573
·0683	0·068300 &c.	68300 &c.	·261342686907
·0068	0·006800 &c.	6800 &c.	·082462112512
9·79	9·7900 &c.	97900 &c.	3·12889756943
97·9	97·9000	979000 &c.	9·89444288

The preceding method may be shortened, as soon as half the decimal places required have been found, by substituting a contracted division.

The rule is, when *half* the number of (decimal and other) places have been obtained, instead of forming a new period, let the remainder stand, strike off a figure from the divisor, and proceed as in contracted division.

The following is the extraction of the square root of 12 to 12 decimal places by this method:

```
          12·00 . . . . . (3·464101615138
           9
 64)       3 00
           2 56
 686)       4400
            4116
   6924)   28400
           27696
   69281)   70400
            69281
   6928201) 11190000
             6928201
   692820|2) 4261799
             4156921
              104878
               69282
               35596
               34641
                 955
                 693
                 262
                 208
                  54
                  55  The nearest.  The 8 not so much too great as 7
                                    would be too small.
```

The student may furnish himself with examples to any amount by the following principle: If A has the square-root B, *four* times A has the square-root *twice* B, *nine* times A has the square-root *three* times B, and so on. Let him then choose a number or fraction, and extract the square-root, say of four times that number, as well as of the number itself. His first result should be twice the second. The last figures only cannot be expected to agree.

The extraction of the cube-root is a long and useless process. When the student becomes acquainted with logarithms, he will always use them for the extraction of all roots, the square-root included.

II.—*Definition and Notation of Powers and Roots.*

Operation.	Denoted by	Commonly called
7×7	7^2	The square or second power of 7.
$7 \times 7 \times 7$	7^3	The cube, or third power of 7.
$7 \times 7 \times 7 \times 7$	7^4	The fourth power of 7.
$7 \times 7 \times 7 \times 7 \times 7$	7^5	The fifth power of 7.
&c.	&c.	&c.

By analogy, 7 is written 7^1 and called the first power of 7.

Condition fulfilled by *p*.	Manner of denoting *p*.	Name of *p*.
$pp = 7$	$\sqrt{7}$ or $7^{\frac{1}{2}}$	Square, or second root of 7, or 7 to the power of one-half.
$ppp = 7$	$\sqrt[3]{7}$ or $7^{\frac{1}{3}}$ . . .	Cube, or third root of 7, or 7 to the power of one-third.
$pppp = 7$	$\sqrt[4]{7}$ or $7^{\frac{1}{4}}$	Fourth root of 7, or 7 to the power of one-fourth.
$ppp = 7^2$	$\sqrt[3]{7^2}$ or $7^{\frac{2}{3}}$ . . .	Cube root of the square of 7, or 7 to the power of two-thirds.
$ppppp = 7^6$	$\sqrt[5]{7^6}$ or $7^{\frac{6}{5}}$	Fifth root of the sixth power of 7, or 7 to the power of six-fifths.
$pp = 7^{11}$	$\sqrt{7^{11}}$ or $7^{\frac{11}{2}}$ . . .	Square root of the eleventh power of 7, or 7 to the power of eleven halves.

Verify the following equations by multiplication:

$$16 = 2^4 = 4^2 = 8^{\frac{4}{3}} = 16^1 = 32^{\frac{4}{5}} = 64^{\frac{2}{3}}$$

$$9 = 3^2 = 9^1 = 27^{\frac{2}{3}} = 81^{\frac{1}{2}} = 243^{\frac{2}{5}} = 729^{\frac{1}{3}}$$

$$32 = 8^{\frac{5}{3}} = 4^{\frac{5}{2}} \qquad 256 = 32^{\frac{8}{5}}$$

In such an equation as $32 = 4^{\frac{5}{2}}$, $\frac{5}{2}$ has two names; one referring it to the 4, the other to the 32.

$\frac{5}{2}$ is called the *exponent* of 4.

$\frac{5}{2}$ is called the *logarithm* of 32 *to the base* 4.

Verify the following assertions:

The number undermentioned	is the logarithm of the corresponding number undermentioned	to the base
2, 3, 4	100, 1000, 10000	10
3, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$	262144, 8, 4, 2	64
6, 5, 4	64, 32, 16	2
$\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{2}{5}$	1024, 32, 16, 256	1048576
$\frac{3}{4}$, $\frac{3}{5}$	32768, 4096	1048576
$\frac{4}{5}$, $\frac{1}{10}$, $\frac{3}{10}$	65536, 4, 64	1048576
$\frac{7}{10}$, $\frac{9}{10}$, $\frac{1}{20}$	16384, 262144, 2	1048576

III.—*Particular cases of Propositions which are proved by Algebraic Reasoning.*

The student should go through the whole of these, adding others similar to them if necessary, until he performs all such operations by habit, without rules.

$$10^6 = 10 \times 10^5 = 10^2 \times 10^4 = 10^3 \times 10^3 = 1000000$$

$$2^8 = 2 \times 2^7 = 2^2 \times 2^6 = 2^3 \times 2^5 \text{ \&c.} = 256$$

$$2^8 \times 2^7 = 2^{15} \qquad 12^9 \times 12^{14} = 12^{23}$$

$$8^3 = \frac{8^4}{8} = \frac{8^5}{8^2} = \frac{8^6}{8^3} = \frac{8^7}{8^4} \text{ \&c.} = 512$$

$$\frac{9^{18}}{9^5} = 9^{13} \qquad \frac{163^{21}}{163^{15}} = 163^6 \qquad \frac{4^{90}}{4^{33}} = 4^{57}$$

$$2^{12} = (2^2)^6 = (2^3)^4 = (2^4)^3 = (2^6)^2 = 4096$$

$$(3^8)^4 = 3^{32} \qquad (7^{11})^5 = 7^{55} \qquad (100^9)^9 = 100^{81} \text{ \&c.}$$

$$2^3 = 2^{\frac{6}{2}}, \text{ or } \sqrt{2^6}, = 2^{\frac{9}{3}} \text{ or } \sqrt[3]{2^9}, = 2^{\frac{12}{4}} \text{ or } \sqrt[4]{2^{12}}$$

$$2^{\frac{1}{2}} = 2^{\frac{2}{4}} = 2^{\frac{3}{6}} = 2^{\frac{4}{8}} = 2^{\frac{5}{10}}$$

$$\text{or, } \sqrt{2} = \sqrt[4]{2^2} = \sqrt[6]{2^3} = \sqrt[8]{2^4} = \sqrt[10]{2^5} \text{ \&c.}$$

$$\sqrt[4]{33^{18}} = \sqrt{33^9} \qquad \sqrt[12]{36^{12}} = 36 \qquad \sqrt[3]{15^{15}} = 15^5$$

$$\sqrt{\sqrt{9}}, \text{ or } (9^{\frac{1}{2}})^{\frac{1}{2}} = 9^{\frac{1}{4}} \qquad \sqrt[3]{\sqrt[3]{9}} \text{ or } (9^{\frac{1}{3}})^{\frac{1}{3}} = 9^{\frac{1}{9}}$$

$$\sqrt[4]{\sqrt[5]{8}} \text{ or } (8^{\frac{1}{5}})^{\frac{1}{4}} = 8^{\frac{1}{20}} \qquad \sqrt[5]{\sqrt[4]{8}} \text{ or } (8^{\frac{1}{4}})^{\frac{1}{5}} = 8^{\frac{1}{20}}$$

$$\sqrt[5]{32^2} = (\sqrt[5]{32})^2 = 4 \qquad \sqrt[5]{32^3} = (\sqrt[5]{32})^3 = 8$$

$$\sqrt[3]{\left\{\sqrt[5]{10^3}\right\}^4} = (10^{\frac{3}{5}})^{\frac{4}{3}} = 10^{\frac{12}{15}} = \left(\left((10^3)^{\frac{1}{5}}\right)^4\right)^{\frac{1}{3}}$$

$$\sqrt[3]{\left\{\sqrt[5]{10^4}\right\}^2} = (10^{\frac{4}{5}})^{\frac{2}{3}} = 10^{\frac{8}{15}} = \left(\left((10^4)^{\frac{1}{5}}\right)^2\right)^{\frac{1}{3}}$$

$$6^{\frac{1}{2}} \times 6^{\frac{1}{3}} = 6^{\frac{1}{2}+\frac{1}{3}} = 6^{\frac{5}{6}}, \text{ or } \sqrt{6} \times \sqrt[3]{6} = \sqrt[6]{6^5}$$

$$7^{\frac{2}{3}} \times 7^{\frac{4}{5}} = 7^{\frac{2}{3}+\frac{4}{5}} = 7^{\frac{22}{15}}, \text{or } \sqrt[3]{7^2} \times \sqrt[5]{7^4} = \sqrt[15]{7^{22}}$$

$$8^{\frac{1}{2}} \div 8^{\frac{1}{3}} = 8^{\frac{1}{2}-\frac{1}{3}} = 8^{\frac{1}{6}}, \text{ or } \sqrt{8} \div \sqrt[3]{8} = \sqrt[6]{8}$$

$$9^{\frac{4}{5}} \div 9^{\frac{2}{3}} = 9^{\frac{4}{5}-\frac{2}{3}} = 9^{\frac{2}{15}}, \text{or } \sqrt[5]{9^4} \div \sqrt[3]{9^2} = \sqrt[15]{9^2}$$

$$10^4 \div 10^9 = \frac{10^4}{10^9} = \frac{10^4 \div 10^4}{10^9 \div 10^4} = \frac{1}{10^5} = \frac{1}{10^{9-4}}$$

$$6^7 \div 6^{18} = \frac{1}{6^{18-7}} = \frac{1}{6^{11}} \qquad 7^{20} \div 7^{33} = \frac{1}{7^{13}}$$

$$3^{\frac{1}{4}} \div 3^{\frac{1}{3}} = \frac{1}{3^{\frac{1}{3}-\frac{1}{4}}} = \frac{1}{3^{\frac{1}{12}}} \qquad 5^{\frac{2}{7}} \div 5^{\frac{5}{7}} = \frac{1}{5^{\frac{3}{7}}}$$

$$(10^{\frac{2}{3}})^{\frac{3}{4}} \times (10^{\frac{1}{2}})^4 \div (10^{\frac{1}{3}})^{\frac{1}{2}} = 10^{\frac{1}{2}} \times 10^2 \div 10^{\frac{1}{6}} = 10^{\frac{1}{2}+2-\frac{1}{6}} = 10^{\frac{7}{3}}$$

$$(5^{\frac{1}{2}})^7 \times (5^3)^{\frac{1}{2}} \div (5^4)^{\frac{1}{8}} = 5^{\frac{7}{2}} \times 5^{\frac{3}{2}} \div 5^{\frac{1}{2}} = 5^{\frac{9}{2}}$$

$$6^{2\frac{1}{2}} \text{ means } 6^{\frac{5}{2}} \qquad 3^{4\frac{1}{5}} \text{ means } 3^{\frac{21}{5}}$$

$$\left(2{\cdot}36\right)^{1{\cdot}01} \text{ means } \left(2{\cdot}36\right)^{\frac{101}{100}} \qquad \left(6^{{\cdot}4}\right)^{{\cdot}5} \text{ is } 6^{{\cdot}2}$$

$$7^{\cdot 1} \text{ means } 7^{\frac{1}{10}} \qquad 8^{\cdot 014} \text{ means } 8^{\frac{14}{1000}}$$

$$2^3 \times 4^3 = 8^3 \qquad 3^4 \times 10^4 = 30^4 \qquad 6^{\frac{1}{2}} \times 7^{\frac{1}{2}} = 42^{\frac{1}{2}}$$

$$2^{\frac{1}{2}} \times 3^{\frac{1}{2}} \times 4^{\frac{1}{2}} = \overline{2 \times 3 \times 4}|^{\frac{1}{2}} = 24^{\frac{1}{2}} \text{ or } \sqrt{2} \times \sqrt{3} \times \sqrt{4} = \sqrt{24}$$

$$7^{\frac{2}{3}} \times 8^{\frac{2}{3}} = \overline{7 \times 8}|^{\frac{2}{3}} \text{ or } \sqrt[3]{7^2} \times \sqrt[3]{8^2} = \sqrt[3]{56^2}$$

$$\left(\frac{1}{2}\sqrt{2}\right)^2 = \left(\frac{1}{2}\right)^2 \times 2 = \frac{1}{2} \qquad \left(\frac{2}{3}\sqrt{\frac{13}{14}}\right)^2 = \frac{4}{9} \times \frac{13}{14} = \frac{26}{63}$$

$$\left(4\sqrt{10}\right)^2 = 160 \qquad \left(\frac{1}{2}\sqrt{8}\right)^2 = 2 \qquad \left(\frac{3}{7}\sqrt{\frac{14}{3}}\right)^2 = \frac{6}{7}$$

$$\sqrt{81} = 9 \qquad \sqrt{81 \times 2} = \sqrt{81} \times \sqrt{2} = 9\sqrt{2} \quad ; \quad \sqrt{156} = 2\sqrt{39}$$

$$\sqrt{\frac{2}{3}} \times \sqrt{\frac{15}{8}} = \frac{\sqrt{5}}{2} \qquad \sqrt{18} \times \sqrt{20} = 2\sqrt{90}$$

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2} \quad ; \quad \sqrt{32} = 4\sqrt{2}$$

$$\sqrt{44} = 2\sqrt{11} \qquad \sqrt{160} = 4\sqrt{10} \qquad \sqrt{1836} = 6\sqrt{51}$$

$$10\sqrt{9} = \sqrt{900} \qquad 7\sqrt{3} = \sqrt{147} \qquad 12\sqrt{12} = \sqrt{1728}$$

$$\sqrt[3]{56} = \sqrt[3]{8 \times 7} = \sqrt[3]{8} \times \sqrt[3]{7} = 2\sqrt[3]{7} \qquad \sqrt{168} = 2\sqrt[3]{21}$$

$$\sqrt[4]{288} = \sqrt[4]{16 \times 18} = \sqrt[4]{16} \times \sqrt[4]{18} = 2\sqrt[4]{18} \quad , \quad \sqrt[4]{6144} = 4\sqrt[4]{24}$$

$$\sqrt[4]{6966} = 3\sqrt[4]{86} \quad , \quad 4\sqrt[4]{3} = \sqrt[4]{768} \qquad \sqrt[7]{256} = 2\sqrt[7]{2}$$

$$\sqrt{\frac{3}{7}} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{3} \times \sqrt{3}}{\sqrt{7} \times \sqrt{3}} = \frac{3}{\sqrt{21}} \qquad \sqrt{\frac{6}{11}} = \frac{6}{\sqrt{66}}$$

$$\sqrt{\frac{3}{7}} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{3} \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{\sqrt{21}}{7} \qquad \sqrt{\frac{6}{11}} = \frac{\sqrt{66}}{11}$$

$$\sqrt{\frac{5}{49}} = \frac{\sqrt{5}}{\sqrt{49}} = \frac{\sqrt{5}}{7} \qquad \sqrt{\frac{3}{64}} = \frac{\sqrt{3}}{8} = \sqrt{\frac{121}{7}} = \frac{11}{\sqrt{7}}$$

$$\sqrt[3]{\frac{3}{5}} = \frac{\sqrt[3]{3}}{\sqrt[3]{5}} = \frac{\sqrt[3]{3} \times \sqrt[3]{3} \times \sqrt[3]{3}}{\sqrt[3]{5} \times \sqrt[3]{3} \times \sqrt[3]{3}} = \frac{3}{\sqrt[3]{5 \times 3 \times 3}} = \frac{3}{\sqrt[3]{45}}$$

$$\sqrt[3]{\frac{3}{5}} = \frac{\sqrt[3]{3}}{\sqrt[3]{5}} = \frac{\sqrt[3]{3} \times \sqrt[3]{5} \times \sqrt[3]{5}}{\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}} = \frac{\sqrt[3]{75}}{5} = \frac{1}{5}\sqrt[3]{75}$$

$$\sqrt[3]{\frac{4}{7}} = \frac{\sqrt[3]{196}}{7} = \frac{4}{\sqrt[3]{112}} \quad , \quad \sqrt{\frac{9}{10}} = \frac{\sqrt[3]{900}}{10} = \frac{9}{\sqrt[3]{810}}$$

The preceding operations occur perpetually in the higher applications of arithmetic; the student should repeat them on low numbers, till he is perfectly familiar with all of them. The following are the rules under which they may all be reduced; but they should be dispensed with if possible, by mere habit of performing the operations.

Rule 1. All roots may be treated as powers, that is, fall under the same rules as powers, when the fraction which has the order of the root in its denominator is used as the exponent. Call them *fractional* powers, so that the word power shall mean both power and root.

$$\sqrt[n]{a} \text{ is } a^{\frac{1}{n}} \qquad \sqrt[n]{a^m} \text{ is } a^{\frac{m}{n}}$$

Rule 2. To raise a power to a power, multiply together the exponents for a new exponent.

$$\left(2^{\frac{3}{2}}\right)^{\frac{3}{4}} = 2^{\frac{9}{8}} \qquad \left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = a^{\frac{mp}{nq}}$$

Rule 3. To raise a product, or quotient, or the result of several multiplications and divisions, to any power, raise every multiplier and divisor to the same power.

$$\left(\frac{a\,b}{c\,d}\right)^n = \frac{a^n\,b^n}{c^n\,d^n}$$

Rule 4. When several powers are raised successively, it is indifferent in what order the operations are performed.

$$\left((a^p)^q\right)^r = \left((a^q)^r\right)^p = a^{qrp}$$

Rule 5. To multiply together two powers of the same quantity, add the exponents for a new exponent.

$$a^m \times a^n = a^{m+n}$$

Rule 6. To divide one power of a quantity by another power of the same, take the difference of the exponents for a new exponent, and place the result in the numerator or denominator, according as the dividend or divisor has the greater exponent.

m greater than n $\quad \dfrac{a^m}{a^n} = a^{m-n}$

m less than n $\quad \dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$

IV. *Various Combinations of the preceding Propositions applied to the Use of the Square-Root.*

Rule 1. To square the sum of two quantities, square each of them, and to the sum of the squares *add* twice the product of the quantities. To square the difference of two quantities, *subtract* twice the product, instead of *adding*.

$$(a+b)^2 = a^2 + b^2 + 2\,a\,b \qquad (a-b)^2 = a^2 + b^2 - 2\,a\,b$$

$$(6+4)^2 = 36 + 16 + 2 \times 24 \qquad (6-4)^2 = 36 + 16 - 2 \times 24$$

$$(6+\sqrt{3})^2 = 6 + 3 + 12\sqrt{3} = 9 + 12\sqrt{3}$$

$$(6-\sqrt{3})^2 = 6 + 3 - 12\sqrt{3} = 9 - 12\sqrt{3}$$

$$(\sqrt{7}+\sqrt{3})^2 = 10 + 2\sqrt{21} \qquad \begin{cases} (\sqrt{14}-\sqrt{2})^2 = 16 - 2\sqrt{28} \\ \qquad\qquad\quad\;\; = 16 - 4\sqrt{7} \end{cases}$$

$$(\sqrt{12}+\sqrt{10})^2 = 22 + 2\sqrt{120}$$

$$\left(\frac{3}{2}+\sqrt{2}\right)^2 = \frac{17}{4} + 3\sqrt{2} \qquad \left(2\tfrac{1}{2}-\sqrt{3}\right)^2 = \frac{37}{4} - 5\sqrt{3}$$

$$\left(\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{7}}\right)^2 = \frac{1}{2}+\frac{2}{7} - 2\sqrt{\frac{1}{2}\times\frac{2}{7}} = \frac{11}{14} - \frac{2}{7}\sqrt{7}$$

$$\left(\sqrt{\frac{3}{8}}-\sqrt{\frac{5}{12}}\right)^2 = \frac{3}{8}+\frac{5}{12} - 2\sqrt{\frac{3}{8}\times\frac{5}{12}} = \frac{19}{24} - \frac{1}{4}\sqrt{10}$$

$(\sqrt{2\tfrac{1}{2}}+\sqrt{3\tfrac{1}{2}})^2 = 6+\sqrt{35}$ $\quad$ $(\sqrt{7\tfrac{1}{2}}-\sqrt{1\tfrac{1}{4}})^2 = 8\tfrac{3}{4}-\tfrac{1}{2}\sqrt{150}$

$(2\sqrt{2}+3\sqrt{7})^2 = 71+12\sqrt{14}$ $\quad$ $(3\sqrt{2}-2\sqrt{3})^2 = 30-12\sqrt{6}$

$\left(\frac{1}{2}\sqrt{3}-1\right)^2 = \frac{7}{4}-\sqrt{3}$ $\quad$ $\left(\frac{1}{3}\sqrt{2}+\frac{1}{2}\sqrt{3}\right)^2 = \frac{35}{36}+\frac{1}{3}\sqrt{6}$

$\left(\frac{3}{7}\sqrt{5}-\frac{1}{7}\sqrt{10}\right)^2 = \frac{55}{49}-\frac{6}{49}\sqrt{50}$ $\quad$ $\left(\frac{2}{3}\sqrt{\frac{1}{2}}-\frac{1}{4}\sqrt{\frac{1}{3}}\right)^2 = \frac{35}{144}-\frac{1}{18}\sqrt{6}$

$(1{\cdot}1-\sqrt{{\cdot}1})^2 = 1{\cdot}31-2{\cdot}2\sqrt{{\cdot}1}$ $\quad$ $(\sqrt{{\cdot}6}-\sqrt{{\cdot}7})^2 = 1{\cdot}3-2\sqrt{{\cdot}42}$

Rule 2. The product of the sum, and difference of two quantities, is the difference of their squares:

$$(a+b)\times(a-b) = a^2-b^2$$
$$(6+4)\times(6-4) = 36-16$$

Factors given.		Product.
$\sqrt{5}+\sqrt{3}$	$\sqrt{5}-\sqrt{3}$	2
$\frac{1}{2}\sqrt{\frac{1}{7}}+\frac{1}{10}$	$\frac{1}{2}\sqrt{\frac{1}{7}}-\frac{1}{10}$	$\frac{9}{350}$
$6+\sqrt{\frac{1}{2}}$	$6-\sqrt{\frac{1}{2}}$	$35\frac{3}{4}$
$\sqrt{\frac{8}{3}}+\sqrt{\frac{3}{8}}$	$\sqrt{\frac{8}{3}}-\sqrt{\frac{3}{8}}$	$\frac{55}{24}$
$\sqrt{\frac{1}{2}}+\frac{1}{2}$	$\sqrt{\frac{1}{2}}-\frac{1}{2}$	$\frac{1}{4}$
$\sqrt{10}+3$	$\sqrt{10}-3$	1

Section 4. *Miscellaneous Questions involving the Use of Fractions.*

1. If $\frac{2}{3}$ of a shilling buy $\frac{1}{4}$ of a gallon, how much will $\frac{3}{5}$ of a shilling buy?

If $\frac{2}{3}s.$ buy $\frac{1}{4}$ gall.
Then $2s.$ buy $\frac{3}{4}$ gall.
$1s.$ buys $\frac{3}{8}$ gall.
$3s.$ buy $\frac{9}{8}$ gall.
$\frac{3}{5}s.$ buys $\frac{9}{40}$ gall.

2. If $\frac{3}{17}$ of a pound sterling is paid for $\frac{2}{15}$ of a yard, how much must be paid for $3\frac{1}{4}$?

If $\frac{2}{15}$ yd. cost £$\frac{3}{17}$
2 yds. „ £$\frac{45}{17}$
1 yd. costs £$\frac{45}{34}$
13 yds. cost £$\frac{585}{34}$
$\frac{13}{4}$ yd. „ £$\frac{585}{136}$

3. If £$2\frac{1}{2}$ buy $3\frac{1}{3}$ gallons, how much will £$4\frac{1}{7}$ buy? *Answer*, $5\frac{11}{21}$.

4. If $3\frac{1}{5}$ acres let for £$10\frac{1}{4}$, how much will $11\frac{1}{3}$ acres let for? *Answer*, £$36\frac{29}{96}$.

5. If £$\frac{2}{3}$ be worth $\frac{1}{5}$ of a sheep, and $\frac{3}{7}$ of a sheep be worth $\frac{1}{14}$ of an ox, how much must be given for 100 oxen? *Answer*, £2000.

6. If 12 oxen be worth 29 sheep, 15 sheep worth 25 hogs, 17 hogs worth 3 loads of wheat, and 8 loads of wheat worth 13 loads of barley; how many loads of barley must be given for 20 oxen? *Answer*, 23 $\frac{41}{408}$ loads.

7. If 12 of A count for 13 of B, 6 of B for 18 of C, and 13 of C for 2 of D; how many of A count for 100 of D? *Answer*, 200.

8. A. is indebted $\frac{1}{14}$ of his whole property, and loses $\frac{7}{8}$ of it. He recovers as much as amounts to adding $\frac{1}{5}$ to what he then has, and afterwards loses $\frac{1}{2}$ of what he has got. Can he then pay his debts? *Answer*, Yes; after which $\frac{1}{280}$ of his original property will remain to him.

9. A. gains 3 per cent. (3 parts out of a hundred) on what he already has, and B. 7 per cent. But A. gains £100 less than B., and they started with the same sums. What were those sums? *Answer*, £2500 each.

10. There is a number to which 3 is added, and $\frac{1}{10}$ of the result taken. To this 5 is added, and $\frac{1}{15}$ of the result taken. The produce is then $1\frac{1}{4}$. What was the number? *Answer*, 172.

11. A woman bought 150 apples at three a-penny, and 100 at two a-penny, and found she neither lost nor gained by selling the whole lot at five for twopence. But on doing the same with a couple of other lots of 150 apples each, she found she was a loser. What was the reason of this?

12. How much per cent. is £62 of £75: that is, how many times does $\frac{62}{75}$ contain $\frac{1}{100}$? Answer, $82\frac{2}{3}$ per cent.

13. What decimal fraction of a pound is one farthing? Ans., ·001041666....

14. How many pounds are there in a hundred million of farthings? (See last question.)

15. What fraction is one pound avoirdupoise of a hundred weight, one day of a year, and one second of a day? Answer, ·00892857, ·002739726, ·000011574, nearly.

16. A cistern $\frac{2}{3}$ full has two cocks, which alone would empty *the whole cistern* in 7 and 5 minutes. How soon will they empty it together? (Show that this amounts to asking how often $\frac{1}{7}+\frac{1}{5}$ is contained in $\frac{2}{3}$.) Answer, in $1\frac{17}{18}$ minutes.

17. The tenth part of a number is increased by 1, the tenth part of the result by 1, and so on in succession five times, the result of which is 6·79829. What was the number? Answer, 568719.

18. Show that if any number be treated in the preceding manner a sufficient number of times, the result may be brought as near to $\frac{10}{9}$ as we please.

19. What is the reason of the following series of equations, the law of which will be immediately perceived:

$$1+\frac{1}{2}+\frac{1}{4}=\frac{4-1}{4} \qquad 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{8-1}{8}$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{16-1}{16} \text{ \&c.}$$

20. From $\sqrt{3} = 1\cdot7320508$ deduce $\frac{1}{\sqrt{3}} = \cdot5773503$ in the most simple manner; and also

$$\frac{1}{2+\sqrt{3}} = \cdot2679492,$$

without dividing by any decimal fraction.

SECTION 5.—*Useful approximative Rules applicable to cases which frequently occur.*

1. To find how much a certain sum per day amounts to in a year, and the converse.

Rule 1. To the number of pence per day add its half; call the result pounds, and this is the amount in 360 days; add a shilling for every penny in the half-day's allowance just found, and this is the result for a year and a day, or for leap-year. For a common year, diminish the above by one day's allowance.

How much does 3*s.* 4*d.* a-day amount to in a year?

	40 pence			
	20 pence, its half			
Sum	60 pence			
	£60	0	0	
	1			20 shillings
	£61	0	0	in 366 days.

The correct amount is 3*s.* 4*d.* less than this.

How much does $7\frac{1}{2}d.$ a-day amount to in a year?

	$3\frac{3}{4}$ its half			
Sum	$11\frac{1}{4}$ pence			
	£11	5	0	$11\frac{1}{4}$ pounds
		3	9	$3\frac{3}{4}$ shillings
	£11	8	9	for 366 days
			$7\frac{1}{2}$	
	£11	8	$1\frac{1}{2}$	for 365 days.

The undermentioned Sum, per Day,	is the undermentioned, per Year (365 Days).
6*s.* 4*d.*	£115 11 8
2*s.* $3\frac{1}{2}d.$	£ 41 16 $5\frac{1}{2}$
$9\frac{3}{4}d.$	£ 14 16 $6\frac{3}{4}$

Rule 2. Take the nearest pound to the year's allowance, subtract one-third of itself from it, and let the result be pence. If more exactness be required, subtract a penny for every six shillings. The result is within a penny of the sum per day.

How much per day is £100 a-year?

100	
$33\frac{2}{3}$	$\frac{1}{3}$ of 100
$66\frac{1}{3}$	pence is
5*s.* 6*d.* nearly;	

therefore 5*s.* 5*d.* is nearly the answer.

Per Year.	Per Day, about
£357	19*s.* 7*d.*
£ 27	1*s.* 6*d.*
£493	£1. 7*s.* 1*d.*

Rule 3. A number of shillings per *week* taken twice, and a half, and a tenth, is the number of pounds per year.

Thus, 16 shillings a week is $2 \times 16 + \frac{1}{2}$ of $16 + \frac{1}{10}$ of 16, or $32 + 8 + 1{\cdot}6$, or $41{\cdot}6$ pounds per year, or £$41\frac{6}{10}$, or £41. 12*s*.

Rule 4. A number of pounds per year is very nearly* one-third and one-twentieth in shillings per week.

Thus, £60 a-year is $\frac{1}{3}$ of 60*s*. + $\frac{1}{20}$ of 60*s*., or 23*s*. a-week (exactly 23*s*. 0*d*. $\frac{12}{13}$). Similarly, £37 a-year is $\frac{1}{3}$ of 37*s*. + $\frac{1}{20}$ 37*s*., or 12*s*. 4*d*. + 1*s*. 10*d*., or 14*s*. 2*d*. per week.

II.—*To reduce Shillings, &c., to the Decimal of £1, and the converse.*

Rule 1. Annex two ciphers to the shillings, and halve the result. Turn the pence and farthings into farthings, adding one if that gives 24 or upwards. Add and make three decimal places.

What decimals of £1 are 15*s*. 9¾*d*., 6¼*d*., and 1*s*. 2¼*d*.?

```
2)1500                     0s. 6¼                  1s. 2¼d.
  750                      27  6¼ × 4 + 1          5)100
   40  9¾ × 4 + 1        ·027  Answer                 50
  790                                                  9 2¼ × 4
 ·790  Answer                                         59
                                                    ·059  Answer.
```

4*s*. 11¼*d*.	is	£ ·247
5*s*. 11½*d*.	is	£ ·297
1*s*. 0*d*.	is	£ ·050
2*s*. 0*d*.	is	£ ·100
16*s*. 7¼*d*.	is	£ ·830
17*s*. 4½*d*.	is	£ ·868
£15 7 6¼	is	£15·376

Rule 2. Given a decimal of a pound: take the three first places, double the first figure, and add one, if the second be 5 or upwards, for the shillings; take the second and third places, throwing out 5 from the second, if that be 5 or upwards, and 1 from the third, if the result of the last give 25 or upwards. This is the number of farthings, which must be turned into pence and farthings.

What shillings, pence, and farthings are there in £·177?

```
First figure × 2                  = 2
Add 1 for 5 in second figure      = 1
                                    3 shillings
Second and third figures,
  with 5 struck out from
  the second                       27
Take away 1, this being up-
  wards of 24                       1
                                   26 farthings
                                 = 6½d.
                          3s. 6½d. Answer.
```

£ ·019	is	4¾*d*.
£ ·076	is	1*s*. 6¼*d*.
£ ·342	is	6*s*. 10¼*d*.
£ ·969	is	19*s*. 4¾*d*.
£1·118	is	£1. 2*s*. 4½*d*.

* Within less than a penny in a pound.

The results of the last two rules are approximations which are sufficient for common purposes. The student should repeat them until he can solve both cases mentally. They give immediately the price of 10 things within threepence, of 100 within about two shillings, and of 1000 within a pound, when the price of one is known. For example, if one thing cost £2. 14*s.* $4\frac{1}{2}$*d.*, or £2·718, ten cost £27·18, or £27·180, or £27. 3*s.* 7*d* (within a penny or two), one hundred cost 271·8., or £271*l.* 16*s.* (within a few shillings), and 1000 cost £2718. (within a pound). The correct answers to the preceding cases are 27*l.* 3*s.* 9*d.*, 271*l.* 17*s.* 6*d.* and 2718*l.* 15*s.* *Observe, that in the case of shillings and sixpences, without odd pence and farthings, these rules are exact.*

What is the interest on 157*l.* 17*s.* 6*d.* for one year, at 5 per cent.?

$$\begin{array}{r} 157\cdot875 \\ 5 \\ \hline 100)789\cdot375 \\ \hline 7\cdot89375 \end{array} \quad £7.\ 17s.\ 11d.$$

III.—*To reduce Miles per Hour to Feet per Second, and the converse.*

Rule 1. Half as much again as the number of miles per hour is, with sufficient exactness for common purposes, the number of feet per second. To be perfectly exact use the following:

$$\text{Number of miles} + \tfrac{1}{2}\text{ the number} - \tfrac{1}{30}\text{ the number.}$$

Thus 6 miles an hour is $6 + \frac{1}{2}$ of $6 - \frac{1}{30}$ of 6, or $8\frac{4}{5}$ feet per second.

Rule 2. Add half of the feet per second to its fifth (and if perfect accuracy be necessary, subtract one eleventh of the last); the result is the number of miles per hour. Thus 22 feet per second gives $\frac{1}{2}$ of $22 + \frac{1}{5}$ of 22, or $11 + 4\cdot4$, or 15·4 miles per hour nearly; $15\cdot4 - \frac{1}{11}$ of 4·4, or 15 miles exactly.

The preceding rules have been given because they frequently apply in practice. In no other case is it worth while to learn a special rule. But in every sort of occupation which has any reference to arithmetic, the necessity for multiplying or dividing by some particular decimal fraction will frequently occur. A calculator who does not meet with any one particular fraction oftener than another, will not need to take any other than common rules, since the trouble of learning and recollecting a particular rule will more than counterbalance its convenience, in the few instances in which he will have need to apply it; but where one particular fraction occurs frequently, the following hints may be useful.

1. The labour of calculation will be saved, and the chance of error almost destroyed, by a table, which may be more or less extensive according to circumstances. For example, a reader of French works of geography, travels, architecture, &c., will continually be obliged to convert metres into feet, and the converse: he should, therefore, make on a card such a table* as the following:—

Metres.	Feet.	Metres.	Feet.	Metres.	Feet.
1	3·2809	10	32·809	100	328·09
2	6·5618	20	65·618	200	656·18
3	9·8427	30	98·427	300	984·27
4	13·1236	40	131·236	400	1312·36
5	16·4045	50	164·045	500	1640·45
6	19·6854	60	196·854	600	1968·54
7	22·9663	70	229·663	700	2296·63
8	26·2472	80	262·472	800	2624·72
9	29·5281	90	295·281	900	2952·81

* In forming such tables, avoid, as much as possible, the necessity of altering what is taken from the table. An expert calculator needs only the first column; but of these there are not many.

This table is calculated from the following:

1 metre is 3·2809 feet,

and its use is as follows:—For example, what is 867·41 metres?

800 metres are	2624·72	feet.
60 - - -	196·85	„
7 - - -	22·97	„
·4 - - -	1·31	„
·01 - - -	·03	„
867·41 - -	2845·88	

2. For a less exact method, to be used when tables are not at hand, or when a great degree of correctness is not required, lay down the number of decimal places which are to be retained, and endeavour to separate these places into simpler fractions, somewhat in the manner followed in the rule of *Practice* in commercial arithmetic. For instance, in the preceding case, suppose that the metre is 3·281 feet, the error of which is less than one ten-thousandth part of a foot, that is, giving in the multiplication an error of less than one foot in ten thousand. The preceding is $3·25 + ·03 + ·001$, which gives the following rule:—To turn A metres into feet, take three times A, the hundredth part of this, and the quarter and thousandth of A, and add the results together. For instance, what number of feet are in 867·41 metres?

$$A = 867·41$$

$$3A = 2602·23$$

$$\tfrac{3}{100}A = 26·02 \text{ nearly enough}$$

$$\tfrac{1}{4}A = 216·85 \quad - \quad - \quad -$$

$$\tfrac{1}{1000}A = ·87 \quad - \quad - \quad -$$

2845·97 nearly as before.

The student may employ himself in endeavouring to simplify other cases. All must depend on his expertness in separating the fractions.

3. Look for such simplifications as may be made by making the multiplier the sum or difference of two numbers or fractions. Thus a degree is $69\frac{1}{2}$ statute miles, or thereabouts. To turn degrees into miles, multiply the degrees by 70 and subtract one-half their number, instead of multiplying by 69 and adding one-half.

4. Multiplication by a number which often comes into use may be more safely done by division. Take the preceding instance of multiplication by 3·2809. Now,

$$3·2809 = \frac{1}{·3048} = \frac{10000}{3048}$$

very nearly. Hence, any one who has often occasion to turn metres into feet, should keep by him the following table of multiples of 3048.

1	3048	4	12192	7	21336
2	6096	5	15240	8	24384
3	9144	6	18288	9	27432

Hence the rule is, multiply by 10,000 and divide by 3048; which latter part, with the assistance of the table, is nothing but inspection and subtraction, as follows:—What is 867·41 metres in feet?

$$867·41 \times 10000 = 8674100$$

```
3048 ) 8674100 ( 2845·83
       6096
       ----
       25781
       24384
       -----
        13970
        12192
        -----
         17780
         15240
         -----
          25400
          24384
          -----
          10160
```

The advantage of this method is, that with the table it is less liable to error than multiplication, and the figures of the result which are most wanted are first found.

We shall proceed in the next treatise to the use of Logarithms.

SECTION 6. *Meaning of Logarithms. Rules. Arrangement of Tables in common use. Method of taking out Logarithms, and Numbers to Logarithms.*

In the preceding treatise (page 23) we have said that if $a^b = c$, then b, which is the *exponent* of a, is called the *logarithm* of c or of a^b, to the base a. Thus $10^3=1000$, whence 3, the exponent of 10, is called the logarithm of 1000 to the base 10. Hence it follows that 3 may be the logarithm of all sorts of numbers, according to the base chosen. Thus:

$$\begin{array}{lll} 2^3 = 8 & 3 = & \text{log. } 8 \text{ (base 2.)} \\ 3^3 = 27 & 3 = & \text{log. } 27 \text{ (base 3.)} \\ 4^3 = 64 & 3 = & \text{log. } 64 \text{ (base 4.)} \\ \&\text{c.} & \&\text{c.} & \&\text{c.} \end{array}$$

But as, in the practice of logarithms, no other base is used, except only 10, we shall, in this treatise, suppose no other base; and logarithms to this base are called *common* logarithms, *tabular* logarithms, or *Brigg's* logarithms. And because we have nothing to do with the method of constructing logarithms, but only with the use to be made of them when they have been found, we shall refer to works on algebra for the former part of the subject, and proceed to the latter, after we have stated in what the difficulty of finding them consists.

According to the common language of algebra, if we raise the mth power of 10, and extract the nth root of the result, we have what is called the $\frac{m}{n}$th power of 10, or

$$\sqrt[n]{10^m} = 10^{\frac{m}{n}}$$

We shall now simply write down some results, not expecting the student to verify them; because, though that might possibly be done by ordinary arithmetic, yet the process would be of very great length and trouble:

$$\sqrt[10]{10^3} \quad \text{or } 10^{\frac{3}{10}} \quad \text{or } 10^{\cdot 3} \quad = 1\cdot 9952623150 \text{ nearly.}$$

$$\sqrt[1000]{10^{301}} \quad \text{or } 10^{\frac{301}{1000}} \quad \text{or } 10^{\cdot 301} \quad = 1\cdot 9998618696 \text{ nearly.}$$

$$\sqrt[100000]{10^{30103}} \quad \text{or } 10^{\frac{30103}{100000}} \quad \text{or } 10^{\cdot 30103} \quad = 2\cdot 0000000200 \text{ nearly.}$$

So that we may get a result as near to 2 as we please; that is, we may find a decimal fraction x, which shall, as nearly as we please, satisfy the equation $10^x = 2$. The answer is $x = \cdot 30103$ nearly. And in the same way we may find an *approximate* logarithm for any other number or fraction. These approximate logarithms are arranged in tables, with certain modifications derived from the following fundamental rules, which are proved* in works on the subject.

1. *The logarithm of a product must be the sum of the logarithms of the factors.* Thus, 6, 8, and 10, *multiplied* together, give 480; the logarithms of 6, 8, and 10, *added* together, give the logarithm of 480. The following instances may be immediately verified from any tables:

$$\begin{array}{ll} 2 \times 5 = 10 & \\ \text{Log. } 2 + \text{Log. } 5 = \text{Log. } 10 & \\ \text{Log. } 2 = & \cdot 3010300 \\ \text{Log. } 5 = & \underline{\cdot 6989700} \\ \text{Log. } 10 = & 1\cdot 0000000 \end{array} \qquad \begin{array}{ll} 4 \times 7 = 28 & \\ \text{Log. } 4 + \text{Log. } 7 = \text{Log. } 28 & \\ \text{Log. } 4 = & \cdot 6020600 \\ \text{Log. } 7 = & \underline{.8450980} \\ \text{Log. } 28 = & 1\cdot 4471580 \end{array}$$

2. *To find the logarithm of a quotient, subtract the logarithm of the divisor from the logarithm of the dividend.* Thus, 20 divided by 5 gives 4; the logarithm of 20, diminished by the logarithm of 5, is the logarithm of 4:

$$\begin{array}{ll} 100 \div 40 = 2\cdot 5 & \\ \text{Log. } 100 = & 2\cdot 0000000 \\ \text{Log. } 40 = & \underline{1\cdot 6020600} \\ \text{Log. } 2\cdot 5 = & 0\cdot 3979100 \end{array} \qquad \begin{array}{ll} 64 \div 16 = 4 & \\ \text{Log. } 64 = & 1\cdot 8061800 \\ \text{Log. } 16 = & \underline{1\cdot 2041200} \\ \text{Log. } 4 = & 0\cdot 6020600 \end{array}$$

3. *The logarithm of a power, root, or combination of power and root, which is*

* The reader must recollect throughout, that we here lay down rules only, not demonstrations.

denoted in algebra by a fractional exponent, is found by multiplying the logarithm of the number given by the exponent in question. The following equations will set this in a clearer light:

$$\text{Log. } aa \text{ or Log. } a^2 = 2 \text{ Log. } a.$$
$$\text{Log. } aaa \text{ or Log. } a^3 = 3 \text{ Log. } a.$$
$$\text{Log. } \sqrt{a} \text{ or Log. } a^{\frac{1}{2}} = \tfrac{1}{2} \text{ Log. } a.$$
$$\text{Log. } \sqrt[3]{a} \text{ or Log. } a^{\frac{1}{3}} = \tfrac{1}{3} \text{ Log. } a.$$
$$\text{Log. } \sqrt[4]{a^3} \text{ or Log. } a^{\frac{3}{4}} = \tfrac{3}{4} \text{ Log. } a.$$
$$\text{Log. } \sqrt[n]{a^m} \text{ or Log. } a^{\frac{m}{n}} = \tfrac{m}{n} \text{ Log. } a.$$

What is the logarithm of the square root of 156?

Log. 156 = 2·1931246
$\frac{1}{2}$ Log. 156 = 1·0965623 *Ans.*

What is the logarithm of the fifth root of the fourth power of 2097?

$$\begin{array}{rr} \text{Log. } 2097 = & 3\cdot3215984 \\ & 4 \\ \hline 5) & 13\cdot2863936 \\ \hline & 2\cdot6572787 \quad \textit{Ans.} \end{array}$$

4. *The logarithm of* 1 *is* 0; *that of the base* (which is here 10) *is* 1; *that of the square of the base* (here 100) *is* 2; *that of the cube of the base* (here 1000) *is* 3; and so on: or

Log. 1 = 0	Log. 1000 = 3
Log. 10 = 1	Log. 10000 = 4
Log. 100 = 2	Log. 100000 = 5
	&c.

5. *As the number increases the logarithm increases, and the greater the number the greater the logarithm: but the rate at which the logarithm increases is perpetually diminishing as the number increases.* Thus we see that, as the number passes from 10,000 to 100,000 (through *ninety thousand* units) the logarithm passes from 4 to 5, receiving no greater increase than takes place while the number passes from 1 to 10 (through *nine* units only).

6. *In any logarithm* (4·6183 for instance) *the whole number* (4) *is called the* characteristic, *and the remainder* (·6183) *the* decimal part *of the logarithm.*

7. *In any number* (368·414 for instance) *the figures which precede the decimal point* (the 3, the 6, and the 8,) *are called* integers, *and those which follow the point are called* decimals. *And figures, when opposed to ciphers, are called* significant. Thus, in 864000, 4 is the last significant figure; in ·000193, 1 is the first significant figure.

8. A fraction less than unity (·5 for instance) has none but a *negative* logarithm: but that students may use logarithms who have not studied algebra, we affix a meaning to the term negative, for this subject only. The term multiplication is extended in arithmetic to whole numbers and fractions, so that multiplication, in its extended meaning, includes the first meaning of division: thus, to *multiply* by $\frac{1}{10}$ is to *divide* by 10. But from the connection which exists between *multiplication* of numbers and *addition* of logarithms, and also between *division* of numbers and *subtraction* of logarithms, we cannot use the word *multiplication* in an extended sense, which includes *division, and keep rules* (1) *and* (2) *at the same time*,* unless we also use the word *addition* in an extended sense, which includes *subtraction*. And this is done as follows: by $\bar{1}$ we mean a unit, with a warning, that in all operations performed upon this $\bar{1}$, we are to subtract where we should have added if the bar had been absent, and to add where we should have subtracted. And with this we say, that 1 being the logarithm of 10, $\bar{1}$ is the logarithm of $\frac{1}{10}$; 2 being the logarithm of 100, $\bar{2}$ is the logarithm of $\frac{1}{100}$: the following are instances of the use of this sign, with the corresponding real operations:—

Multiply	*Divide*
1000 by $\frac{1}{10}$	1000 by 10
Log. 1000 = 3	Log. 1000 = 3
Log. $\frac{1}{10}$ = $\bar{1}$	Log. 10 = 1
Add $\bar{2}$	*Subtract* $\bar{2}$

And 2 is log. 100

$1000 \times \frac{1}{10} = 100, \quad 1000 \div 10 = 100$

* The choice is, between making two rules, and using the words of one rule in a sense which will make that one include both. The latter is the more difficult at first, but the more convenient in the end.

Divide	*Multiply*
1000 by $\frac{1}{100}$	1000 by 100
Log. 1000 = 3	Log. 1000 = 3
Log. $\frac{1}{100} = \bar{2}$	Log. 100 = 2
Subtract 5	*Add* 5

And 5 is log. 100,000.

$$1000 \div \frac{1}{100} = 100{,}000 \qquad 1000 \times 100 = 100{,}000$$

When a subtraction appears which is impossible, invert the subtraction, and place the bar over the result. The following are instances, with the corresponding operations, the first line of each set containing logarithms, and the second the numbers and operations corresponding:—

$$\left\{\begin{array}{l} 3 - 5 = \bar{2} \\ 1000 \div 100{,}000 = \frac{1}{100} \end{array}\right.$$

$$\left\{\begin{array}{l} 2 - 3 = \bar{1} \\ 100 \div 1000 = \frac{1}{10} \end{array}\right.$$

$$\left\{\begin{array}{l} 0 - 3 = \bar{3} \\ 1 \div 1000 = \frac{1}{1000} \end{array}\right.$$

In all other cases, combinations of the preceding rules may be used: and it must be considered that $\bar{1}$ and $\bar{1}$ added make $\bar{2}$, and so on: the following instances will contain all the cases:—

$$\left\{\begin{array}{l} \bar{1} + \bar{1} = \bar{2} \\ \frac{1}{10} \times \frac{1}{10} = \frac{1}{100} \end{array}\right.$$

$$\left\{\begin{array}{l} \bar{2} + \bar{3} = \bar{5} \\ \frac{1}{100} \times \frac{1}{1000} = \frac{1}{100{,}000} \end{array}\right.$$

$$\left\{\begin{array}{l} \bar{2} - \bar{3} \text{ or } \bar{2} + 3 \\ \text{or } 3 + \bar{2} \text{ or } 3 - 2 = 1 \\ \frac{1}{100} \div \frac{1}{1000} = 10 \end{array}\right.$$

$$\left\{\begin{array}{l} \bar{2} - 2 \text{ or } \bar{2} + \bar{2} = \bar{4} \\ \frac{1}{100} \div 100 = \frac{1}{10{,}000} \end{array}\right.$$

$$\left\{\begin{array}{l} 1 + \bar{2} + \bar{3} = 1 - 2 - 3 = \bar{4} \\ 10 \times \frac{1}{100} \times \frac{1}{1000} = \frac{1}{10{,}000} \end{array}\right.$$

$$\left\{\begin{array}{l} \bar{4} - \bar{4} = 0 \\ \frac{1}{10{,}000} \div \frac{1}{10{,}000} = 1 \end{array}\right.$$

$$\left\{\begin{array}{l} 0 - \bar{3} = 3 \\ 1 \div \frac{1}{1000} = 1000 \end{array}\right.$$

What are the results of the following, and what are the corresponding operations in the numbers to which the terms are logarithms?

$$4 + \bar{3} - \bar{2} - \bar{1}$$

$$\bar{2} - \bar{3} + 5 - 1$$

What is the logarithm of ·5?

Log. 5 = ·69897
Log. 10 = 1·00000

Subtract. Impossible, therefore invert the subtraction, and place a bar over the whole; as follows,

$\overline{\cdot 30103}$

What is 20 × ·5?

Log. 20 = 1·30103
Log. ·5 = $\overline{\cdot 30103}$
Add 1·00000 *Ans.* 10.

What is 100 ÷ ·5?

Log. 100 = 2·00000
Log. ·5 = $\overline{\cdot 30103}$
Subtract 2·30103 which is log. 200 } *Ans.* 200.

But the necessity of using decimal places with a negative sign, can always be avoided, and the characteristic only made negative, as follows: for

log ·5 or log. $\frac{5}{10}$ or log. 5 − log. 10 or

·69897 − 1 write ·69897 + $\bar{1}$

or $\bar{1}$·69897;

in which the first figure only is to be used as a negative quantity. We repeat the preceding instances.

What is 20 × ·5?

Log. 20 = 1·30103
Log. ·5 = $\bar{1}$·69897
Add 1·*0000 *Ans.* 10.

Here the 1, which is carried after adding 1, 6, and 3, (where we have placed an asterisk instead of a cipher to mark the place) instead of increasing the $\bar{1}$, destroys it.

What is 100 ÷ ·5?

Log. 100 = 2·00000
Log. ·5 = $\bar{1}$·69897
Subtract 2·30103 as before.

To make any logarithm which is entirely negative, negative in the characteristic only, make that characteristic greater by 1, and subtract the decimal part from 1.

What is $\overline{40\cdot41372}$?

$$1 - \cdot41372 = \cdot58628$$

Answer $\overline{41}\cdot58628$

$$\overline{0\cdot3} = \bar{1}\cdot7 \quad \overline{1\cdot21} = \bar{2}\cdot79 \quad \overline{0\cdot1} = \bar{1}\cdot9$$

$$\overline{1\cdot6141982} = \bar{2}\cdot3858018$$

In the practice of logarithms, it will be necessary to appear to subtract the greater from the less, which is done by subtracting in the usual way till we come to the last place, inverting the subtraction which there occurs, and placing the negative sign over the result.

$$\begin{array}{lll} \text{From} & 1\cdot6936 & 20\cdot414 \\ \text{Take} & 3\cdot0177 & 29\cdot666 \\ \hline \textit{Ans.} & \bar{2}\cdot6759 & \overline{10}\cdot748 \end{array}$$

$$\begin{array}{cccc} 6\cdot4 & 12\cdot8 & 6\cdot6 & 0\cdot00 \\ 7\cdot9 & 14\cdot4 & 6\cdot7 & 4\cdot28 \\ \hline \bar{2}\cdot5 & \bar{2}\cdot4 & \bar{1}\cdot9 & \bar{5}\cdot72 \end{array}$$

$$\begin{array}{cc} 0\cdot0000 & 0\cdot00000 \\ 2\cdot1896 & 0\cdot12345 \\ \hline \bar{3}\cdot8104 & \bar{1}\cdot87655 \end{array}$$

In the following examples the negative characteristic is treated in the manner already described, namely, as to be subtracted in addition, and added in subtraction. The figure *carried* is always to be added, and, therefore, makes a negative characteristic less: thus 2 carried to $\bar{5}$ makes it $\bar{3}$.

$$\begin{array}{ccc} \text{Add.} & \text{Add.} & \text{Add.} \\ 1\cdot48 & 9\cdot83 & \bar{2}\cdot18 \\ \bar{2}\cdot56 & \bar{1}\cdot47 & \bar{6}\cdot00 \\ \bar{3}\cdot41 & \bar{4}\cdot66 & 9\cdot14 \\ \hline \bar{3}\cdot45 & 5\cdot96 & 1\cdot32 \end{array}$$

$$\begin{array}{llll} \text{From} & \bar{1}\cdot616 & 8\cdot413 & \bar{4}\cdot17 \\ \text{Take} & 2\cdot929 & \bar{1}\cdot097 & \bar{5}\cdot28 \\ \hline & \bar{4}\cdot687 & 9\cdot316 & 0\cdot89 \end{array}$$

$$\begin{array}{llll} \text{From} & 0\cdot000 & 0\cdot00 & \bar{2}\cdot66 \\ \text{Take} & \bar{2}\cdot147 & \bar{1}\cdot42 & \bar{3}\cdot44 \\ \hline & 1\cdot853 & 0\cdot58 & 1\cdot22 \end{array}$$

As the difficulty lies entirely in the line of characteristics, we give some examples of that line only, the figure carried from the preceding line being written in Roman figures at the top.

$$\begin{array}{lccccc} & \text{I} & \text{I} & \text{II} & \text{IV} & \text{III} \\ \text{Add} & 1 & \bar{2} & \bar{3} & 2 & \bar{1} \\ & \bar{3} & 10 & \bar{8} & 6 & \bar{2} \\ & \bar{7} & \bar{5} & 7 & \bar{9} & \bar{3} \\ & 4 & \bar{8} & 1 & 3 & 4 \\ \hline & \bar{4} & \bar{4} & \bar{1} & 6 & 1 \end{array}$$

$$\begin{array}{lcccc} & \text{O} & \text{I} & \text{I} & \text{I} \\ \text{From} & 2 & 0 & \bar{3} & \bar{1} \\ \text{Take} & \bar{3} & \bar{2} & \bar{4} & 0 \\ \hline & 5 & 1 & 0 & \bar{2} \end{array}$$

To multiply a logarithm with a negative characteristic by a whole number, proceed in all respects as in common multiplication, exeept only in *subtracting*, instead of *adding* the figures which are to be carried, so soon as the characteristic comes to be multiplied.

$$\begin{array}{cccc} \bar{1}\cdot61 & \bar{2}\cdot55 & \bar{4}\cdot1 & \bar{4}\cdot6 \\ 4 & 3 & 8 & 2 \\ \hline \bar{2}\cdot44 & \bar{5}\cdot65 & \overline{32}\cdot8 & \bar{7}\cdot2 \end{array}$$

When the multiplier exceeds 12, and the process is not performed in one line, the better way is to omit the characteristic altogether, at first, and subtract the product arising from it afterwards, as in the following multiplication of $\bar{2}\cdot136$ by 15.

$$\begin{array}{rr} & \cdot136 \\ & 15 \\ \hline & 680 \\ & 136 \\ \hline & 2\cdot040 \\ 4 \times 15 = & 60 \\ \hline \text{Subtract} & \overline{58}\cdot040 \end{array}$$

To divide a logarithm with a negative characteristic by a whole number, begin by *increasing* the characteristic until it is divisible by the whole number, make the quotient a negative characteristic for the result, and use the augment which was found necessary, as if it had been a remainder. Thus, to divide $\bar{1}\cdot4$ by 2, increase the first 1, and make it $\bar{2}$ (necessary augment, 1) and 2 being contained in 2 once, $\bar{1}$ is the characteristic of the quotient. Then, taking the augment 1, prefix it to the 4, giving 14, which contains 2 seven times. Therefore $\bar{1}\cdot7$ is the quotient.

$2)\bar{3}\cdot010$	$4)\bar{1}\cdot11$	$10)\bar{6}\cdot52$
$\bar{2}\cdot505$	$\bar{1}\cdot78$	$\bar{1}\cdot45$
$7)\bar{8}\cdot10$	$3)\bar{9}\cdot12$	$4)\bar{4}\cdot13$
$\bar{2}\cdot87$	$\bar{3}\cdot04$	$\bar{1}\cdot03$

As divisions by higher numbers rarely occur, we shall only give one instance, that the student may exercise himself in reconciling the process as it here appears, with the rule given. The asterisks mark where the process differs from common division.

$$\begin{array}{l} 13)\overline{165}\cdot61(\overline{13}\cdot35... \\ \quad\;\; 13 \\ \quad\;\; \overline{\;\;35} \\ {*}\;\; \underline{\;\;39} \\ {*}\;\;\;\; 46 \\ \quad\;\;\;\; \underline{39} \\ \quad\;\;\;\; 71 \end{array}$$

We can now give a logarithm, by help of the tables, to any number or fraction, and can, by the above conventions, make the rules marked (1) (2) and (3) include all cases of logarithmic operations, by help of the following rules.

(*a*) An alteration in the position of the decimal point, alters only the characteristic, and not the decimal part of the logarithm, if the significant figures remain the same: thus all the following numbers and fractions have the same decimal part in their logarithms, with different characteristics.

·000256	2·56	25600
·00256	25·6	256000
·0256	256	2560000
·256	2560	25600000

(*b*) In every whole number, let a decimal point be understood after the unit's place. Thus 58 is 58·, or 58·0, or 58·00, &c.

(*c*) When there are figures before the decimal point, let the characteristic be *one* less than the number of places of those figures. Thus the logarithm of 26861·5 has the characteristic 4; so also has that of 26861 (or 26861·).

The decimal places of the logarithm of 21925 are ·3409396; hence

Log. 21925000	=	7·3409396
Log. 2192500	=	6·3409396
Log. 219250	=	5·3409396
Log. 21925	=	4·3409396
Log. 2192·5	=	3·3409396
Log. 219·25	=	2·3409396
Log. 21·925	=	1·3409396
Log. 2·1925	=	0·3409396

(*d*) When there are no figures (or only ciphers) before the decimal point, let the characteristic be negative, and let it tell in what place following the decimal point, the first significant figure is found. Thus, in ·0000136, the first significant figure being in the fifth place following the decimal point, the characteristic of the logarithm is $\bar{5}$. The decimal places in the logarithm of 324 being ·510545, we have

Log. ·324	=	$\bar{1}\cdot510545$
Log. ·0324	=	$\bar{2}\cdot510545$
Log. ·00324	=	$\bar{3}\cdot510545$
Log. ·000324	=	$\bar{4}\cdot510545$
Log. ·0000324	=	$\bar{5}\cdot510545$

The decimal places in the logarithm of 1 being 000, &c., we have the following logarithms, which consist entirely of characteristics:

Log. 1000	=	3·0000...
Log. 100	=	2·0000..
Log. 10	=	1·0000
Log. 1	=	0·0000
Log. ·1	=	$\bar{1}\cdot0000$
Log. .01	=	$\bar{2}\cdot0000$
Log. ·001	=	$\bar{3}\cdot0000$ &c.

and these are the only numbers to which logarithms can be exactly found; the decimal places of all others being approximations only.

Tables of logarithms (generally) contain the decimal part of the logarithm, which is evidently all that is necessary, as the characteristic can be found by the preceding rule. Being approximations, they are more or less correct according to the greater or smaller number of places which they give. Modern tables never have fewer than *four*, or more than *seven* decimal places. The following is the rule by which the power of a table of logarithms is to be judged.

The number of places of figures which may be obtained in a result derived from any table of logarithms, is the same as the number of decimals to which the logarithms are carried. But towards the end of the table, the last place thus obtained cannot always be depended upon within a unit.

We shall proceed to the description of the arrangements of several tables, such as are most likely to fall in the reader's way.

I. The tables which run to seven places of decimals are all arranged in one form, of which the following is a specimen.

No.	0	1	2	3	4	5	6	7	8	9
4550	6580114	0209	0305	0400	0496	0591	0687	0782	0877	0973
1	1068	1164	1259	1355	1450	1545	1641	1736	1832	1927
2	2023	2118	2213	2309	2404	2500	2595	2690	2786	2881
--	- - -	--	--	--	--	--	--	--	--	--
9	8696	8791	8886	8982	9077	9172	9267	9363	9458	9553
4560	9648	9744	9839	9934	0029	0125	0220	0315	0410	0506
1	6590601	0696	0791	0886	0982	1077	1172	1267	1362	1458

Diff.	
95	
1	10
2	19
3	29
4	38
5	48
6	57
7	67
8	76
9	86

The first column contains the first four places of the number, and over the head of the page is the fifth place of the number. The first three places of the logarithm (which throughout the speimen are either 658 or 659,) are not repeated with every logarithm, but only inserted at (or as near as may be to) the place where a change of the third figure takes place. But the best way to explain this table will be to destroy arrangement and abbreviation, and begin to write it down at full length. The student must account for every figure of the following out of the specimen. The characteristic need not be inserted, as what we here take out is merely the decimal part of the logarithm.

Log. 45500	·6580114
Log. 45501	·6580209
Log. 45502	·6580305
Log. 45503	·6580400
- - - - -	- - - - -
- - - - -	- - - - -
Log. 45509	·6580973
Log. 45510	·6581068
Log. 45511	·6581164
- - - - -	- - - - -
- - - - -	- - - - -
Log. 45602	·6589839
Log. 45603	·6589934
Log. 45604	·6590029 { change of 3rd fig.
Log. 45605	·6590125

It would break the page to show that 658 becomes 659 in the middle of it; and various methods are used to remind the computer that the change has taken place. In different works, the line 4560, *after the change*, is varied thus:—

	$_{0}029$	$_{0}125$	&c.
or	$\bar{0}029$	$\bar{0}125$	&c.
or	0029	0125	&c.

all serving to remind that the first three places must be looked for *immediately below*, instead of more or less above, the line of the last four.

The column marked Diff. (for *difference*) shows how to find the logarithm of a number of six or seven places of figures. For instance, what is the logarithm of 4551132? Take out the decimal part of log. 45511; to this *add* what comes opposite to the *sixth* place in the column Diff.; (the sixth place is 3, and 29 is opposite to 3 in column Diff.); *add* the nearest number of tens in the number opposite to the seventh place (the seventh place is 2; opposite to 2 in col. Diff. is 19, nearest number of tens, 2 tens) and the result is the decimal part of the logarithm required: thus—

Log. 45511..	·6581164
3.	29
2	2
Log. 4551132	·6581195

$$\text{Log. } 455{\cdot}2008 = 2{\cdot}6582031$$
$$\text{Log. } 45603{\cdot}97 = 4{\cdot}6590027^{*}$$
$$\text{Log. } {\cdot}4560444 = \bar{1}{\cdot}6590071$$

In the earlier part of the tables, where columns of differences occur more thickly, several for the same line of logarithms, it is almost immaterial which is used; but for safety, take that column of differences which is headed by the difference between the logarithm taken out and the next following it.

To find the number corresponding to a given logarithm, look in the table for the decimal places, which are nearest below those of the given logarithm; take out this logarithm, and the *five* places of the number, subtract the logarithm taken out from the given logarithm. Look in the second column of the

* The change in the third figure takes place in the process.

differences for the number next below the result of subtraction just found, opposite to it will be found the *sixth* place of the number. Subtract the number used in the second column of the differences from the result of subtraction above-mentioned, annex a cipher, and repeat the process with the column of differences, taking the nearest this time, whether above or below; the result is the *seventh* place of the number. For instance, what is the number to the logarithm 1·6582554?

	·6582554
Nearest log. below No. 45525	6582500
	54
Opposite to 5 is	48
Subt. and annex 0	60

Opp. to 6 is 57, the nearest.

The first five places are 45525, the sixth is 5, and the seventh is 6, so that 4552556 is the number required; and because 1 is the characteristic, there must be two places before the decimal point; that is, 45·52556 is the answer.

The following useful numbers are mostly taken from the list at the end of Mr. Babbage's logarithms. They will serve as exercises, either in taking the logarithm to a number, or the converse.

	No.	Log.
Circumference of circle (diam. being 1)	3·141593	·4971499
Area of circle (do. do.)	·7853982	$\bar{1}$·8950899
Content of sphere (do. do.)	·5235988	$\bar{1}$·7189986
No. of seconds in 360°	1296000	6·1126050
No. of arcs of 1″ in the radius	206264·8	5·3144251
No. of arcs of 1′ in the radius	3437·747	3·5362739
No. of arcs of 1° in the radius	57·29578	1·7581226
Base of Naperian logarithms	2·718282	·4342945
Modulus of common logarithms	·4342945	$\bar{1}$·6377843
Metres in a toise	1·94904	·2898200
Yards in a toise	2·131531	·3286916
Feet in a toise	6·394593	·8058129
Yards in a metre	1·093633	·0388716
Feet in a metre	3·280899	·5159929
Inches in a metre	39·37079	1·5951741
Feet in a French foot	1·065765	·0276616
Acres in an are (French)	·02471143	$\bar{2}$·3928979
Lbs. troy in a gramme	·00268098	$\bar{3}$·4282928
Lbs. avoir. in a gramme	·00220606	$\bar{3}$·3436173
Cwts. in a kilogramme	·0196969	$\bar{2}$·2943993
Gallons in a litre	·2200969	$\bar{1}$·3426139
Seconds in 24 hours	86400	4·9365137
Diurnal acceleration of stars in mean solar seconds	235·9093	2·3727451
Common tropical year in mean solar days	365·2422	2·5625810
Grains in a cubic inch of water (barom. 30 inch, therm. 62, Fahr.)	252·458	2·4021891
Inches in the pendulum, which vibrates seconds in a vacuum in the latitude of London	39·1393	1·5926130

II. The second set of tables, which it will be worth while to describe, has five places of figures in the logarithms, and four places in the number, with a difference to find a fifth. We have not described logarithms of six places, partly because they are arranged much in the manner of those which have

seven places, and partly because tables of six places are of comparatively little use. For most practical purposes out of astronomy, and for very many of the details of calculation connected with the latter science, five places are amply sufficient: and where five are not sufficient, seven are much more frequently wanted than six; besides which, the arrangement of most tables of six places which we have seen is so defective, that those of seven are, in our opinion, more easily used.

The best tables of five places (though with a very singular and awkward defect, presently to be noticed), are those of Lalande*. The following is a specimen:—

Nomb.	0. 21' 30" Logarit.	D.
1290	3·11059	34
1291	3·11093	33
1292	3·11126	34
1293	3·11160	33
1294	3·11193	34
1295	3·11227	34
1296	3·11261	34
1297	3·11294	33
1298	3·11327	

The defect alluded to is the characteristic, which is inserted as if the logarithms of whole numbers of four places were always required, to the exclusion of all others. Thus, though the characteristic above given is correct for the logarithm of 1292, it is not so for those of 129·2, 12·92, &c. The best way for the student who uses this work, is never to think of the characteristic as anything but an addition to the boundary line; that is, to look upon the numbers as separated from the decimal part of their logarithms by a fanciful boundary, like

|3
|3 instead of simply |
|3

To find the logarithm of any four places, simply look in the table and choose the right characteristic. Thus:

Log. ·1295 = $\bar{1}$·11227

To find the logarithm of ·12956, take the difference which comes next under 11227, namely 34; multiply it by the new figure 6, but instead of writing down the first place, carry the *nearest number of tens* to the next place. Say, 6 times 4 is 24, carry 2; 6 times 3 is 18 and 2 is 20. So that

Log. ·1295· = $\bar{1}$·11227
6   20
Log. ·12956 = $\bar{1}$·11247

Log. ·1295· = $\bar{1}$·11227
7   24
Log. ·12957 = $\bar{1}$·11251

To find the number to a given logarithm, take out of the table the decimal part next below the given decimal part, and the four places opposite to it. Annex a cipher to the difference, and divide by the number in the column of differences, taking the nearest quotient of one figure. That one figure is the fifth figure of the number. For instance, what is the number to the logarithm $\bar{2}$·11178?

$\bar{2}$·11178
1293   ·11160
33)180(5
Ans. ·012935

Given Log.   8·11153
1292   11126
34)270(8
Ans. 129280000 nearly.

We cannot fill up the remaining places out of this table, and must place ciphers instead. The real number to the log. 8·11153 is

129279600·09 *very* nearly.

The numbers given in page 39 may be made exercises; but the nearest five significant figures of the number must be taken, and the nearest five decimal figures of the logarithm will be found.

Example. What is Log. 3·1416

Log. 3·141·   0·49707
6   8
Log. 3·1416   0·49715 *Ans.*

* The title-page of the best edition is as follows:—"*Tables des Logarithmes pour les nombres et pour les sinus. Avec les explications, &c. &c. &c. Edition Stéréotype gravée fondue et imprimée*, par FIRMIN DIDOT. A PARIS, &c. 1805. (*tirage de* 1831)." The last four words should be particularly looked at.

III. There are logarithms of four places on the table given in the Treatise on Arithmetic and Algebra, which are sufficient for many purposes. These tables are arranged somewhat after the manner of those of seven places, with the exception of the column of differences being placed horizontally, with a common heading. A few examples of the method of taking out logarithms will suffice:

$$\begin{array}{lrl} \text{Log. } 16{\cdot}8 \cdot\cdot & = & 1{\cdot}2253 \\ \quad 4 & & \quad 11 \\ \hline \text{Log. } 16{\cdot}84 & & 1{\cdot}2264 \end{array}$$

$$\begin{array}{lrl} \text{Log. } 1{\cdot}69 \cdot\cdot & = & 0{\cdot}2279 \\ \quad 1 & & \quad 3 \\ \hline \text{Log. } 1{\cdot}691 & = & 0{\cdot}2282 \end{array}$$

The number to a logarithm might be found by the reverse process. Thus:

$$\begin{array}{ll} \text{Given Log.} & \bar{1}{\cdot}2687 \\ 185 \cdot\cdot & \underline{2672} \\ \quad 6 \text{ or } 7 & \quad 15 \\ \textit{Ans.} & {\cdot}1856 \text{ or } {\cdot}1857 \end{array}$$

But these tables are accompanied by an anti-logarithmic table, in which the numbers and logarithms change places; so that a number is found from its logarithm by the same process as that which finds the logarithm from the number. For instance, in the preceding example,

$$\begin{array}{ll} 268 \cdot\cdot & 1854 \\ \underline{\quad 7} & \underline{\quad 3} \\ \bar{1}{\cdot}2687 \text{ is Log. of} & {\cdot}1857 \end{array}$$

The table of anti-logarithms is more trustworthy than the inverse process with the table of logarithms.

IV. We subjoin a table of logarithms and anti-logarithms to three places only; partly because there is considerable power in such a table, and partly because it will be a guide to the beginner in consulting larger tables, as he may thus (while new to the subject) find out to what part of the larger table to turn, in either of the operations of taking out logarithms or numbers.

In the following table the differences between successive logarithms are placed in smaller figures in the intermediate space. As these tables are only an abbreviation of those already described, the first gives the log. of two places to three places; the second gives the proportional parts of the differences; the third gives the number of a log. of two places to three places. When the difference is 12 or under, the proportional part can be found at once by the process described in speaking of tables of five places:

Given No. 16·3

$$\begin{array}{lrl} \text{Log. } 16 \cdot\cdot & = & 1{\cdot}204 \text{ D } 26 \\ \quad 3 & & \quad 8 \\ \text{Log. } 16{\cdot}3 & = & \overline{1{\cdot}212} \end{array}$$

Given No. 109

$$\begin{array}{lrl} \text{Log. } 10 \cdot\cdot & = & 1{\cdot}000 \text{ D } 41 \\ \quad 9 & & \quad 37 \\ & & \overline{1{\cdot}037} \end{array}$$

Given Log. $\bar{2}{\cdot}308$

$$\begin{array}{ll} 30 \cdot\cdot & 200 \text{ D } 4 \\ \underline{\quad 8} & \underline{\quad 3} \\ \bar{2}{\cdot}308 & {\cdot}0203 \end{array}$$

Given Log. 1·496

$$\begin{array}{ll} 49 \cdot\cdot & 309 \text{ D } 7 \\ \underline{\quad 6} & \underline{\quad 4} \\ 1{\cdot}496 & 31{\cdot}3 \end{array}$$

V. Finally, we recommend the student to commit to memory the following table of logarithms to two places:

No.	Log.	No.	Log.	No.	Log.
1	00	4	60	7	85
2	30	5	70	8	90
3	48	6	78	9	95

LOGARITHMS.

1	2	3	4	5	6	7	8	9
0 000	0 301	0 477	0 602	0 699	0 778	0 845	0 903	0 954
41	21	14	11	9	7	6	5	5
1 041	1 322	1 491	1 613	1 708	1 785	1 851	1 908	1 959
38	20	14	10	8	7	6	6	5
2 079	2 342	2 505	2 623	2 716	2 792	2 857	2 914	2 964
35	20	14	10	8	7	6	5	4
3 114	3 362	3 519	3 633	3 724	3 799	3 863	3 919	3 968
32	18	12	10	8	7	6	5	5
4 146	4 380	4 531	4 643	4 732	4 806	4 869	4 924	4 973
30	18	13	10	8	7	6	5	5
5 176	5 398	5 544	5 653	5 740	5 813	5 875	5 929	5 978
28	17	12	10	8	7	6	5	4
6 204	6 415	6 556	6 663	6 748	6 820	6 881	6 934	6 982
26	16	12	9	8	6	5	6	5
7 230	7 431	7 568	7 672	7 756	7 826	7 886	7 940	7 987
25	16	12	9	7	7	6	4	4
8 255	8 447	8 580	8 681	8 763	8 833	8 892	8 944	8 991
24	15	11	9	8	6	6	5	5
9 279	9 462	9 591	9 690	9 771	9 839	9 898	9 949	9 996
22	15	11	9	7	6	5	5	4

PROPORTIONAL PARTS.

	41	38	35	32	30	28	26	25	24	22	21	20	19	18	17	16	15	14	13	
1	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2.	2	2	1	1	1
2	8	8	7	6	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3	2
3	12	11	11	10	9	8	8	8	7	7	6	6	6	5	5	5	5	4	4	3
4	16	15	14	13	12	11	10	10	10	9	8	8	8	7	7	6	6	6	5	4
5	21	19	18	16	15	14	13	13	12	11	11	10	10	9	9	8	8	7	7	5
6	25	23	21	19	18	17	16	15	14	13	13	12	11	11	10	10	9	8	8	6
7	29	27	25	22	21	20	18	18	17	15	15	14	13	13	12	11	11	10	9	7
8	33	30	28	26	24	22	21	20	19	18	17	16	15	14	14	13	12	11	10	8
9	37	34	32	29	27	25	23	23	22	20	19	18	17	16	15	14	14	13	12	9
	41	38	35	32	30	28	26	25	24	22	21	20	19	18	17	16	15	14	13	

ANTI-LOGARITHMS.

·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
0 100	0 126	0 158	0 200	0 251	0 316	0 398	0 501	0 631	0 794
2	3	4	4	6	8	9	12	15	19
1 102	1 129	1 162	1 204	1 257	1 324	1 407	1 513	1 646	1 813
3	3	4	5	6	7	10	12	15	19
2 105	2 132	2 166	2 209	2 263	2 331	2 417	2 525	2 661	2 832
2	3	4	5	6	8	10	12	15	19
3 107	3 135	3 170	3 214	3 269	3 339	3 427	3 537	3 676	3 851
3	3	4	5	6	8	10	13	16	20
4 110	4 138	4 174	4 219	4 275	4 347	4 437	4 550	4 692	4 871
2	3	4	5	7	8	10	12	16	20
5 112	5 141	5 178	5 224	5 282	5 355	5 447	5 562	5 708	5 891
3	4	4	5	6	8	10	13	16	21
6 115	6 145	6 182	6 229	6 288	6 363	6 457	6 575	6 724	6 912
2	3	4	5	7	9	11	14	17	21
7 117	7 148	7 186	7 234	7 295	7 372	7 468	7 589	7 741	7 933
3	3	5	6	7	8	11	14	18	22
8 120	8 151	8 191	8 240	8 302	8 380	8 479	8 603	8 759	8 955
3	4	4	5	7	9	11	14	17	22
9 123	9 155	9 195	9 245	9 309	9 389	9 490	9 617	9 776	9 977
3	3	5	6	7	9	11	14	18	23

SECTION 7.—*Application of Logarithms worked at Length.*

In most of the following examples, we shall use the tables of seven places; those who employ smaller tables can produce the same result, as far as their tables go. The following is an instance of the way in which the same question must be treated, according to different tables:

What is ·1234567 × 26813·92?

1. *With tables of seven places:*

Log. ·12345..	=	$\bar{1}$·0914911
6		212
7		25
Log. ·1234567	=	$\bar{1}$·0915148
Log. 26813..	=	4·4283454
9		146
2		3
Log. 26813·92	=	4·4283603
		$\bar{1}$·0915148
Add		3·5198751
33103.. ·		5198674
		77
5		66
8		110

Ans. 3310·358.

2. *With five places:* or what is ·12346 × 26814?

Log. ·1234.	=	$\bar{1}$·09132
6		21
Log. ·12346	=	$\bar{1}$·09153
Log. 2681.	=	4·42830
4		6
Log. 26814	=	4·42836
		$\bar{1}$·09153
		3·51989
3310·		51983
		13) 60 (5

Ans. 3310·5.

3. *With four places:* ·1235 × 26810:

Log. ·123	$\bar{1}$·0899
5	17
Log. ·1235	$\bar{1}$·0916
Log. 26800	4·4281
1	2
Log. 26810	4·4283
	$\bar{1}$·0916
	3·5199
·519·	3304
9	7
·5199	3311 *Ans.*

4. *With three places:* ·123 × 26800:

Log. ·12·	$\bar{1}$·079	D 35
3	11	
Log. ·123	$\bar{1}$·090	
Log. 26000	4·415	
8	13	
Log. 26800	4·428	
	$\bar{1}$·090	
	3.518	
51··	324	
8	6	
3·518	3300 *Ans.*	

In future we shall give the logarithms to seven places, but without going through the detail of using the table of differences to find the sixth and seventh places, either of a number to a logarithm or of a logarithm to a number, except in a few particular cases.

Question 1. Find $\frac{1}{1084\cdot9}$

Log. 1	=	0·0000000
Log. 1084·9	=	3·0353897
·000921744		$\bar{4}$·9646103

Question 2. Find $\sqrt{\cdot 1}$ and $\sqrt[5]{97.65625}$.

Log. ·1	2) $\bar{1}$·0000000
	$\bar{1}$·5000000
31622	4999893
	107
7	96
8	110

or $\sqrt{\cdot 1}$ = ·3162278

Log. 97·65625	5) 1·9897000
2·5 *Ans.*	·3979400

In the last result, the exact coincidence (to seven places) of the answer with 2·5 may induce a supposition that 97·65625 is the exact fifth power of 2·5, which is really the case; but nothing can be inferred from the tables, except that the fifth root of 97·65625 lies between 2·4999995 and 2·5000005. It might be

2·499999576........
or 2·500000214......

and the answer of the tables would still be 2·5.

Question 3. $\dfrac{\sqrt[5]{(32\cdot92416 \times 10\cdot27251)^6}}{1084\cdot9}$

Log. 32·92416		1·5175147
Log. 10·27251		1·0116766
		2·5291913
		6
	5)	15·1751478
		3·0350296
Log. 1084·9		3·0353897
Ans. ·9991712		$\bar{1}$·9996399

Question 4. Find a fourth proportional to 1234, 2345, and 3456; or find 2345 × 3456 ÷ 1234:

Log. 2345	3·3701428
Log. 3456	3·5385737
	6·9087165
Log. 1234	3·0913152
Ans. 6567·518	3·8174013

We shall hereafter give a more expeditious way of solving this question.

Question 5. What is the thousandth power of 2?

Log. 2 =	·3010300
	1000
Multiply	301·0300000

Now, this is the logarithm, as nearly as our tables will tell, of

107151900000.....

the number of ciphers being 294; that is, apparently, the thousandth power of 2 is a number of 302 places of figures, the first seven of which are 1071519. But it must be recollected that a thousand times ·3010300 is 301·0300, and that we only annex four more ciphers because we do not know with what figures to fill up the vacant places. We cannot, therefore, depend upon more than four places of the result, and should say that 2^{1000} is a number of 302 figures, of which the first four are 1071. If we would have the first seven places correct, we must go to a table of ten places at least. This gives

Log. 2	=	·3010,299957
Log. 2^{1000}	=	301·0299957
1071508		0299922
		35

so that the first seven figures are 1071508.

Let us here observe, that by mere inspection of a logarithm, we answer questions which would take years of calculation. For instance, from the above logarithm of 2, we see that the tenth power of 2 has 4 figures (3+1); the hundredth power has 31 figures (30+1); the millionth power has 301,030 figures, and so on. Hence the simplest method *in theory*, of calculating a logarithm to seven places, is by the following formula:—

$$\text{Log. } x = \frac{\left\{\begin{matrix}\text{No. of fig. in ten-}\\ \text{millionth power of } x\end{matrix}\right\} - 1}{\text{ten million}}$$

but this, of course, would be practically impossible to use.

Question 6. What whole number is that which has 256 places of figures in its 70th power. The logarithm of that 70th power must be between

255·000.... and 255·999....

that is, the logarithm of the number itself must lie between

$$\frac{255\cdot000....}{70} \text{ and } \frac{255\cdot999....}{70}$$

or 3·6428571 and 3·6571428

Answer: All whole numbers between 4394 and 4540, both inclusive.

Question 7. What is the value of

$$\sqrt{\frac{5\sqrt[3]{138}}{\sqrt[5]{\cdot01}}}$$

Log.	138	3)2·1398791
		·7132930
Log.	5	·6989700
Log. $5\sqrt[3]{138}$		1·4122630
Log.	·01	5)$\bar{2}$·0000000
		$\bar{1}$·6000000
		1·4122630
		2)1·8122630
Ans. 8·056224		·9061315

Operations with logarithms may be divided into—1. Those in which a number need never be found to a logarithm

until the end of the process; 2. Those in which numbers must be found to logarithms as a subordinate part of the process. All the instances hitherto given, and all which involve only multiplication, division, raising of powers, and extraction of roots, fall under the first case; while all which contain addition or subtraction fall under the second. For instance, to find

$$\sqrt{\sqrt[3]{4} + \sqrt[3]{5}}$$

we must first find $\sqrt[3]{5}$, then $\sqrt[3]{4}$, then make the addition indicated, and find the square root of the sum.

Log. 4.	Log. 5.
3)·6020600	3)·6989700
·2006867	·2329900
1·587401	1·709976
1·709976	
3·297377...	2)·5151686
Ans. 1·809607	·2575843

Question 8.

$$\sqrt[10]{\left(\cdot 01 \sqrt[10]{(\cdot 01 \sqrt[10]{\cdot 01})}\right)}\,?$$

Log. ·01	10)$\bar{2}$·0000000
	$\bar{1}$·8000000
	$\bar{2}$·0000000
	10)$\bar{3}$·8000000
	$\bar{1}$·7800000
	$\bar{2}$·0000000
	10)$\bar{3}$·7800000
Ans. ·5997911	$\bar{1}$·7780000

Repeat the process until the tenth root has been extracted seven times, and show that the result will then be very nearly equal to the ninth root of ·01.

Question 9. Supposing the earth to be 7916, and the moon 2160 miles in diameter, how many times does the bulk of the former contain the latter? [Spheres are to one another as the cubes of their diameters; that is, if one diameter contain another x times, the sphere on the first contains that on the second $x\,x\,x$ times.] The question is, what is $(7916 \div 2160)^3$?

Log. 7916	3·8985058
Log. 2160	3·3344538
	0·5640520
	3
Ans. 49·22163	1·6921560

Answer—About $49\frac{1}{4}$ times.

Question 10. What is the number of cubic miles in the earth and moon, the diameters being as in the last question? [To find the cubic miles in a sphere, multiply the cube of the diameter by the cubical content of a sphere of one mile in diameter, page 39.]

Log. 7916		Log. 2160
3·8985058		3·3344538
3		3
11·6955174		10·0033614
$\bar{1}$·7189986		$\bar{1}$·7189986
11·4145160		9·7223600
259726400000	*Answers nearly*	5276671000

Question 11. To how much will 15*l.* 7*s.* $3\frac{1}{2}$*d.* amount in fifty years, at 3 per cent. compound interest; or what is

$$\pounds 15 \quad 7 \quad 3\tfrac{1}{2} \times (1\cdot 03)^{50}$$

The sum mentioned is £15·364.

Log.	1·03	·0128372
		50
		·6418600
Log.	15·364	1·1865043
	67·354	1·8283643

Answer—£67 7 1—very nearly.

Question 12. How many feet are there in 867·41 metres [page 39, log. No. of feet in metre = ·5159929.]

Log. 867·41	2·9382244
Log. (feet in metre)	·5159929
	3·4542173

Answer—2845·885

Question 13. Taking it for granted, as is proved in a higher branch of mathematics, that when x is a large number, the product

$$1 \times 2 \times 3 \times 4 \ldots\ldots \times (x-1) \times x$$

is very nearly equal to

$$\sqrt{6\cdot 2831854 \times x} \times \left\{\frac{x}{2\cdot 7182818}\right\}^x$$

what is (nearly) the product of the first thousand numbers? is it greater or less than would be obtained by substituting the average for every one of the numbers, and how many times does the greater contain the less? Also how many figures are in each product?

[The average of 1, 2, 3......1000 is 500·5, and the products to be compared are therefore

$1 \times 2 \times 3 \times \ldots\ldots \times 1000$ & $(500\cdot 5)^{1000}$]

Log. 6·283185	·7981799
Log. 1000	3·0000000
	2)3·7981799
	1·8990899*

Log. 1000	3·0000000
Log. 2·718282	·4342945
	2·5657055
	1000
	2565·7055
	1·8991*
	2567·6046

Hence the product of the first thousand numbers contains 2568 figures, of which the first four are 4023; and the best approximation we can make is—

4023000....(2564 ciphers).

Log. 500·5	=	2·6994041
		1000
		2699·4041
		2567·6046
		131·7995

It appears that the second product has 2700 figures, the first four of which are 2535: it is incomparably the greater of the two, and contains the first a number of times, having 132 figures, the first four of which are 6302. As some further examples of the preceding formula, let $[x]$ signify the product of all the numbers up to x inclusive; then—

Log. [1010]	=	2597·6284
Log. [1020]	=	2627·6952
Log. [1030]	=	2657·8046
Log. [1040]	=	2687·9561
Log. [1050]	=	2718·1493
Log. [1100]	=	2869·7278
Log. [1150]	=	3022·2933
Log. [1200]	=	3175·8028

Question 14. What whole power of 2 is nearer than any other to 100,000,000? That is, how many times does the logarithm of 100,000,000 contain the logarithm of 2?

Log. 2 Log. 100,000,000
·30103) 8·00000...... (26·57
Ans.—The 27th power.

To get examples by which the student may ascertain whether he has acquired the highest degree of accuracy in taking out logarithms, &c., the verification of cases such as those in page 27 (rule 2) will be useful. For instance:—

Question 15. Verify to seven places of figures, (if the logarithms to seven places will serve) the equation

$$\sqrt{18} + \sqrt{11} = \frac{7}{\sqrt{18} - \sqrt{11}}$$

Log. 18.		Log. 11.
2)1·2552725		2)1·0413927
·6276363		·5206964
4·242641		3·316625
3·316625		
7·559266 Sum	its log.	·8784795
0·926016 Diff.	its log.	$\bar{1}$·9666185
		0·8450980

which is correctly the logarithm of 7.

At the beginning of the tables (1000...) an alteration of a unit in the seventh figure of the number makes an alteration of 4 units in the seventh figure of the logarithm; so that two logarithms, which differ only in the seventh decimal, by less than 4, are for every practical purpose the same. But in the last half of the tables, a unit of difference in the seventh figure of the number causes less than a unit of difference in the seventh place of the logarithm, which renders the tables not so safe in the latter part as in the former. To illustrate this, we form the following table from the extreme end of the table to seven places, repeating only the figures which change.

No.	Dec. Part of Log.
9999900	·9999957
1	57
2	58
3	58
4	59
5	59
6	60
7	60
8	60
9	61

From this it appears that ·9999960 may belong to 999990, followed either by 6, 7, or 8, so that the number cannot be found within two units in the seventh place. But this is the extreme point; and, generally speaking, the results may be depended upon within one unit in the seventh place, which is always more than sufficient for practical purposes.

Question 16. What is the value of x in the equation?

$$(20)^x = 100$$

* It is useless to retain more than four places of this.

This is the same as asking, what is the logarithm of 100 to the base 20? Taking the logarithms of both sides, we have

$$\text{Log. } 20^x \text{ or } x \times \text{Log. } 20 = \text{Log. } 100$$

$$x = \frac{\text{Log. } 100}{\text{Log. } 20} = \frac{2}{1 \cdot 30103}$$

$$= 1 \cdot 537244$$

Question 17. At what rate of compound interest will money double itself in ten years? That what is the solution of

$$(1 + x)^{10} = 2$$

$$x = \sqrt[10]{2} - 1 = \cdot 071773$$

or 7·177 per cent.; that is £7 3 $6\frac{1}{2}$ per cent.

In working questions of compound interest for long periods of time*, it is sometimes necessary to have certain logarithms to more than seven places. The following will be sufficient.

No.	Dec. part of Log.	Rate per cent. in which this Log. is used.
10025	00108 43813	$\frac{1}{4}$
10050	00216 60618	$\frac{1}{2}$
10075	00324 50548	$\frac{3}{4}$
10100	00432 13738	1
10125	00539 50319	$1\frac{1}{4}$
10150	00646 60422	$1\frac{1}{2}$
10175	00753 44179	$1\frac{3}{4}$
10200	00860 01718	2
10225	00966 33167	$2\frac{1}{4}$
10250	01072 38654	$2\frac{1}{2}$
10275	01178 18305	$2\frac{3}{4}$
10300	01283 72247	3
10325	01389 00603	$3\frac{1}{4}$
10350	01494 03498	$3\frac{1}{2}$
10375	01598 81054	$3\frac{3}{4}$
10400	01703 33393	4
10425	01807 60636	$4\frac{1}{4}$
10450	01911 62904	$4\frac{1}{2}$
10475	02015 40316	$4\frac{3}{4}$
10500	02118 92991	5
10525	02222 21045	$5\frac{1}{4}$
10550	02325 24596	$5\frac{1}{2}$
10575	02428 03760	$5\frac{3}{4}$
10600	02530 58653	6

Question 18. What is the amount of one farthing, for 500 years, at 3 per cent. compound interest?

One farthing is £.·001041667, and the quantity to be found is

$$£.\ (1 \cdot 03)^{500} \times \cdot 001041667$$

$$\begin{array}{rl} \text{Log. } 1 \cdot 03 & \cdot 0128372247 \\ & \underline{\qquad\qquad 500} \\ & 6 \cdot 41861235 \\ \text{Log. } \cdot 001041667 & \underline{3 \cdot 0177286} \\ 2731 \cdot 121 & \underline{3 \cdot 4363410} \end{array}$$

Ans. £2731 2*s.* $5\frac{1}{4}$*d.*

Question 19. Given the common logarithm to find the hyperbolic or Naperian logarithm.

This may be done by the following table:

1	023 025 851
2	046 051 702
3	069 077 553
4	092 103 404
5	115 129 255
6	138 155 106
7	161 180 957
8	184 206 807
9	207 232 658

This table is intended to abbreviate the operation of multiplying by 2·3025851, and its use will be evident from the following examples. What is, first, the Naperian logarithm of 56?

* Such, for example, as Dr. Price's celebrated problem about a farthing put out to compound interest at the beginning of the world.

Common Log. 56.

In table, we find opposite to									
	1·	7	4	8	1	8	8	0	
1	0	2	3	0	2	5	8	5	1
7		1	6	1	1	8	0	9	6
4			0	9	2	1	0	3	4
8				1	8	4	2	0	7
1					0	2	3	0	3
8						1	8	4	2
8							1	8	4
		4	0	2	5	3	5	1	7

Make seven decimal places, and the answer is 4·0253517, the Naperian logarithm of 56.

What is Nap. Log. 9828 ?

Common Log. 9828.

```
3· 9 9 2 4 6 5 1
0 6 9 0 7 7 5 5 3
    2 0 7 2 3 2 6 6
      2 0 7 2 3 2 7
        0 4 6 0 5 2
          0 9 2 1 0
            1 3 8 2
              1 1 5
                0 2
    9 1 9 2 9 9 0 7
```

Ans. 9·1929907.

Examples for practice :

Number.	Nap. Log.
3·141593	1·1447299
2349	7·7617450
156·3	5·0517778

When there is no characteristic, use one place less, and make seven places. When there is a negative characteristic, neglect it, and proceed as in last sentence; but subtract at the end the number opposite to the characteristic in the table with all its places, attending to page 35.

What is Nap. Log. ·008 ?

Common Log. ·008.

```
 3̄·9030900
 20723266
   069078
     2072
 20794416
 069077553
 5̄1716863
```

Ans. 5̄·1716863

[N. B. As a check upon this rule, remember that the Naperian logarithm must be something more than twice the common logarithm.]

Question 20. To reduce the Naperian logarithm to the common logarithm, use the following table in the same manner:

1	0434 2945
2	0868 5890
3	1302 8834
4	1737 1779
5	2171 4724
6	2605 7669
7	3040 0614
8	3474 3559
9	3908 6503

The Naperian logarithm being 9·1929907, what is the common logarithm ?

```
 9·1929907
 39086503
  0434295
   390865
    08686
     3909
      391
        3
 39924652
```

Ans. 3·9924652

SECTION 8.—*Examples of the Application of Logarithms for Practice.*

Before proceeding to give any examples, we shall explain why we have deviated from the usual practice, and in a manner which some of our readers will consider rather singular. In working rules by examples, which are presumed to be quite correctly answered in the book, the student is apt to work by the answer—that is, to look at the answer from time to time, and judge, or at least guess, whether he is proceeding correctly. Very few have the resolution to shut the book, and not look at the answer until they have produced their own. The consequence is, that no confidence is gained, and the student has to learn how to be independent after he has left his elementary treatise, and has to solve questions which occur in practice. To give no answers at all, would be depriving him of an assistance which, to a certain extent, is useful, and even necessary. We have therefore made some figure or figures, or positions of the decimal point (perhaps many or all, but the student must find this out), intentionally incorrect; so that while there will be enough to

assist the student who is disposed to learn how to shift for himself, there will be enough to perplex the one who has no assurance of being correct, except what he derives from the printed answer. When a figure or figures of the answer are found to differ from those here given, let the process be thoroughly re-examined, until the student is satisfied that he has obtained the correct answer.*

$$\frac{441{\cdot}5059}{89{\cdot}72584} = 4{\cdot}921610 \qquad \frac{1}{6{\cdot}729} = {\cdot}1486105$$

$$\frac{13052{\cdot}62}{{\cdot}9914449} = 13169{\cdot}25 \qquad \frac{1}{51{\cdot}88} = {\cdot}01927535$$

$$\frac{6115{\cdot}27}{79122{\cdot}35} = {\cdot}0772889 \qquad \frac{1}{291{\cdot}2} = {\cdot}003444066$$

$$\frac{7{\cdot}466382}{66{\cdot}52304} = {\cdot}1122380 \qquad \frac{1}{{\cdot}1239} = 8{\cdot}071025$$

$$\frac{{\cdot}9326154}{1-({\cdot}4663077)^2} = 1{\cdot}191654$$

$$\sqrt{100+(8{\cdot}09784)^2} = 12{\cdot}86759$$

$$\sqrt{(629{\cdot}3203)^2+(777{\cdot}144)^2} = 999{\cdot}9998$$

$$\sqrt{{\cdot}00001215} = {\cdot}003485685$$

$$\sqrt{.027} = {\cdot}1653168 \qquad \sqrt{{\cdot}27} = {\cdot}5196162$$

$$\sqrt[3]{4{\cdot}355} = 1{\cdot}633137 \qquad \sqrt[3]{436} = 7{\cdot}582787$$

$$(13{\cdot}22869)^{\frac{3}{8}} = 48{\cdot}11445 \qquad \sqrt[100]{6} = 1{\cdot}0181$$

$$\frac{{\cdot}018594 \times 763^{\frac{13}{15}}}{7654{\cdot}3 \times 794} = {\cdot}000002337$$

$$\sqrt[7]{8} = 1{\cdot}3859 \qquad \sqrt[4]{3{\cdot}5246} = 1{\cdot}371179$$

$$\sqrt[8]{172\tfrac{5}{8}} = 1{\cdot}904169 \qquad \sqrt[13]{\frac{3348}{569}} = 1{\cdot}146156$$

$$\left(\frac{9}{8}\right)^{21} = 11{\cdot}87322 \qquad \left(\frac{643}{637}\right)^{100} = 31{\cdot}69104$$

$$\left(\frac{167}{53}\right)^{{\cdot}32} = 1{\cdot}44378 \qquad \left(\frac{5}{7}\right)^{{\cdot}0537} = {\cdot}982693$$

$$\frac{(52072)^{13} \times \sqrt{({\cdot}000734)^3}}{(255608)^3} = 9930{\cdot}834$$

$$\left(\frac{42666}{1147}\right)^{12} \times \left(\frac{765}{19432}\right)^{13} = 627568{\cdot}8$$

$$\sqrt[5]{\left(\frac{7}{3}\sqrt[4]{6}\right)} = 1{\cdot}215695$$

$$\sqrt[3]{({\cdot}26\sqrt{\tfrac{2}{3}})} = {\cdot}596544$$

$$\sqrt[5]{\left(\frac{3425\sqrt[7]{136}}{{\cdot}00034}\right)} = 28{\cdot}94619$$

$$253\sqrt[3]{\frac{716{\cdot}5}{\sqrt{2}}} = 2016{\cdot}014$$

$$\sqrt{\frac{132\,(7{\cdot}356)^9}{\sqrt{(3{\cdot}25)^5}}} = 144{\cdot}5972$$

* Many of the examples are taken, *mutatis mutandis*, from Meier Hirsch's *Sammlung von Beispielen, Formeln*, &c., Berlin, 1816; others from trigonometrical tables, &c.

$$\frac{(466871)^{\frac{6}{7}} \times (3576)^{\frac{16}{9}}}{(996003) \times (\cdot 0071)^{\frac{1}{6}}} = 1780845$$

$$\sqrt[8]{(21 + \sqrt[6]{19})} = 1\cdot 470075$$

$$\sqrt[3]{5\cdot 03 + \sqrt[5]{\cdot 2})} = 1\cdot 792929$$

$$\sqrt[5]{(9\cdot 921 - 3\sqrt{5\cdot 02})} = 1\cdot 261866$$

$$\frac{1 + \sqrt{3}}{2\sqrt{2}} = \cdot 9659258 \qquad \frac{\sqrt{(5 + \sqrt{5})}}{2\sqrt{2}} = \cdot 9510565$$

$$\frac{1 + \sqrt{3}}{8\sqrt{2}}(\sqrt{5} + 1) - \frac{\sqrt{3} - 1}{8}\sqrt{(5 - \sqrt{5})} = \cdot 6293204$$

$$\frac{\sqrt{3} - 1}{8\sqrt{2}}(\sqrt{5} - 1) + \frac{\sqrt{3} + 1}{8}\sqrt{(5 + \sqrt{5})} = \cdot 9986295$$

$$\sqrt[16]{\left(\frac{43 + 5\sqrt[3]{278}}{\sqrt[5]{17}}\right)} = 1\cdot 264848$$

$$\sqrt[7]{\cdot 01\sqrt[6]{(\cdot 02\sqrt[5]{\cdot 03})}} = \cdot 4640688$$

What is the diameter of the sphere which shall have the same content as a cube of 21·16 yards in length? *Ans.* 26·25 yards.

Find the number of cubic *feet* in a cube of 15 *inches* long. *Ans.* 1·953.

What is the diameter of a circle whose circumference is 25000 miles? *Ans.* 7958 miles nearly.

What is the circumference of a circle whose diameter is 7958 miles? *Ans.* 25000$\frac{4}{5}$ miles.

What is the area of a circle whose circumference is 22 feet? *Ans.* 38·517 square feet.

What is the area of a circle whose diameter is 15·25 inches? *Ans.* 20·2949 square inches.

The surface of a sphere being four times the area of its largest circle, what is the surface of a sphere of 4·5 feet in diameter? *Ans.* 63·6174 square feet.

A sphere of six feet in diameter is painted at the rate of a halfpenny per square inch, what is the cost? *Ans.* £33 18*s.* 7*d.*

The diameter of a sphere being 7 feet, what is the side of the cube of equal solidity? *Ans.* 5·64228.

If a sphere be 3 feet in diameter, how long is the side of a square of the same surface? *Ans.* 5·31736 feet.

The squares of the times of revolution of different planets being as the cubes of their mean distances from the sun, and the mean distances of Saturn and Jupiter being in the proportion of 9538786 to 5202776, and the time of revolution of Jupiter 4332·585 days, what is that of Saturn? *Ans.* 10759·22.

The diameter of a sovereign being ·87 of an inch, how many miles would 600,000,000 sovereigns extend, if placed side by side? *Ans.* 8238·64.

Section 9.—*Arithmetical Complement. Trigonometrical Tables: their Use in common Calculations.*

The arithmetical complement of a number is the number by which it falls short of the unit of the next higher denomination. It is abbreviated into *Ar. co.* Thus:

Ar. co.	6	= 10	−	6	=	4
Ar. co.	893	= 1000	−	893	=	107
Ar. co.	·669	= 1	−	·669	=	·331

The lowest denomination considered is the unit. Thus:

Ar. co. ·0094 = 1 − ·0094
not ·01 − ·0094

The most expeditious way of finding the arithmetical complement is as follows:—Begin from the left, subtract every figure from 9, *up to the lowest significant figure, which subtract from* 10. Repeat the ciphers at the end, if any.

No.	156·142	·0013754
Ar. co.	843·858	·9986246
No.	1798000	4009000
Ar. co.	8202000	5991000

When there is a negative characteristic, *add* it to 9, instead of subtracting it from 9.

No.	$\bar{1}$·439	$\bar{2}$·33	$\bar{3}$·108
Ar. co.	10·561	11·67	12·892

The student should now practise taking out from the tables, not the logarithms there written, but their arithmetical complements, without first taking out the logarithms themselves. The operation above described can be correctly performed in the head, with a little practice. For instance, looking in the table, and seeing ·6123180, he should say—6 and 3 make 9, put down 3; 1 and 8 make 9, put down 8, &c., up to 8 and 2 make *ten*, which will give ·3876820.

To subtract a number, add its arithmetical complement; the result will be too great by a unit of the kind which was used in making the arithmetical complement. Thus, 9 − 4 may be thus found:

$$9 + \text{Ar. co. } 4 - 10$$

and subtractions may be reduced to the subtractions of single units from the results of an addition. In the following examples, the first is the common method, the second the one just described.

From	9·66813	9·66813
Take	3·44210	6·55790
	6.22603	16·22603 − 10
From	2·30746	2·30746
Take	$\bar{3}$·42815	12·57185
	4·87931	14·87931 − 10
From	$\bar{1}$·21769	$\bar{1}$·21769
Take	$\bar{2}$·30999	11·69001
	0·90770	10·90770 − 10

The better way will be always to write, after an arithmetical complement, the unit which must be subtracted, after addition has been substituted for subtraction by means of that complement.

What is 1835 + 968 − 1036, and 21648 − 9763 − 144?

1835	21648
968	237 − 10,000
8964 − 10,000	856 − 1000
11767	22741
10000	11000
1767	11741

Required a fourth proportional to 117·1097, 17·36482, 9510·565:

Log. 9510·565	3·9782063
Log. 17·36482	1·2396702
Ar. co. Log. 117·1097	7·9314071 − 10
	13·1492836 − 10

Ans. 1410·209.

The student may now try any of the preceding examples, with addition of arithmetical complements instead of subtraction. But we recommend him rather to avoid this method, which is very subject to error, except in the hands of a practised computer.

The trigonometrical tables have already been described (for trigonometrical purposes) in the treatise on that science (page 51). The logarithms there given are generally made too great by ten; that is, instead of the subtractive characteristic $\bar{1}$, we have the characteristic 9, &c.; or, instead of subtracting 1, we add 9, which makes the result too great by 10. In trigonometrical operations this is convenient; but principally because the extraction of roots very seldom occurs. If we had, for example, to extract the square root of the sine of 46°, which we find in the tables to be ·7193398, and the tabular logarithm of which is 9·8569341 (but, in reality, $\bar{1}$ ·8569341), the following process will be wrong in the characteristic:

$$\begin{array}{r} 2)\ 9·8569341 \\ \hline 4·9284671 \end{array}$$

for the dividend being 10 too much, the quotient will be 5 too much; or, rather, the addition of the dividend being intended to be followed by a subtraction of 10, the addition of the quotient must be followed by a subtraction of 5. In extracting the cube root, the following process gives characteristic and decimal part both wrong:

$$\begin{array}{r} 3)\ 9·8569341 \\ \hline 3·2836447 \end{array}$$

for, the dividend being too great by 10, the quotient is too great by $3\frac{1}{3}$, or 3·3333333, and must be set right by subtracting this. But, to reduce the result to the tabular logarithm, the logarithm should be made 20, 30, 40, &c., too great before dividing by 2, 3, 4, &c., as in those cases the results will severally be 10 too great. But, perhaps, the better way is to restore the proper negative characteristic, and proceed in the way already described (page 37).

What we have here to do with the trigonometrical tables is to observe that they may be considered as registers of the value of certain expressions, which, being already calculated, may be referred to, and thus the trouble of fresh calculation saved. We shall proceed to explain the following table:

Sine.	Cosine.	Tangent.	Cotangent.	Secant.	Cosecant.
a	$\sqrt{1-a^2}$	$\frac{a}{\sqrt{1-a^2}}$	$\frac{\sqrt{1-a^2}}{a}$	$\frac{1}{\sqrt{1-a^2}}$	$\frac{1}{a}$
$\sqrt{1-a^2}$	a	$\frac{\sqrt{1-a^2}}{a}$	$\frac{a}{\sqrt{1-a^2}}$	$\frac{1}{a}$	$\frac{1}{\sqrt{1-a^2}}$
$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{\sqrt{1+a^2}}$	a	$\frac{1}{a}$	$\sqrt{1+a^2}$	$\frac{\sqrt{1+a^2}}{a}$
$\frac{1}{\sqrt{1+a^2}}$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{a}$	a	$\frac{\sqrt{1+a^2}}{a}$	$\sqrt{1+a^2}$
$\frac{\sqrt{a^2-1}}{a}$	$\frac{1}{a}$	$\sqrt{a^2-1}$	$\frac{1}{\sqrt{a^2-1}}$	a	$\frac{a}{\sqrt{1-a^2}}$
$\frac{1}{a}$	$\frac{\sqrt{a^2-1}}{a}$	$\frac{1}{\sqrt{a^2-1}}$	$\sqrt{a^2-1}$	$\frac{a}{\sqrt{1-a^2}}$	a

By this we mean, that if a be the sine of an angle, or if we can find a in the table of sines, we find $\sqrt{1-a^2}$ in the corresponding line of the table of cosines, $\frac{a}{\sqrt{1-a^2}}$ in that of the table of tangents: if a be found in the table of tangents, we have opposite to it $\frac{a}{\sqrt{1+a^2}}$ in that of sines, and $\frac{1}{\sqrt{1+a^2}}$ in that of cosines. If the tables only give *logarithms** of sines, &c., we must look for the logarithm of a, and we find the logarithms of the above quantities.

To use this method with great accuracy would give very nearly if not quite as much trouble as the common logarithmic process, but by mere inspection a few places of any result may be obtained; so that when very considerable accuracy is not required, calculation is altogether saved. For instance—

$$a \text{ being } \cdot 6346, \text{ what is } \frac{a}{\sqrt{1+a^2}}$$

if the first be a tangent, the second is the corresponding sine: we look in the table of tangents (Hutton's), and find as follows, under 32° 24′, that the tangent being ·6346193, the sine is ·5358268, so that ·5358 will be very near the answer. To get a nearer answer, we must use the method which we proceed to describe.

Definition. When a number is to be found in a table opposite to another number, the second, by means of which we know where to go in the table, is called the *argument*. Thus, in finding the logarithm of 56, we *enter* the column of numbers with the argument 56. In the last example, we *enter* the table of tangents with the *argument* ·6346. The result obtained we shall call the *resultant*†.

Let a be the given argument not exactly to be found in the tables. Let M and m (M being the greater) be the arguments in the tables, between which a is found to lie. Let R and r be the corresponding resultants to M and m. Thus,

Column of arguments.	Column of resultants.
M	R
a	ρ
m	r

or the following—

m	r
a	ρ
M	R

according as the arguments decrease or increase the resultant of the argument a is

$$r + \frac{(a-m)(R-r)}{M-m}$$

when arguments and resultants both increase or both decrease together; and

$$r - \frac{(a-m)(r-R)}{M-m}$$

* Hutton's tables, which give *sines*, &c., as well as their logarithms, are decidedly the best of which we know for the engineer or mechanic. There are more useful tables for the astronomer, to which we need not here allude. For logarithms of numbers only, we believe those of Mr. Babbage to be best for general use, of all those which contain seven places.

† There is no correlative to *argument* in common use. The reader may take his choice of *resultant*, *inference*, *extract*, *function*, &c.; either of which would be better than no word at all.

when one of the two increases and the other decreases. This process is called *interpolation.*

In the above example, we are to find the sine to the tangent ·6346000 (the argument). Looking in the tables, we find as follows—

Arguments. Tangents.	Resultants. Sines.
$m = 6342113$	$5355812 = r$
$a = 6346000$	?
$M = 6346193$	$5358268 = R$

Arguments and resultants increase together.

$M - m = 4080$ $a - m = 3887$ $R - r = 2456$

$\dfrac{3887 \times 2456}{4080} = 2340$ (nearest whole No.)

$$\begin{array}{r} 5355812 \\ 2340 \\ \hline \cdot 5358152 \end{array}$$ *Answer.*

This is within a unit, in the last place, of the truth.

An unlimited number of examples of the preceding process may be obtained by taking an argument and resultant from the tables themselves, and finding the latter by means of those which come before and after. Suppose, for instance, that the cotangent of 19° 31′ had been erased or blotted so as not to be visible, and that it were required to fill it up by using the table of tangents, the process would be as follows:—

Arguments. Tangents.	Resultants. Cotangents.
$m = 3541186$	$28239129 = r$
$a = 3544460$	?
$M = 3547734$	$28187003 = R$

Arguments increase and resultants decrease.

$M - m = 6548$ $a - m = 3274$ $r - R = 52126$

$\dfrac{3274 \times 52126}{6548} = 26063$

$$\begin{array}{r} 2 \cdot 8239129 \\ 26063 \\ \hline 2 \cdot 8213066 \end{array}$$ *Ans.*

which is incorrect by a unit in the sixth decimal place.

This process may be applied to tables of logarithms; or, in fact, to any tables. It is substantially what is done in finding the logarithm of a number intermediate to two numbers in the tables, or *vice versa*, and also in finding the logarithmic sine, &c. of an angle intermediate to two angles in the table, as the following examples will prove, if the student compare them with the usual process.

Required log. 6416958. We shall only consider the decimal part.

Arguments. Numbers.	Resultants. Logarithms.
$m = 6416900$	$8073253 = r$
$a = 6416958$	?
$M = 6417000$	$8073320 = R$

Arguments and resultants increase together.

$M - m = 100$ $a - m = 58$ $R - r = 67$

$\dfrac{58 \times 67}{100} = 39$ (nearest whole No.)

$8073253 + 39 = 8073292$

Answer ·8073292

Required the logarithm of the sine of 36° 18′ 47″·6.

Arguments. Angles.	Resultants. Log. sines.
$m = 36° \, 18' \, 0''$	$9 \cdot 7723314 = r$
$a = 36° \, 18' \, 47'' \cdot 6$	?
$M = 36° \, 19' \, 0''$	$9 \cdot 7725033 = R$

Arguments and resultants increase together.

$M - m = 60''$ $a - m = 47'' \cdot 6$ $R - r = 1719$

$\dfrac{47 \cdot 6 \times 1719}{60} = 1364$ (nearest wh. No.)

$$\begin{array}{r} 9 \cdot 7723314 \\ 1364 \\ \hline 9 \cdot 7724678 \end{array}$$ *Answer.*

What is the angle whose logarithmic sine is 9·9475008?

Arguments. Log. sines.	Resultants. Angles.
$m = 9 \cdot 9474674$	$62° \, 23' \, 0'' = r$
$a = 9 \cdot 9475008$	?
$M = 9 \cdot 9475335$	$62° \, 24' \, 0'' = R$

Arguments and resultants increase together.

$M - m = 661$ $a - m = 334$ $R - r = 60$

$\dfrac{334 \times 60}{661} = 30'' \cdot 3$ (nearest tenth)

Answer 62° 23′ 30″ ·3

What is the logarithm of the cosine of 57° 5′ 9″ ·8?

Arguments. Angles.	Resultants. Log cosines.
$m = 57° \, 5' \, 0''$	$9 \cdot 7351345 = r$
$a = 57° \, 5' \, 9'' \cdot 8$	?
$M = 57° \, 6' \, 0''$	$9 \cdot 7349393 = R$

Arguments increase and resultants decrease.

$M - m = 60''$ $a - m = 9'' \cdot 8$ $r - R = 1952$

$\dfrac{9 \cdot 8 \times 1952}{60} = 319$ (nearest wh. No.)

$$\begin{array}{r} 9 \cdot 7351345 \\ 319 \\ \hline 9 \cdot 7351026 \end{array}$$ *Answer.*

What is the angle whose logarithmic cosine is 8·8852331 ?

Arguments. Log. cosines.	Resultants. Angles.
m = 8·8849031	85° 36′ 0″ = r
a = 8·8852331	?
M = 8·8865418	85° 35′ 0″ = R

Arguments increase and resultants decrease.

$\text{M}-m=16387 \quad a-m \; 3300 \quad r-\text{R}=60''$

$$\frac{3300 \times 60}{16387} = 12''\cdot 1 \text{ (nearest tenth)}$$

$$85^\circ\, 36' - 12''\cdot 1 = 85^\circ\, 35'\, 47''\cdot 9$$

The accuracy of the preceding method depends upon the tabular differences continuing the same, or nearly the same, for the resultants of several successive arguments. This is the case for the most part throughout the tables; but where it is not so, a small error is committed. For instance, in the last example, if we look at the table, we find—

Log. cos. 85° 34′ − log. cos. 85° 35′ = ·0016325
Log. cos. 85° 35′ − log. cos. 85° 36′ = ·0016387
Log. cos. 85° 36′ − log. cos. 85° 37′ = ·0016450

An error of some tenths of a second in the answer is the consequence. In the tables of sines and tangents of angles under 2°, or of cosines of angles above 88°, the disparity of the differences is so great, that it is useless to apply the above process; and it is usual, therefore, to give a separate table of sines and tangents for the first two degrees, in which the angles increase by seconds.

In all other applications of the trigonometrical tables to calculation of common algebraical formulæ, no very great advantage is gained, where much accuracy is required, by any tables which give the logarithms to minutes only. The reason is the length of the processes of interpolation. Where extreme accuracy is not required, the common tables, which go to minutes, are often advantageous, as in the following instance :—

To find $\sqrt{a^2+b^2}$, let $\tan \theta = \dfrac{b}{a}$ then

$$\sqrt{a^2+b^2} = \frac{a}{\cos\theta}.$$

Example. What is

$$\sqrt{(92\cdot 736)^2 + (64\cdot 018)^2}\,?$$

Log. 64·018	= 1·8062343
Log. 92·736	= 1·9672484
Log. tan. 34° 37′	9·8389859 *
	1·9672484
Log. cos. 34° 37′	$\bar{1}$·9153846 †
Ans. 112·68	2·0518638

The more exact process, beginning from the third line of the above, is the following :—

Log. tan. 34° 36′ 51″·8	9·8389859
	1·9672484
Log. cos. 34° 36′ 51″·8	$\bar{1}$·9153965
Ans. 112·6813	2·0518519

The following method of verifying any term in a table may be useful when an error is suspected, and no other table is at hand for comparison.

Let z be the suspected term, and let A, B, C, &c., be those which precede, and a, b, c, &c. those which follow; so that the table proceeds thus :

$$\cdots \text{D, C, B, A}, z, a, b, c, d \cdots$$

When the tabular differences appear uniform in the neighbourhood of the term z, use the first of the formulæ at the head of next page; but where this is not the case, count the number of places of the differences which alter at each step, and use the formula opposite to that number in the next page.

For instance, suppose we wish to verify the logarithm of the sine of 1° 12′, we have—

A = 8·3149536	B = 8·3087941
a = 8·3270163	b = 8·3329243
16·6419699	16·6417184
15	6
249·6295485	99·8503104
16·6412989	
266·2708474	C = 8·3025460
99·8503104	c = 8·3387529
20) 166·4205370	16·6412989
8·32102685	
8·3210269 in the tables.	

* 9 is written for $\bar{1}$, that this result may be too great by 10, as are all the logarithms in the table of tangents.

† It is more convenient here to restore the real characteristic.

0.	$z = \frac{1}{2} \quad (A + a)$
1, 2	$z = \frac{1}{6}\{4 \ (A + a) - \quad (B + b)\}$
3, 4	$z = \frac{1}{20}\{15\ (A + a) - 6 \ (B + b) + C + c\}$
5, 6	$z = \frac{1}{70}\{56\ (A + a) - 28\ (B + b) + 8\ (C + c) - (D + d)\}$

The trigonometrical tables may be made to furnish a solution of equations of the second degree*, in cases where the coefficients are too complicated to admit of easy multiplication and division. The rules are as follows:—

$$\left.\begin{array}{l} 1\ ax^2 + bx + c = 0 \\ 2\ ax^2 - bx + c = 0 \end{array}\right\} \text{Let } \sin\theta = \frac{2\sqrt{ac}}{b}$$

$$\left.\begin{array}{l} 3\ ax^2 + bx - c = 0 \\ 4\ ax^2 - bx - c = 0 \end{array}\right\} \text{Let } \tan\theta = \frac{2\sqrt{ac}}{b}$$

Then the two *numerical* values of x will be

$$\frac{\sqrt{ac}}{a}\tan\frac{\theta}{2} \quad \text{and} \quad \frac{\sqrt{ac}}{a}\cot\frac{\theta}{2}$$

and the signs will be as follows:—

1. Both negative.
2. Both positive.
3. Greater numerical root negative.
4. Lesser numerical root negative.

The instance on the opposite side will admit of easy verification in the common way.

The process which we have put in brackets is one of those little artifices of calculation, which occur in hundreds, but which it is impossible to reduce to rule, and which nothing but practice can teach. We have got the first root by means of the

$$3x^2 - 8x + 4 = 0$$

Log. a	$0{\cdot}4771213$
Log. c	$0{\cdot}6020600$
Log. ac	$2)\overline{1{\cdot}0791813}$
Log. $\sqrt{ac}$	${\cdot}5395907$
Log. 2	${\cdot}3010300$
Ar. Co. log. b	$9{\cdot}0969100 - 10$
Log. sin $60^\circ\ 0'\ 0''{\cdot}0$	$9{\cdot}9375307 - 10$

$$\theta = 60^\circ\ 0'\ 0''{\cdot}0$$
$$\tfrac{1}{2}\theta = 30^\circ\ 0'\ 0''{\cdot}0$$

Log. tan $\frac{1}{2}\theta$ =	$9{\cdot}7614394 - 10$
Log. $\sqrt{ac}$ =	${\cdot}5395907$
Ar. co. log. a =	$9{\cdot}5228787 - 10$
${\cdot}6666667$	$19{\cdot}8239088 - 20$†
[2 Log. tan $\frac{1}{2}\theta$	$19{\cdot}5228788 - 20$
2	$0{\cdot}3010300$]

formula $\frac{\sqrt{ac}}{a}\tan\frac{1}{2}\theta$; and which is ${\cdot}6666667$, as far as seven places of decimals. [In fact, the real root is $\frac{2}{3}$.] It remains to find the second root from the formula $\frac{\sqrt{ac}}{a}\cot\frac{1}{2}\theta$, so that, in like manner as we found the first root from the calculation of

$$\text{Log. tan.} \tfrac{1}{2}\theta + \text{log.} \sqrt{ac} + \text{Ar. co. log.}\ a - 10$$

we should proceed to find the second from

$$\text{Log. cot.} \tfrac{1}{2}\theta + \text{log.} \sqrt{ac} + \text{Ar. co. log.}\ a - 10$$

But since the tangent and cotangent of the same angle are reciprocals, or

$$\cot. \tfrac{1}{2}\theta = \frac{1}{\tan. \frac{1}{2}\theta}$$

we have

$$\text{log. cot.} \tfrac{1}{2}\theta = -\text{log. tan.} \tfrac{1}{2}\theta$$

consequently the second formula is the same as

$$\text{log.} \sqrt{ac} + \text{Ar. co. log.}\ a - 10 - \text{log. tan} \tfrac{1}{2}\theta$$

which, calling P the result of the first, is

$$P - 2\ \text{log. tan.} \tfrac{1}{2}\theta$$

and it is thus that we have proceeded.

The preceding example, given merely for the sake of illustrating the rule, is one which might have been more easily done by the usual method. But the following is a case in which much trouble will be saved by the trigonometrical method.

$$\overset{a}{1{\cdot}0082}\,x^2 + \overset{b}{6{\cdot}4347}\,x - \overset{c}{2{\cdot}4566} = 0$$

* In this process we suppose the student to understand algebra, as far as the solution of equations of the second degree.

† 19—20 gives the characteristic $\overline{1}$.

Log. a	0·0035467
Log. c	0·3903344
Log. ac	2) 0·3938811
Log. $\sqrt{ac}$	0·1969406
Log. 2	0·3010300
Ar. co. log. b	9·1914717−10
Log. tan. 26° 3′ 56″1·	9·6894423−10
θ =	26° 3′ 56″ ·1
$\frac{1}{2}\theta$ =	13° 1′ 58″·1
Log. tan. $\frac{1}{2}\theta$	9·3644973−10
Log. $\sqrt{ac}$	0·1969406
Ar. co. log. a	9·9964533−10
·3613193	19·5578912−20
	18·7289946−20
6·743675	0·8288966

The numericsl roots being therefore ·3613193 and 6·743675, we find, by the rule of signs before given, that the real roots are

·3613193 and −6·743675.

To form examples of this method, let the student proceed as follows: choose any two numbers for roots; for instance, 1·308 and −·486, and any valuc of a, for instance, 2·709. Take the algebraical sum and product of the roots; that is

$$1·308 - \quad ·406 = \quad ·902$$
$$1·308 \times -·486 = -·635688$$

Multiply these by 2·709, the chosen value of a, giving 2·443518 and −1·722079. Change the sign of the first; then the roots of

$$2·709\, x^2 - 2·443518\, x - 1·722079$$

should be 1·308 and −·486.

The only way to obtain any security in the use of logarithms is to work many examples, beginning with simple cases. Repeated failures will take place at first; but the student will finally acquire that sort of habit which suggests the right method independently of rules.

We shall now proceed to examples of algebraical operations.

Section 10. *Division of Algebraical Operations. Algebraical Reduction, Addition, and Subtraction of Integral Quantities.*

The operations of algebra may be divided into two classes:—1. Those which present no forms distinct from the results of ordinary arithmetic. 2. Those which require the use of the negative or other symbol in the manner peculiar to algebra.

The operations of algebra may be subdivided again as follows:—1. Those which are usually left to the student in the higher parts of the subject. 2. Those which are usually inserted at length. Among the infinite number of classes of examples which might be chosen, we shall confine ourselves to those which it is most necessary the student should be able to perform, in order to read any work on the higher parts of algebra, or on the differential calculus. We suppose the student to gain the demonstrations of the rules here exemplified from some other source.

I. *Questions illustrative of the meaning of the fundamental symbols + and −, to be answered without rules.*

1. By how much does $a + b$ exceed a? By how much does $a + b$ exceed $a - 1$? By how much does $a + b$ exceed $a - m$, $a - b$, and $a - 2b$?

2. On what does it depend whether $a + b$ or $a + c$ is the greater? When $a + b$ is greater than $a + c$, by how much does the first exceed the second? By how much does $a + 2b$ exceed $a + b$?

3. On what does it depend whether $a + b - c$ is greater than, equal to, or less than a? By how much does $a + b - c$ fall short of $a + b + 1$, and of $a + b + c$?

4. On what does it depend whether $a - b$ is greater than, equal to, or less than $a - c$?

5. If a lie between b and c, c being the greater, what must a be in order that $c - a$ and $b + a$ may be equal?

6. If a be greater than b, and x less than y, which of the following must be true, which must be false, and which may be either true or false?

$a + x$	is greater than	$b + y$
$a + y$	. . .	$b + x$
$a - x$	. . .	$b - y$
$a + x$	is less than	$b + y$
$a + y$	. . .	$b + x$

II.—*On the Use of Brackets.* Distinguish between the meanings of—

$a + (b + c)$	and	$a + b + c$
$a - (b + c)$	and	$a - b + c$
$a - (b - c)$	and	$a - b - c$
$(a - b) - c$	and	$a - (b - c)$.
$2(a + b)$	and	$2a + b$.

$$\begin{cases} a - \{b - (c - d)\} \\ a - b - c - d. \\ a - (b - c) - d \text{ and} \\ a - (b - c - d). \end{cases}$$

III.—*Simple Reductions.* How is an expression altered if $8a$, and afterwards $5a$ be added to it? *Answer,* $13a$ is added to it, which is thus expressed—

$$P + 8a + 5a = P + 13a$$

Establish the following, and express the question asked, in words, as above.

$$P + a + a = P + 2a$$
$$P + a - a = P$$
$$P - a + a = P$$
$$P + 2a - a = P + a$$
$$P - 3a + 4a = P + a$$
$$P - 16a - 3a = P - 19a$$
$$P + 16a - 20a = P - 4a$$
$$P - 21a + a = P - 20a$$

Give all the ways of expressing $P + 3a$ derived from the preceding.

$P + a + a + a$	$P - a + 4a$
$P + 2a + a$	$P - 2a + 5a$
$P + a + 2a$	$P - 3a + 6a$ &c.

$P + 4a - a$	$4a + (P - a)$
$P + 5a - 2a$	$5a + (P - 2a)$
$P + 6a - 3a$	$6a + (P - 3a)$
&c.	&c.
$4a - (a - P)$	$5a - (2a - P)$ &c.

In the last two sets of the preceding, point out the cases in which they are possible, or impossible, and why. Repeat the whole process with $P - a$, $P + 6a$, &c.

IV.—*General expressions derived from the preceding.*

$$P + ma + na = P + (m + n)a$$
$$P + ma - na = P + (m - n)a$$
$$\text{or } = P - (n - m)a$$

$$P - ma + na = P - (m - n)a$$
$$\text{or } = P + (n - m)a$$

V.—*Simple Reductions including Fractions.*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

$$\frac{1}{2}a + \frac{1}{3}a + \frac{1}{4}a = \frac{13}{12}a \text{ or}$$

$$\frac{a}{2} + \frac{a}{3} + \frac{a}{4} = \frac{13a}{12}$$

$$x + 6x - \frac{1}{2}x + 2x = 8\tfrac{1}{2}x = \frac{17x}{2}$$

$$3ab - ab + \frac{6}{7}ab = \frac{20ab}{7}$$

$$x^3 + 2x^3 + 3x^3 - \tfrac{1}{2}x^3 = \frac{11}{2}x^3$$

$$xyz - \frac{1}{2}xyz + \frac{1}{3}xyz = \frac{5}{6}xyz$$

VI.—*Reductions in cases where the arrangement of the terms makes the operations appear impossible.*

$a - 2a$ is impossible, but $a - 2a + 3a$ is to be considered as a misplacement of $a + 3a - 2a$, which is $2a$.

$$6a - 10a + 4a = 0$$
$$2a - 9a - 3a + 16a = 6a$$

$$xy - 4xy + 10xy = 7xy$$
$$\tfrac{1}{2}p^3 - p^3 + \tfrac{3}{4}p^3 = \tfrac{1}{4}p^3$$
$$a^2c - 3a^2c + 100a^2c = 98a^2c$$
$$m + 4m - 11\tfrac{1}{2}m + 12m = \frac{11}{2}m$$

VII.—*Generalization of the two preceding Articles.*

$$m a + n a = (m + n) a$$
$$m a - n a = (m - n) a$$
$$m a + n a + p a = (m + n + p) a$$
$$m x + n x - p x = (m + n - p) x$$
$$\frac{m x}{n} + \frac{p x}{q} - \frac{r x}{s} = \left(\frac{m}{n} + \frac{p}{q} - \frac{r}{s}\right) x$$

VIII.—*Additions in which subsequent Reduction is impossible.*

Add together $a + b$, $c - d$, $e - f$, and $a b - c f$. *Answer*, $a + b + c - d + e - f + a b - c f$.

Add together $a - b + c$, $a c - 1 + f$, $c f + 6 d$, and $p + q^2$. *Answer*, $a - b + c + a c - 1 + f + c f + 6 d + p + q^2$.

IX.—*Additions in which subsequent Reduction is possible.*

Add $a - b$, $a - c$, $c - 7$, and $c + 6$. *Answer*, $a - b + a - c + c - 7 + c + 6$, or $2 a - b + c - 1$.

Add $a + b$ and $a - b$. *Answer*, $a + b + a - b$, or $2 a$.

Add

$$\begin{array}{r} a b c - a + x y + 12 - p^2 - q^2 \\ 2 a b c - 4 p^2 - x y - q^2 + 20 + a \\ 10 p^2 - 100 + 40 a - 14 x y \\ 4 q^2 - 4 p^2 + a b c \\ \hline 4 a b c + 40 a - 14 x y - 68 + p^2 + 2 q^2 \end{array}$$

Add

$$\begin{array}{r} 13 x^3 + 20 x^2 - 45 x + \frac{1}{2} \\ \frac{1}{2} x^3 - x^2 + 50 x + 3\frac{1}{4} \\ 30 x^2 - 5 x^3 + 15 x - 8 \\ 2 x - x^2 + x^3 - 10 \\ \hline 9\frac{1}{2} x^3 + 48 x^2 + 22 x - 14\frac{1}{4} \end{array}$$

Further examples are deferred for the present.

X.—*Rule of Addition.*

Let the first term of each expression be considered as having the sign +

Annex all the expressions with their signs.

Make such reductions as are practicable.

Rule of Subtration.

Let the first terms of both expressions have the sign +

Change the signs in the expression which is to be subtracted.

Annex the expressions, and make reductions.

XI.—*Subtractions in which Reduction is impracticable.*

From a take $b - c$. *Ans.* $a - b + c$.

From $2 a - c$ take $4 b - d$. *Ans.* $2 a + d - c - 4 b$.

From $p - q + r - s$ take $z - p q$. *Ans.* $p - q + r - s - z + p q$.

XII. *Subtractions in which Reduction is practicable.*

From a take $a - c$. *Ans.* $a - a + c$, or c.

From $a + b$ take $a - b$. *Ans.* $a + b - a + b$, or $2 b$.

From $p - a$ take $q - a$. *Ans.* $p - a - q + a$, or $p - q$.

$$\begin{array}{lr} \text{From} & a b + 3 a - 4 b + 16 - z \\ \text{Take} & 6 a b - 12 a - 5 b + 12 z - 8 \\ \hline \textit{Ans.} & 15 a - 5 a b + b + 24 - 13 z \end{array}$$

$$\begin{array}{lr} \text{From} & 6 a^2 b - 12 a + \frac{1}{2} - 3 z - 22 a b \\ \text{Take} & a^2 b + 100 - 40 a b + a - 3 z \\ \hline \textit{Ans.} & 5 a^2 b - 13 a - 99\frac{1}{2} + 18 a b \end{array}$$

$$\begin{array}{lr} \text{From} & 3 x^3 - 18 x^2 + 6 x - 150 \\ \text{Take} & 12 x^3 - 40 x^2 + 5 x + 40 \\ \hline \textit{Ans.} & 22 x^2 - 9 x^3 + x - 190 \end{array}$$

$$\begin{array}{lr} \text{From} & a - b + c - d + e - f + g - h \\ \text{Take} & a - 2 b + 3 c + 4 d - 5 e + f + g - 8 h \\ \hline \textit{Ans.} & b - 2 c - 5 d + 6 e - 2 f + 7 h \end{array}$$

$$(\tfrac{1}{2} a + b) - (a + \tfrac{1}{2} b) = \tfrac{1}{2} b - \tfrac{1}{2} a$$
$$(\tfrac{1}{2} a - b) - (a - b) + c = c - \tfrac{1}{2} a$$
$$(2 a + b) - (a - 3 b) = a + 4 b$$

Let there be any series of quantities, a, b, c, d, e, f, &c., and let another series be obtained by taking the first from the second, the second from the third, the third from the fourth, and so on. Let this process be repeated with the second series, giving a third, with the third series, giving a fourth, and so on. What will be the several series ?

1st Series.	2nd Series.	3rd Series.
a	$b - a$	$c - 2b + a$
b	$c - b$	$d - 2c + b$
c	$d - c$	$e - 2d + c$
d	$e - d$	$f - 2e + d$
e	$f - e$	$g - 2f + e$
f	$g - f$	$h - 2g + f$
&c.	&c.	&c.

4th Series.	5th Series.
$d - 3c + 3b - a$	$e - 4d + 6c - 4b + a$
$e - 3d + 3c - b$	$f - 4e + 6d - 4c + b$
$f - 3e + 3d - c$	$g - 4f + 6e - 4d + c$
$g - 3f + 3e - d$	$h - 4g + 6f - 4c + d$
&c.	&c.

1st term of the Series.	
6	$f - 5e + 10d - 10c + 5b - a$
7	$g - 6f + 15e - 20d + 15c - 6b + a$
&c.	&c.

One way to form results which shall give examples of addition and subtraction combined, is as follows:—Take any number of quantities at pleasure, subtract each one from that which follows, add all the differences *and the first of the quantities* together; the result should be the *last of the quantities.*

For instance, let the quantities be $a - b$, $2a + b - c$, $3a + 2b - 4c$, and $7a + 6b$. From the second take the first, which gives $a + 2b - c$; from the third take the second which gives $a + b - 3c$; from the fourth take the third, which gives $4a + 4b + 4c$. Add together the following—

$$\begin{array}{l} a - b \\ a + 2b - c \\ a + b - 3c \\ \underline{4a + 4b + 4c} \end{array}$$

which gives $7a + 6b$, the last of the quantities first mentioned.

As questions of mere addition and subtraction are of little use by themselves, we shall now proceed to another point.

SECTION 11. *Exercises in the use of the Algebraical symbols + and − as distinguished from Arithmetical* symbols.*

In whatever way the symbols $+a$ and $-a$ may be explained, the method of using them is as follows:—

$+(+a)$ is $+a$ $\quad +(-a)$ is $-a$
$-(-a)$ is $+a$ $\quad -(+a)$ is $-a$
or, *like signs produce* +, *unlike signs* −.

$$a + (-b) = a - b = -b + (+a)$$
$$a - (-b) = a + b = b - (-a)$$
$$+\{+\{+a\}\} = +a \qquad -\{-\{-a\}\} = -a$$
$$a - b = -(b - a) = -b - (-a)$$
$$1 - 2 = -1 \qquad 2 - 3 = -1 \qquad a - 2a = -a$$
$$6 - 15 = -9 \qquad 4 - 7 - 8 = -11$$
$$x - (x + y) = x - x - y = -y$$
$$(2a + 36) - (4a + 76) = -(2a + 40)$$
$$a - b + 3c - d = -(b + d - a - 3c)$$
$$-a - 4a - 7a - 12a = -24a$$
$$3a - 7b - 4a + 5b = -a - 2b$$
$$+a \times +b = +ab \qquad +a \times -b = -ab$$
$$-a \times -b = +ab \qquad -a \times +b = -ab$$
$$+a \times +b \times -c \times +d = -abcd$$
$$-a \times -b \times -c \times +d = -abcd$$

* The student may omit this section until he has read an explanation of the negative sign.

$$-3 \times -4 = +12 \qquad -3 \times +4 = -12$$
$$ab \times -c = ac \times -b = bc \times -a$$
$$\frac{a}{b} = \frac{a \times -1}{b \times -1} = \frac{-a}{-b}, \text{ If } a = -b,\ b = -a$$
$$\frac{p-q}{x-y} = \frac{-(p-q)}{-(x-y)} = \frac{q-p}{y-x}$$
$$\frac{+a}{+b} = +\frac{a}{b} \qquad \frac{-a}{+b} = -\frac{a}{b}$$
$$\frac{-a}{-b} = +\frac{a}{b} \qquad \frac{+b}{-b} = -\frac{a}{b}$$
$$\frac{p-q}{x-y} = \frac{-(q-p)}{x-y} = -\frac{q-p}{x-y}$$
$$\frac{p-q}{x-y} = \frac{p-q}{-(y-x)} = -\frac{p-q}{y-x}$$
$$a - b + c = a - (b - c) = a + (c - b)$$
$$= -(b - a - c) = -(b - a) + c = -(b - c) + a$$
$$= -(-a + b - c) = a + (-b) + c$$

What suppositions will make the following expressions identically the same?

$$ax^2 + bxy + cy^2 + dy + e \qquad ax - by + cz$$
$$3x^2 - 4xy - 2y^2 + 7y - 6 \qquad 3x + y + z$$

Ans. $a = 3 \quad b = -4 \quad c = -2 \quad d = 7 \quad e = -6;$ | $a = 3 \quad b = -1 \quad c = 1$

and the following—

$$px^2 + qx + r$$
$$\tfrac{1}{2}x^2 - x + 1$$

Ans. $p = \frac{1}{2} \quad q = -1 \quad r = 1$

$$\text{If } x = -y \qquad x^4 = y^4 \qquad x^7 = -x^7$$
$$x^2 = y^2 \qquad x^5 = -y^5 \qquad x^8 = y^8$$
$$x^3 = -y^3 \qquad x^6 = y^6 \qquad x^9 = -y^9$$

Which of the following pairs are the same, and which differ in sign?

$$-x^8 \text{ and } (-x)^8 \qquad -x^{13} \text{ and } (-x)^{13}$$
$$x^6 \text{ and } (-x)^6 \qquad x^7 \text{ and } (-x)^7$$
$$x^3 \times (-y)^4 \times (-z)^5 \text{ and } (-x)^3 \times (-y)^4 \times z^5$$

Change the sign of x in the following expression, that is, write $-x$ instead of $+x$, and $+x$ instead of $-x$.

$$a + bx + cx^2 + dx^3 - \frac{1}{x}$$

it becomes $a + b(-x) + c(-x)^2 + d(-x)^3 - \dfrac{1}{-x}$

or, $a - bx + cx^2 - dx^3 + \dfrac{1}{x}$

Of the following expressions, the second in each set arising from changing the sign only of some letter in the first, let the student find out which letter it is, and account for every change.

1. $$a + bx + \frac{1-x}{1+x} - x^2(1 + x^3)$$
$$a - bx + \frac{1+x}{1-x} - x^2(1 - x^3)$$

2. $$ax(x - x^3) + bx^2(x^2 - x^3) + \frac{x(1-x)}{b + x^2}$$

$$a\,x\,(x - x^3) + b\,x^2\,(x^2 + x^3) - \frac{x\,(1 + x)}{b + x^2}$$

3. $$(2\,a\,x + b)^2 + 4\,a\,c - b^2 + \frac{x - a\,x}{1 + a^2}\left\{x + a\,y\right\}$$

$$(2\,a\,x - b)^2 - 4\,a\,c - b^2 + \frac{x + a\,x}{1 + a^2}\left\{x - a\,y\right\}$$

4. $$a - b\,x + c\,x^2 - d\,x^3 + \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}$$

$$a + b\,x + c\,x^2 + d\,x^3 - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3}$$

When the signs of two or more letters are changed, the general rule is, every term changes sign in which an *odd* number of factors changes sign. Thus, in

$$a^3 + a\,b\,c - a\,c^2\,b - b^2\,c^3$$

if b and c both change sign, that is, if $-b$ be written for b, and $-c$ for c, &c., the term which changes sign is $-a\,c^2\,b$, or, $a\,c\,c\,b$, (in which an odd number of factors, c, c, b, change their signs) and the expression becomes

$$a^3 + a\,b\,c + a\,c^2\,b - b^2\,c^3$$

In the following examples one *or more* letters have their signs changed in the second and succeeding ones of each. The student must ascertain which letters they are.

1. $$\frac{a + b}{a - b} + \frac{a - a^2}{b - a\,b} + a\,b^2 c - a\,x\,y$$

$$-\frac{b - a}{a + b} - \frac{a + a^2}{b + a\,b} + a\,b^2\,c + a\,x\,y$$

2. $$\frac{a - b + c}{a^2 - b^2 + c^2}\,,\quad \frac{a + b - c}{a^2 - b^2 + c^2}\,,\quad \frac{a - b + c}{b^2 - a^2 - c^2}$$

Observe, that no even power of a ($a^2\,a^4\,a^6$ &c.) changes sign when a changes sign; how then can the first expression become the third by changes of sign only?

3. $$(p + q)\quad (q + r)\quad (r + p)\quad ,\quad (p + q)\quad (q - r)\quad (r - p)$$

$$-(p + q)\quad (q + r)\quad (r + p)\quad ,\quad (p - q)\quad (q + r)\quad (p - r)$$

4. $$(b^2 - 4\,a\,c)\quad (a - b)\quad (b - c)\quad ,\quad -(b^2 + 4\,a\,c)\quad (a + b)\quad (b - c)$$

$$(b^2 + 4\,a\,c)\quad (a + b)\quad (c - b)\quad ,\quad -(b^2 - 4\,a\,c)\quad (a + b)\quad (b + c)$$

SECTION 12. *Multiplication and Division of Single terms.*

$$a\,b = b\,a = \frac{a\,b\,c}{c} = \frac{a^2\,b}{a} = \frac{a\,b^2}{b} = \frac{a^2\,b^2}{a\,b}$$

$$a\,b \times a\,c = a^2\,b\,c \qquad 3\,a\,b \times 6\,a\,c = 18\,a^2\,b\,c$$

$$p^2\,q^3 \times q^7 = p^2\,q^{10} \qquad a \times a \times a^2 = a^4$$

$$a^5 \times a^7 = a^{12} \qquad a^m \times a^n = a^{m+n}$$

$$2\,(a + b) = 2\,a + 2\,b \qquad 3(a - b) = 3\,a - 3\,b$$

$$2\,a\,(a + b + c - d) = 2\,a^2 + 2\,a\,b + 2\,a\,c - 2\,a\,d$$

$$3\,a\,b\,(4\,a - a\,b + x) = 12\,a^2\,b - 3\,a^2\,b^2 + 3\,a\,b\,x$$

$$\frac{1}{2}\left(\frac{1}{2}\,a - 2\,b\right) = \frac{1}{4}\,a - b \qquad \frac{2}{3}\left(\frac{3}{4}\,a - \frac{1}{5}\,a\,b\right) = \frac{1}{2}\,a - \frac{2}{15}\,a\,b$$

$$4\,a\,b\,c\left(2\,a\,b + \frac{3}{2}\,b\,c - 6\,a\,c\right) = 8\,a^2\,b^2\,c + 6\,a\,b^2\,c^2 - 24\,a^2\,b\,c^2$$

$$3 a^2 z (z^3 y - a^3 z + z - 1) = 3 a^2 z^4 y - 3 a^5 z^2 + 3 a^2 z^2 - 3 a^2 z$$

$$\frac{x}{y}\left(\frac{a}{b} + \frac{c}{d} + \frac{e}{f}\right) = \frac{x}{y} \times \frac{a}{b} + \frac{x}{y} \times \frac{c}{d} + \frac{x}{y} \times \frac{e}{f}$$

$$= \frac{a x}{b y} + \frac{c x}{d x} + \frac{e x}{f y}$$

$$\frac{2 m}{n}\left(3 a b + \frac{m}{n}\right) = \frac{6 a b m}{n} + \frac{2 m^2}{n^2}$$

$$\frac{3 a^2}{a} = 3 a \qquad \frac{14 a^2 b}{2 a b} = 7 a \qquad \frac{210 a^{10} b^{15}}{70 a^5 b^4} = 3 a^5 b^{11}$$

$$\frac{2 a b}{2 a c} = \frac{b}{c} \qquad \frac{x y}{z z} = \frac{y}{z} \qquad \frac{x^2}{x z} = \frac{x}{z} \qquad \frac{y^2}{y^3 x} = \frac{1}{y x}$$

$$\frac{4 a^2 b}{8 a^3 c} = \frac{b}{2 a c} \qquad \frac{12 a}{8 b} = \frac{3 a}{2 b} \qquad \frac{13 a}{13 a^2} = \frac{1}{a} \qquad \frac{a}{a} = 1$$

$$\frac{6 a b c}{9 a^2 b^2} = \frac{2 c}{3 a b} \qquad \frac{x y^2 z^3 v^2}{x^2 y^2 z^2 v^2} = \frac{z}{x} \qquad \frac{2 a b c^2 m}{8 a^2 b c^3 m^6} = \frac{1}{4 a c m^5}$$

$$\frac{v t}{2 m^2 v t} = \frac{1}{2 m^2} \qquad \frac{p^7 q^{14}}{3 p^4 q^{18}} = \frac{p^3}{3 q^4} \qquad \frac{m^2 a b^2}{m a b^3} = \frac{m}{b}$$

$$\frac{x y (x + y)^2}{x^2 (x + y)^3} = \frac{y}{x (x + y)} \qquad \frac{(p - q)^4 z}{f x (p - q)^2} = \frac{(p - q)^2 z}{f x}$$

$$\frac{m a + n a}{a} = \frac{m a}{a} + \frac{n a}{a} = m + n$$

$$\frac{m a^2 - n a}{a} = \frac{m a^2}{a} - \frac{n a}{a} = m a - n$$

$$\frac{6 a v^2 + 3 a v^3 - 12 a^3 v}{3 a v} = 2 v + v^2 - 4 a^2$$

$$\frac{12 x^4 - 9 x^3 + 3 x^2 - 3 a x^2}{3 x^2} = 4 x^2 - 3 x + 1 - a$$

$$\frac{2 m n^2 + 8 a m n - 6 a^2 m^2 n^2 - 2 m n}{2 m n} = n + 4 a - 3 a^2 m n - 1$$

$$\frac{2 a + 2}{4} = \frac{a + 1}{2} \qquad \frac{3 x^2 - 6 x}{3 a x} = \frac{x - 2}{a} \qquad \frac{3}{3 + 3 a} = \frac{1}{1 + a}$$

$$\frac{p^2 + 2 p x}{p^2 + 2 p y} = \frac{p + 2 x}{p + 2 y} \qquad \frac{a^2 b - a b^2}{a^2 b^2 + a b} = \frac{a - b}{a b + 1}$$

$$\frac{y^2 - y + 3 a v y}{y - v y + 2 a m y} = \frac{y - 1 + 3 a v}{1 - v + 2 a m} \qquad \frac{x^2}{x y + x} = \frac{x}{y + 1}$$

$$\frac{a x^4 y^3 - a^2 x^3 y^2 + a^3 x^2 y}{2 a x^5 y^4 + 3 a^2 x^4 y^3 - 2 a^3 x^3 y^2} = \frac{x^3 y^2 - a x^2 y + a^2 x}{2 x^4 y^3 + 3 a x^3 y^2 - 2 a^2 x^2 y}$$

$$\frac{\frac{1}{2} a + \frac{1}{3} b}{2 a - \frac{1}{4} b} = \frac{(\frac{1}{2} a + \frac{1}{3} b) 12}{(2 a - \frac{1}{4} b) 12} = \frac{6 a + 4 b}{24 a - 3 b}$$

$$\frac{a + \frac{b}{2}}{b + \frac{c}{2}} - \frac{\frac{1}{3} a}{\frac{1}{8} b} = \frac{2 a + b}{2 b + c} - \frac{8 a}{3 b}$$

$$\frac{a+\frac{b}{c}}{a-\frac{b}{c}}=\frac{ac+b}{ac-b} \qquad \frac{2ax+\frac{1}{2}}{\frac{x}{a}-\frac{1}{2}}=\frac{4a^2x+a}{2x-a}$$

$$\frac{\frac{1}{1+x}-x}{1-\frac{x}{1+x}}=\frac{1-x-x^2}{1+x-x}=1-x-x^2$$

SECTION 13. *Separation of Factors.*

$$a+b=a\left(1+\frac{b}{a}\right)=b\left(\frac{a}{b}+1\right)=ab\left(\frac{1}{b}+\frac{1}{a}\right)$$

$$=c\left(\frac{a}{c}+\frac{b}{c}\right)=\frac{1}{c}\left(ac+bc\right)=\frac{a}{b}\left(b+\frac{b^2}{a}\right)$$

When any expression A is to be put in a form of which m shall be a factor, then $\frac{A}{m}$ must be the other factor: for

$$A=m\times\frac{A}{m}$$

$$b^2-4ac=b^2\left(1-\frac{4ac}{b^2}\right)=4\left(\frac{b^2}{4}-ac\right)=\frac{a}{c}\left(\frac{cb^2}{a}-4c^2\right)$$

$$x^2-2xy=x\left(x-2y\right)=\frac{x}{y}\left(xy-2y^2\right)=\frac{x^2}{y^2}\left(y^2-\frac{2y^3}{x}\right)$$

$$=xy\left(\frac{x}{y}-2\right)=\frac{y^2}{x^2}\left(x^4-\frac{2x^3}{y}\right)=zx\left(\frac{x}{z}-\frac{2y}{z}\right)$$

$$ax+b=a\left(x+\frac{b}{a}\right)=x\left(a+\frac{b}{x}\right)=b\left(\frac{a}{b}x+1\right)$$

$$x^2+xy+y^2=x^2\left(1+\frac{y}{x}+\frac{y^2}{x^2}\right)=y^2\left(\frac{x^2}{y^2}+\frac{x}{y}+1\right)$$

$$x^3-3x^2y+3xy^2-y^3=x^3\left\{1-3\frac{y}{x}+3\frac{y^2}{x^3}-\frac{y^3}{x^3}\right\}$$

$$(a+b)^2-c=(a+b)\left(a+b-\frac{c}{a+b}\right)=(a+b)^2\left(1+\frac{c}{(a+b)^2}\right)$$

$$\frac{x+y}{x-y}=\frac{y\left(\frac{x}{y}+1\right)}{x\left(1-\frac{y}{x}\right)}=\frac{y}{x}\,\frac{\frac{x}{y}+1}{1-\frac{y}{x}}=\frac{y}{x}\,\frac{\frac{x^2}{y}+x}{x-y}$$

$$ab+bc+ca=a\left(b+c+\frac{bc}{a}\right)=ab\left(1+\frac{c}{a}+\frac{c}{b}\right)$$

$$=abc\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)=bc\left(\frac{a}{c}+1+\frac{a}{b}\right)$$

$$x+y-z=\overline{z-y}\left(\frac{x}{z-y}-1\right)=\overline{x+y}\left(1-\frac{z}{x+y}\right)$$

$$=\overline{x+y+z}\left(1-\frac{2z}{x+y+z}\right)=\overline{z-x}\left(\frac{y}{z-x}-1\right)$$

$$p + q = \overline{p - z}\left(1 + \frac{z + q}{p - z}\right) = \overline{p + z}\left(1 - \frac{z - q}{p + z}\right)$$

$$= \overline{p + z}\left(1 + \frac{q - z}{p + z}\right) = \overline{p + q - z}\left(1 + \frac{z}{p + q - z}\right)$$

$$a^2 + 2\,a\,b = \overline{a^2 + b^2}\left(1 + \frac{2\,a - b}{a^2 + b^2}b\right) = \overline{a^2 - b^2}\left(1 + \frac{(2\,a + b)\,b}{a^2 - b^2}\right)$$

$$\begin{aligned}
a^2 + 2\,a\,b + b^2 &= a^2 + a\,b + a\,b + b^2 = (a + b)\,(a + b)\\
a^2 - 2\,a\,b + b^2 &= a^2 - a\,b - (a\,b - b^2) = (a - b)\,(a - b)\\
a^2 - b^2 &= a^2 - a\,b + a\,b - b^2 = (a - b)\,(a + b)\\
a^3 + b^3 &= a^3 + a^2\,b - a^2\,b - a b^2 + a\,b^2 + b^3\\
&= \overline{a + b}\,(a^2 - a\,b + b^2)\\
a^3 - b^3 &= a^3 - a^2\,b + a^2\,b - a\,b^2 + a\,b^2 - b^3\\
&= \overline{a - b}\,(a^2 + a\,b + b^2)
\end{aligned}$$

SECTION 14. *Examples of Multiplication.*

$$\begin{array}{r}
x + 1\\
x - 3\\
\hline
x^2 + x\\
-\,3\,x - 3\\
\hline
x^2 - 2\,x - 3
\end{array}
\qquad
\begin{array}{r}
x - 1\\
x - 3\\
\hline
x^2 - x\\
-\,3\,x + 3\\
\hline
x^2 - 4\,x + 3
\end{array}
\qquad
\begin{array}{r}
x - 1\\
x + 3\\
\hline
x^2 - x\\
+\,3\,x - 3\\
\hline
x^2 + 2\,x - 3
\end{array}$$

$$\begin{array}{l}
x^2 - 2\,a\,x + 3\,a^2\\
x^2 + 3\,b\,x - 2\,b^2\\
\hline
x^4 - 2\,a\,x^3 + 3\,a^2\,x^2\\
\quad + 3\,b\,x^3 - 6\,a\,b\,x^2 + 9\,a^2\,b\,x\\
\qquad - 2\,b^2\,x^2 + 4\,a\,b^2\,x - 6\,a^2\,b^2\\
\hline
x^4 - (2\,a - 3\,b)\,x^3 + (3\,a^2 - 6\,a\,b - 2\,b^2)\,x^2 + a\,b\,(9\,a + 4\,b)\,x - 6\,a^2\,b^2
\end{array}$$

$$\begin{array}{r}
a + b\\
a + b\\
\hline
a^2 + a\,b\\
+\,a\,b + b^2\\
\hline
a^2 + 2\,a\,b + b^2
\end{array}
\qquad
\begin{array}{r}
a - b\\
a - b\\
\hline
a^2 - a\,b\\
-\,a\,b + b^2\\
\hline
a^2 - 2\,a\,b + b^2
\end{array}
\qquad
\begin{array}{r}
a + b\\
a - b\\
\hline
a^2 + a\,b\\
-\,a\,b - b^2\\
\hline
a^2 - b^2
\end{array}$$

$$\begin{aligned}
(a + b + c)^2 &= a^2 + b^2 + c^2 + 2\,a\,b + 2\,b\,c + 2\,c\,a\\
(a + b - c)^2 &= a^2 + b^2 + c^2 + 2\,a\,b - 2\,b\,c - 2\,c\,a\\
(a - b - c)^2 &= a^2 + b^2 + c^2 - 2\,a\,b + 2\,b\,c - 2\,c\,a\\
(7\,a\,x + 12\,b\,x^2)^2 &= 49\,a^2\,x^2 + 168\,a\,b\,x^3 + 144\,b^2\,x^4
\end{aligned}$$

$$\left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2 \qquad \left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2$$

$$(a + b + c)\,(a + b - c) = a^2 + b^2 - c^2 + 2\,a\,b$$

If $P = b^2 - 4\,d\,c$, $Q = b\,d - 2\,a\,e$, $R = d^2 - 4\,a\,f$

$$Q^2 - P\,R = 4\,a\,(b^2 - 4\,a\,c)\left(\frac{c\,d^2 + a\,e^2 - b\,d\,e}{b^2 - 4\,a\,c} + f\right)$$

and $(b\,x + d)^2 - 4\,a\,(c\,x^2 + e\,x + f) = P\,x^2 + 2\,Q\,x + R$

$$(2\,a\,x + b)^2 + 4\,a\,c - b^2 = 4\,a\,(a\,x^2 + b\,x + c)$$

$$(m^2 - n^2)^2 + 4\,m^2 n^2 = (m^2 + n^2)^2$$

$$\frac{(a c - q^2)\,(a b - r^2) - (q r - a p)^2}{a} = a\,b c + 2\,p\,q\,r - a\,p^2 - b\,q^2 - c\,r^2$$

$$\text{If}\ \left.\begin{aligned} A &= b\,c - p^2 \\ B &= c\,a - q^2 \\ C &= a\,b - r^2 \end{aligned}\right\}\ \text{and}\ \left\{\begin{aligned} P &= q\,r - a\,p \\ Q &= r\,p - b\,q \\ R &= p\,q - c\,r \end{aligned}\right.$$

then the following six quantities are equal: show this by multiplication.

$$\frac{BC - P^2}{a} \quad \frac{CA - Q^2}{b} \quad \frac{AB - R^2}{c} \qquad \frac{QR - AP}{p} \quad \frac{RP - BQ}{q} \quad \frac{PQ - CR}{r}$$

We give the formation of one of them at length.

$$\begin{aligned} Q &= r p - b\,q & A &= b\,c - p^2 \\ R &= p\,q - c\,r & P &= q\,r - a\,p \\ Q\,R &= p^2 q r - p b q^2 - c p r^2 + b c q r & AP &= b c q r - p^2 q r - a b c p + a p^3 \\ A\,P &= b c q r - p^2 q r - a b c p + a p^3 & & \\ Q\,R - A\,P &= a\,b c p + 2\,p^2 q\,r - a p^3 - b p\,q^2 - c\,p\,r^2 & & \end{aligned}$$

$$\frac{Q\,R - A\,P}{p} = a\,b\,c + 2\,p\,q\,r - a\,p^2 - b\,q^2 - c\,r^2$$

$$(a^2 + b^2 + c^2)\,(p^2 + q^2 + r^2) - (a p + b\,q + c\,r)^2$$
$$= (a\,q - b\,p)^2 + (b\,r - c\,q)^2 + (c p - a\,r)^2$$
$$(a p + b\,q + c r)\,(a\,s + b\,t + c\,v) - (a^2 + b^2 + c^2)\,(p\,s + q\,t + r\,v)$$
$$= (c q - b\,r)\,(b\,v - c\,t) + (a\,r - c\,p)\,(c\,s - a\,v) + (b\,p - a\,q)\,(a\,t - b\,s)$$

$$\text{If}\quad P = a + b + c \qquad Q = a + b - c$$
$$R = b + c - a \qquad S = c + a - b$$

$$P\,Q\,R\,S = 2\,a^2 b^2 + 2\,b^2 c^2 + 2\,c^2 a^2 - a^4 - b^4 - c^4$$
$$Q\,R + R\,S + S\,Q = 2\,b\,c + 2\,c\,a + 2\,a\,b - a^2 - b^2 - c^2$$
$$P^2 + Q^2 + R^2 + S^2 = 4\,a^2 + 4\,b^2 + 4\,c^2$$
$$\begin{aligned} Q\,R^2 + RS^2 + S\,Q^2 = \quad & a^3 + 6\,a\,b\,c - 5\,a\,b^2 + 3\,b\,a^2 \\ + & b^3 \qquad\qquad - 5\,b\,c^2 + 3\,c\,b^2 \\ + & c^3 \qquad\qquad - 5\,c\,a^2 + 3\,a\,c^2 \end{aligned}$$

$$(a^2 + a\,b + b^2)\,(a - b) = a^3 - b^3$$
$$(a^3 + a^2 b + a\,b^2 + b^3)\,(a - b) = a^4 - b^4$$
$$(a^4 + a^3 b + a^2 b^2 + a\,b^3 + b^4)\,(a - b) = a^5 - b^5$$
$$(a^2 - a\,b + b^2)\,(a + b) = a^3 + b^3$$
$$(a^3 - a^2 b + a\,b^2 - b^3)\,(a + b) = a^4 - b^4$$
$$(a^4 - a^3 b + a^2 b^2 - a\,b^3 + b^4)\,(a + b) = a^5 + b^5$$
$$(x - a)\,(x - b) = x^2 - (a + b)\,x + a\,b$$
$$(x + a)\,(x + b) = x^2 + (a + b)\,x + a\,b$$

Examples of Division, when the divisor has more than one term, are of little or no use in the elementary part of algebra.

Section 15. *Operations on Fractions.*

The leading rules of arithmetic, which relate to simple fractions, are embodied in the following formulæ :—

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$$\frac{a}{b} = \frac{m\,a}{m\,b}, \qquad a + \frac{b}{c} = \frac{a\,c + b}{c}, \qquad a - \frac{b}{c} = \frac{a\,c - b}{c},$$

$$\frac{b}{c} - a = \frac{b - a\,c}{c}, \qquad \frac{a}{x} + \frac{b}{x} = \frac{a + b}{x}, \qquad \frac{a}{x} - \frac{b}{x} = \frac{a - b}{x},$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a\,d + b\,c}{b\,d}, \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{b\,d}, \qquad \frac{a}{b} \times x = \frac{a\,x}{b},$$

$$\frac{a}{b} \div x = \frac{a}{b\,x}, \qquad \frac{a}{b} \times \frac{c}{d} = \frac{a\,c}{b\,d}, \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a\,d}{c\,b},$$

$$\frac{a}{b} \times x = \frac{a}{b \div x}, \qquad \frac{a}{b} \div x = \frac{a \div x}{b}.$$

The reduction of fractions which have fractional numerators and denominators, is exemplified in the following formulæ :—

$$\frac{\frac{a}{b}}{\frac{x}{y}} = \frac{a\,y}{b\,x} \qquad \frac{\frac{a}{b} - \frac{c}{d}}{\frac{a}{d} + \frac{b}{c}} = \frac{(a\,d - b\,c)\,c}{(a\,c + b\,d)\,b} \qquad \frac{p + \frac{a}{q}}{r - \frac{b}{s}} = \frac{p\,q\,s + a\,s}{q\,r\,s - b\,q}$$

$$\frac{\frac{m}{n} - \frac{x}{y}}{\frac{p}{q} - \frac{a}{b}} = \frac{m\,b\,q\,y - b\,n\,q\,x}{b\,n\,p\,y - a\,n\,q\,y} \qquad \frac{\frac{a}{b} \times \frac{m}{n}}{\frac{p}{q} \times \frac{x}{y}} = \frac{a\,m\,q\,y}{b\,n\,p\,x}$$

$$\frac{a}{b} + \frac{m}{n} + \frac{p}{q} - \frac{x}{y} = \frac{a\,n\,q\,y + b\,m\,q\,y + b\,n\,p\,y - b\,n\,q\,x}{b\,n\,q\,y}$$

$$x + \frac{1}{x} = \frac{x^2 + 1}{x} \qquad x - \frac{1}{x} = \frac{x^2 - 1}{x} \qquad \frac{1}{x} - \frac{1}{y} = \frac{y - x}{x\,y}$$

$$\frac{a}{q\,b} + \frac{x}{c\,q} = \frac{a\,c + b\,x}{b\,c\,q} \qquad \frac{x}{a} - \frac{a}{x} = \frac{x^2 - a^2}{a\,x} \qquad \frac{1 - x}{x} \times \frac{1 + x}{x^2} = \frac{1 - x^2}{x^3}$$

$$\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} = \frac{x^2 + x - 1}{x^3} \qquad \frac{y}{x} + \frac{y^2}{x^2} - \frac{y^3}{x^3} = \frac{x^2\,y + x\,y^2 - y^3}{x^3}$$

$$\frac{a}{1 - x} + \frac{a}{x} = \frac{a}{x - x^2} \qquad \frac{p + q}{p - q} + \frac{p}{q} = \frac{p^2 + q^2}{p\,q - q^2}$$

$$\frac{a + b}{a - b} + \frac{a - b}{a + b} = \frac{2\,a^2 + 2\,b^2}{a^2 - b^2} \qquad \frac{3\,x}{x - 1} - \frac{2}{x} = \frac{3\,x^2 - 2\,x + 2}{x^2 - x}$$

$$n\frac{n - 1}{2} = \frac{n^2 - n}{2} \qquad n.\frac{n - 1}{2} \,.\, \frac{n - 2}{3} = \frac{n^3 - 3\,n^2 + 2\,n}{6}$$

$$n.\frac{n - 1}{2} \,.\, \frac{n - 2}{3}\;\frac{n - 3}{4} = \frac{n^4 - 6\,n^3 + 11\,n^2 - 6\,n}{24}$$

$$n.\frac{n + 1}{2} + n + 1 = \overline{n + 1}\,\frac{n + 2}{2}$$

$$n.\frac{n + 1}{2} \,.\, \frac{2\,n + 1}{3} + \overline{n + 1}^2 = \overline{n + 1}\,\frac{n + 2}{2} \,.\, \frac{2\,n + 3}{3}$$

$$\frac{x + a}{x + b} - \frac{x + c}{x + e} = \frac{(a + e - c - b)\,x + a\,e - b\,c}{x^2 + (b + e)\,x + e\,b}$$

$$\frac{x + a}{x - b} - \frac{x - a}{x + b} = \frac{2\,(a + b)\,x}{x^2 - b^2}$$

$$\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=\frac{3x^2+2(a+b+c)x+ab+bc+ca}{x^3+(a+b+c)x^2+(ab+bc+ca)x+abc}$$

$$\frac{ax-b}{a-bx}-\frac{x+1}{x-1}=\frac{(a+b)x^2-2ax-(a-b)}{(a+b)x-a-bx^2}$$

$$\frac{1+x}{1+\frac{1}{x}}=x \qquad \frac{\frac{1}{1-x}-\frac{1}{1+x}}{\frac{x}{1-x}+\frac{1}{1+x}}=\frac{2x}{x^2+1} \qquad \frac{3+\frac{1}{x}}{2-\frac{2}{x}}=\frac{3x+1}{2x-2}$$

We shall find further examples of these operations in the next section.

Section 16. *Solution of Equations of the first degree, or Simple Equations.*

Rule 1. Both sides of an equation may be increased, diminished, multiplied, or divided by the same quantity.

Rule 2. Any term of an equation may be removed from one side of the equation to the other, if the sign be changed from + to −, or from − to +.

Let $x-2=7-x$ What is the value of x?

$$x+x=7+2$$
$$2x=9$$
$$x=\frac{9}{2}=4\tfrac{1}{2}$$

Verification $4\tfrac{1}{2}-2=2\tfrac{1}{2}$ $\quad 7-4\tfrac{1}{2}=2\tfrac{1}{2}$

Let $x-a=b-x$

$$2x=a+b$$
$$x=\tfrac{1}{2}(a+b)$$

Verification $\tfrac{1}{2}(a+b)-a=\tfrac{1}{2}(b-a)$

$$b-\tfrac{1}{2}(a+b)=\tfrac{1}{2}(b-a)$$

Let $3x-4=12x-9$

$$9-4=12x-3x$$
$$5=9x$$
$$\frac{5}{9}=x$$

Verification $3\times\frac{5}{9}-4=-\frac{21}{9}$

$$12\times\frac{5}{9}-9=-\frac{21}{9}$$

Before reading the theory of the negative sign, the student must consider this negative result as showing that the equation should have been written

$$4-3x=9-12x$$

which gives $x=\frac{5}{9}$ and can be verified arithmetically.

Let $ax-b=cx-d$

$$ax-cx=b-d$$
$$(a-c)x=b-d$$
$$x=\frac{b-d}{a-c}$$

Verification $a\dfrac{b-d}{a-c} - b = \dfrac{ab - ad - ab + bc}{a-c}$

$$= \frac{bc - ad}{a-c}$$

$$c\frac{b-d}{a-c} - d = \frac{bc-ad}{a-c}$$

Let $\dfrac{x}{2} + \dfrac{x}{3} = 4 - \dfrac{x}{4}$

$$12\frac{x}{2} + 12\frac{x}{3} = 48 - 12\frac{x}{4}$$

$$6x + 4x = 48 - 3x$$

$$6x + 4x + 3x = 48$$

$$13x = 48 \quad x = 3\frac{9}{13} \text{ or } \frac{48}{13}$$

Verification $\dfrac{x}{2} + \dfrac{x}{3} = \dfrac{40}{13} = 4 - \dfrac{x}{4}$

Let $\dfrac{x}{a} + \dfrac{x}{b} = c - \dfrac{x}{d}$

$$abd\frac{x}{a} + abd\frac{x}{b} = abcd - abd\frac{x}{d}$$

$$bdx + adx = abcd - abx$$

$$bdx + adx + abx = abcd$$

$$(bd + ad + ab)x = abcd$$

$$x = \frac{abcd}{bd + ad + ab}$$

Verification $\dfrac{x}{a} = \dfrac{bcd}{bd + ad + ab} \qquad \dfrac{x}{b} = \dfrac{acd}{bd + ad + ab}$

$$\frac{x}{a} + \frac{x}{b} = \frac{bcd + acd}{bd + ad + ab} = c - \frac{x}{d}$$

Let $\dfrac{x-1}{3} - \dfrac{x-4}{5} = 1 - \dfrac{x}{6}$

Multiply by 30, the least common multiple of 3, 5, and 6.

$$10x - 10 - (6x - 24) = 30 - 5x$$

$$10x - 10 - 6x + 24 = 30 - 5x$$

$$9x = 16 \quad x = \frac{16}{9}$$

Verification $\dfrac{x-1}{3} = \dfrac{7}{27}, \quad \dfrac{x-4}{5} = -\dfrac{4}{9}, \quad 1 - \dfrac{x}{6} = \dfrac{19}{27}, \quad \dfrac{7}{27} - \left(-\dfrac{4}{9}\right)$

$$= \frac{7}{7} + \frac{4}{9} = \frac{19}{27}.$$

The negative sign in the verification shows that the equation should have been written

$$\frac{x-1}{3} + \frac{4-x}{5} = 1 - \frac{x}{6}$$

which gives the same value of x.

Let $\dfrac{x-a}{b} - \dfrac{x-c}{d} = e - \dfrac{x}{f}$

Multiply both sides by $b\,d\,f$.

$$d\,f\,x - a\,d\,f - b\,f(x - c) = b\,e\,d\,f - b\,d\,x$$
$$d\,f\,x - a\,d\,f - b\,f\,x + b\,f\,c = b\,e\,d\,f - b\,d\,x$$
$$d\,f\,x - b\,f\,x + b\,d\,x = b\,e\,d\,f + a\,d\,f - b\,f\,c$$
$$x = \frac{b\,e\,d\,f + a\,d\,f - b\,c\,f}{d\,f - b\,f + b\,d}$$

Verification
$$\frac{x - a}{b} = \frac{e\,d\,f - c\,f + a\,f - a\,d}{d\,f - b\,f + b\,d}$$
$$\frac{x - c}{d} = \frac{b\,e\,f + a\,f - c\,f - b\,c}{d\,f - b\,f + b\,d}$$
$$\frac{x - a}{b} - \frac{x - c}{d} = \frac{e\,d\,f - b\,e\,f - a\,d + b\,c}{d\,f - b\,f + b\,d}$$
$$= e - \frac{b\,e\,d + a\,d - b\,c}{d\,f - b\,f + b\,d}$$

Let $\dfrac{a\,x - b}{a} - \dfrac{b\,x - a}{b} = 1 + \dfrac{x}{a}$

Multiply both sides by $a\,b$.
$$a\,b\,x - b^2 - a\,b\,x + a^2 = a\,b + b\,x$$
$$b\,x = a^2 - b^2 - a\,b \qquad x = \frac{a^2}{b} - b - a$$
$$\frac{a\,x - b}{a} = \frac{a^2}{b} - b - a - \frac{b}{a}$$
$$\frac{b\,x - a}{b} = \frac{a^2}{b} - b - a - \frac{a}{b}$$
$$\frac{a\,x - b}{a} - \frac{b\,x - a}{b} = \frac{a}{b} - \frac{b}{a}$$
$$1 + \frac{x}{a} = 1 + \frac{a}{b} - \frac{b}{a} - 1 = \frac{a}{b} - \frac{b}{a}$$

Let $\dfrac{x - a}{b} + \dfrac{x - b}{a} + \dfrac{x - a\,b}{a\,b} = 1$
$$a\,x - a^2 + b\,x - b^2 + x - a\,b = a\,b$$
$$x = \frac{a^2 + 2\,a\,b + b^2}{a + b + 1}$$
$$\frac{x - a}{b} = \frac{a + b - \frac{a}{b}}{a + b + 1} \qquad \frac{x - b}{a} = \frac{a + b - \frac{b}{a}}{a + b + 1}$$
$$\frac{x - a\,b}{a\,b} = \frac{x}{a\,b} - 1 = \frac{\frac{a}{b} + 2 + \frac{b}{a}}{a + b + 1} - 1$$
$$\frac{x - a}{b} + \frac{x - b}{a} + \frac{x - a\,b}{a\,b} = \frac{2\,a + 2\,b + 2}{a + b + 1} - 1 = 1$$

In the succeeding examples, we give only the solution, or value of x, and the value which each side of the equation assumes when the value of x is substituted.

$$\text{Let} \quad x + \frac{x-a}{b} - \frac{c+x}{a\,b} = d - \frac{x-e}{a}$$

The value of x will be found to be

$$\frac{a^2 + a\,b\,d + b\,e + c}{a\,b + a + b - 1}$$

which will give

$$\frac{x-a}{b} = \frac{a\,b\,d - a^2\,b - a\,b + a + b\,e + c}{b\,(a\,b + a + b - 1)}$$

$$\frac{c+x}{a\,b} = \frac{a^2 + a\,b\,d + a\,b\,c + a\,c + b\,c + b\,e}{a\,b\,(a\,b + a + b - 1)}$$

$$\frac{x-e}{a} = \frac{a^2 + a\,b\,d - a\,b\,e - a\,e + c + e}{a\,(a\,b + a + b - 1)}$$

and the common value of the first and second sides of the equation is

$$\frac{a^2\,(b\,d + d - 1) + a\,(b\,e + e - d) - (c + e)}{a\,(a\,b + a + b - 1)}$$

$$\text{Let} \quad \frac{\frac{a}{c}\,x + a}{b} + \frac{x - \frac{2x-a}{b}}{c} = \frac{ax - a^2}{b\,c}$$

the value of x, and the value of each side of the equation, are as follows:—

$$a\,\frac{a+c+1}{2-b} \qquad \frac{a^2}{bc} \qquad \frac{a+b+c-1}{2-b}$$

$$\text{Let} \quad (a + x)\,(b + x) = (c + x)\,d + x)$$

This appears at first sight to be an equation of the second degree, but it is not so, owing to the occurrence of x^2 on both sides of the equation.

$$x = \frac{a\,b - c\,d}{c + d - a - b}$$

$$(a + x)\,(b + x) = (c + x)\,(d + x) = \frac{\overline{c-a}\,.\,\overline{c-b}\,.\,\overline{a-d}\,.\,\overline{b-d}}{c + d - a - b}$$

In verifying equations, and in algebraical operations generally, remember that addition and subtraction seldom or never take place in denominators, so long as they remain denominators, and only occur when, by the rules for fractionsl operations, denominators have been incorporated with numerators. Also remember that, except for the purpose of incorporating two expressions, the indication of the multiplication is simpler than the actual result. For instance, we form a better idea of $a + b$ taken $a + b$ times, which involves one multiplication, and two additions of the most simple character, than of its equal $a^2 + 2\,a\,b + b^2$ involving two additions and three multiplications. Hence, the most convenient plan will be, to let the indications of multiplication remain in denominators, without performing the operations, until, in the course of the process, those denominators become numerators or factors of numerators. Thus,

$$\frac{(a + b)\,(c + d)}{(e + f)\,(g + h)}$$

should be written

$$\frac{a\,c + a\,d + b\,c + b\,d}{(e + f)\,(g + h)}$$

When a denominator occurs which will be written several times in the course of a process, the better way will be to substitute a single letter for it, restoring the original denominator only when the chosen letter comes to appear in a numerator. The following example is worked at full length in every

respect, containing everything which the student would find it necessary to write :—

$$\frac{x-b}{a} - \frac{x-a}{b} = x - \frac{(a+b)\,x}{a\,b}$$

$$b\,x - b^2 - a\,(x-a) = a\,b\,x - (a+b)\,x$$

$$b\,x - b^2 - a\,x + a^2 = a\,b\,x - a\,x - b\,x$$

$$a^2 - b^2 = a\,b\,x - 2\,b\,x$$

$$x = \frac{a^2 - b^2}{a\,b - 2\,b}. \text{ Let } a\,b - 2\,b = \mathrm{P}$$

$$x - b = \frac{a^2 - b^2}{\mathrm{P}} - b = \frac{a^2 - b^2 - a\,b^2 + 2\,b^2}{\mathrm{P}} = \frac{a^2 + b^2 - a\,b^2}{\mathrm{P}}$$

$$x - a = \frac{a^2 - b^2}{\mathrm{P}} - a = \frac{a^2 - b^2 - a^2\,b + 2\,a\,b}{\mathrm{P}}$$

$$\frac{x-b}{a} - \frac{x-a}{b} = \frac{a^2 + b^2 - a\,b^2}{a\,\mathrm{P}} - \frac{a^2 - b^2 - a^2\,b + 2\,a\,b}{b\,\mathrm{P}}$$

$$= \frac{a^2\,b + b^3 - a\,b^3}{a\,b\,\mathrm{P}} - \frac{a^3 - a\,b^2 - a^3\,b + 2\,a^2\,b}{a\,b\,\mathrm{P}}$$

$$= \frac{a^2\,b + b^3 - a\,b^3 - a^3 + a\,b^2 + a^3\,b - 2\,a^2\,b}{a\,b\,\mathrm{P}}$$

$$= \frac{b^3 - a\,b^3 - a^3 + a\,b^2 + a^3\,b - a^2\,b}{a\,b\,\mathrm{P}} \quad \text{(1st side)}$$

Again, $$x - \frac{(a+b)\,x}{a\,b} = x\left(1 - \frac{a+b}{a\,b}\right) = x \,.\, \frac{a\,b - a - b}{a\,b}$$

$$= \frac{a^2 - b^2}{\mathrm{P}} \cdot \frac{a\,b - a - b}{a\,b} = \frac{a^3\,b - a\,b^3 - a^3 + a\,b^2 - a^2\,b + b^3}{a\,b\,\mathrm{P}} \quad \text{(2nd side)}$$

Multiplications should always, unless where they are more than one and complicated, be performed as in the last line of the above, without arranging the multiplier and multiplicand under each other.

When it is evident that the numerator of a result is the product of certain factors, that result should be written both with the multiplications indicated and developed. As in the following example :—

$$\frac{a^2\,x}{b-c} - c\,d = b\,x - a\,c$$

$$x = \frac{c\,(b-c)\,(d-a)}{a^2 - b^2 + b\,c} = \frac{c\,b\,d - c^2\,d - a\,b\,c + a\,c^2}{a^2 - b^2 + b\,c}$$

the sides of the equation become

$$\frac{b^2\,c\,d - b\,c^2\,d - a^2\,c}{a^2 - b^2 + b\,c}$$

In the following examples we shall discuss the cases in which are presented what have been called the *critical* values of the solutions, namely, the forms 0, $\frac{\mathrm{A}}{0}$, and $\frac{0}{0}$, which are treated in all algebraical works, and which are here considered only as connected with the following theorems, which the student is to verify, each as it occurs, upon the examples given.

1. $x = 0$ indicates an equation of the form $a\,x + b = c\,x + b$, in which a and c are not equal: or an equation which may be reduced to that form.

2. $x = \frac{\mathrm{A}}{0}$, indicates an equation which either has or may be reduced to, the form $a\,x + b = a\,x + c$ in which b and c are not equal; and which cannot be true for any value of x.

3. $x = \frac{0}{0}$, indicates an equation which either has or may be reduced to, the form $a\,x + b = a\,x + b$, which is true for all values of x.

$$\frac{x-c}{a} - \frac{x-b}{c} = \frac{px}{c} + \frac{q}{a}$$

$$x = \frac{cq + c^2 - ab}{c - a - ap}$$

The sides of the equation become

$$\frac{ac^2p - a^2bp + c^2q - acq}{ac(c - a - ap)}$$

If $cq + c^2 - ab = 0$, that is, if $q = \frac{ab - c^2}{c}$, the solution becomes $x = 0$, unless it happen that we have also $c - a - ap = 0$, in which case it takes the form $\frac{0}{0}$. Let us suppose the first only, and not the second. The equation itself is the same as

$$\left(\frac{1}{a} - \frac{1}{c}\right)x + \frac{b}{c} - \frac{c}{a} = \frac{p}{c}x + \frac{q}{a}$$

But if $q = \frac{ab - c^2}{c} = \frac{ab}{c} - c$;

then $\frac{q}{a} = \frac{b}{c} - \frac{c}{a}$,

which call B. Then the equation is

$$\left(\frac{1}{a} - \frac{1}{c}\right)x + \mathrm{B} = \frac{p}{c}x + \mathrm{B}$$

which is in the form given in the first theorem.

If at the same time $c - a - ap = 0$,

$$p = \frac{a-c}{a} = 1 - \frac{c}{a},$$

then $\frac{p}{c} = \frac{1}{c} - \frac{1}{a}$, which call A.

The equation is $\mathrm{A}x + \mathrm{B} = \mathrm{A}x + \mathrm{B}$, which has the form given in the third theorem.

If the latter only be true, the solution has the form $\frac{cq^2 + c^2 - ab}{0}$, and the equation itself corresponding to this case is (A meaning as before)

$$\mathrm{A}x + \frac{b}{c} - \frac{c}{a} = \mathrm{A}x + \frac{q}{a}$$

which has the form given in the second theorem.

The student should now go through the various examples which have been given, and should show that the same suppositions which reduce the value of x to either of the forms

$$x = 0 \qquad x = \frac{1}{0} \qquad x = \frac{0}{0}$$

will, when applied to the equation itself, reduce it to the form

$$\mathrm{A}x + \mathrm{B} = \mathrm{C}x + \mathrm{B} \qquad \mathrm{A}x + \mathrm{B} = \mathrm{A}x + \mathrm{D} \qquad \mathrm{A}x + \mathrm{B} = \mathrm{A}x + \mathrm{B}$$

Equations should also be solved, deferring all reductions until the result has been obtained in the form of a complex fraction, as follows. The equation

$$\frac{x}{a} - \frac{1}{c}\left(b - \frac{px}{a}\right) = \frac{ac - x}{a + c}$$

or $\frac{x}{a} - \frac{b}{c} + \frac{px}{ac} = \frac{ac}{a+c} - \frac{x}{a+c}$

or $\frac{x}{a} + \frac{px}{ac} + \frac{x}{a+c} = \frac{ac}{a+c} + \frac{b}{c}$

gives $\left(\frac{1}{a} + \frac{p}{ac} + \frac{1}{a+c}\right)x = \frac{ac}{a+c} + \frac{b}{c}$

$$x = \frac{\frac{ac}{a+c} + \frac{b}{c}}{\frac{1}{a} + \frac{p}{ac} + \frac{1}{a+c}}$$

This should now be reduced by multiplying both the numerator and denominator by $ac(a+c)$, which gives

$$x = \frac{a^2c^2 + a^2b + abc}{(c+p)(a+c) + ac}$$

Let $\frac{ax - b}{c} + \frac{p - \frac{a - bx}{ab}}{a + b} = 1 - \frac{b - x}{a}$

or $\frac{ax}{c} - \frac{b}{c} + \frac{p}{a+b} - \frac{1}{b(a+b)} + \frac{x}{a(a+b)} = 1 - \frac{b}{a} + \frac{x}{a}$

$$x = \frac{1 - \dfrac{b}{a} + \dfrac{b}{c} - \dfrac{p}{a+b} + \dfrac{1}{b(a+b)}}{\dfrac{a}{c} + \dfrac{1}{a(a+b)} - \dfrac{1}{a}}$$

$$= \frac{a\,b\,c\,(a+b) - b^2 c\,(a+b) + a\,b^2 (a+b) - a\,b\,c\,p + a\,c}{a^2 b\,(a+b) + b\,c - b\,c\,(a+b)}$$

One of the things which the student must observe is, that an equation may be, in particular cases, intelligible in some forms and not in others, though the intelligible forms are direct deductions from the unintelligible. For instance suppose

$$\frac{x-a}{c} + b\,x = \frac{1-x}{e} \qquad (1)$$

$$e\,x - a\,e + b\,c\,e\,x = c - c\,x \qquad (2)$$

$$x = \frac{c + a\,e}{b\,c\,e + e + c} \qquad (3)$$

If we suppose $c = 0$, the equation assumes the unintelligible form

$$\frac{x-a}{0} + b\,x = \frac{1-x}{e}$$

while (2) assumes the form

$$e\,x - a\,e = 0, \qquad \text{or } x = a$$

which is also the result of the solution (3) in the case where $c = 0$. We have here nothing to do except with the following circumstance, that the rules for the solution of an equation give solutions to unintelligible as well as intelligible cases, which the student must connect together by means of the usual explanation of the former cases. At present, let all the examples be looked at, and let the following theorem be verified in each case.

When such suppositions are made as to the values of the letters representing known quantities as will make one denominator only equal to nothing, then the same suppositions applied to the solution will give such a value of the unknown quantity as makes the numerator of that denominator equal to nothing.

It is usual to accustom the student to the solution of various problems producing equations of the first degree: these are of no use whatever in themselves, but may be made to furnish illustrations of the several algebraical peculiarities of the results. To these we shall therefore proceed.

SECTION 17.—*Interpretation of the cases of a Problem producing Equations of the first degree.*

We shall solve one problem in general terms, that is, by expressing the known quantities by letters whose values are supposed to be given; and shall then proceed to inquire what particular values of the known quantities will give critical or other peculiar solutions, and what is the meaning of the problem in the cases thus formed.

It is supposed that the student is already acquainted with the usual method of treating the negative sign, and the following examples are for exercise, not for primary instruction.

Problem. There are two pieces of stuff, of l and l' yards in length; of these the owner sells the same number of yards at p and p' shillings a yard, and afterwards selling the remainders at q and q' shillings a yard, finds the same receipts from both pieces. What was the number of yards first sold of each?

Let x be that number of yards. Then

$$p\,x + q\,(l - x)$$

is received from the first piece, and

$$p'\,x + q'\,(l' - x)$$

from the second. The equation of these expressions requires that x should be

$$\frac{q\,l - q'\,l'}{(p' - q') - (p - q)}$$

$$p\,x + q\,(l - x) = \frac{q\,l\,(p' - q') - q'\,l'\,(p - q)}{(p' - q') - (p - q)} = p'\,x + q'\,(l' - x)$$

It is usual to write algebraical expressions in some form which shows symmetry, even where cases may occur in which combinations thus introduced are negative. The rules for the negative sign render such cases as manageable as any others.

Let us first suppose that $p' - q' = p - q$, or that the first and second prices of each stuff exceed each other by the same sum. The solution then takes the form

$$\frac{q\,l - q'\,l'}{0}$$

and the equation becomes $A = p' - q' = p - q$

$$A\,x + q\,l = A\,x + q'\,l'$$

which is never true when $q\,l$ differs from $q'\,l'$. Nevertheless, it may be made to approach as near as we please to the truth by taking x sufficiently great. For it must be remembered, that $x + a$ and $x + b$, for instance, are nearly to equality the greater x is taken. Thus, $1000 + 1$ and $1000 + 2$ are more nearly equal * than $4 + 1$ and $4 + 2$. But at first it cannot be supposed that x is greater than the least of l and l'. Consequently, keeping the literal meaning of the problem, it is here impossible. But we will now state another problem, of which the one just given is a particular case, and which leads to the same equation.

Problem. A man has two pieces of stuff, of l and l' yards in length; he engages to furnish the same number of yards of each at p and p' shillings a yard, and having made this bargain, he finds the prices in the market to be q and q' shillings a yard. He makes such new bargains as enable him to fulfil the first, and leave him without any stuff of either sort. He then finds his receipts (or deficits if he have lost) to be the same from both. What number of yards of each did he first engage to supply?

This is the problem in its most general terms. It meets the case that he may first have engaged to supply more than he has got of one or both; and also that in making up the stipulated quantities, he may be obliged to buy at such a price, that on the whole he will lose instead of gain.

Let us first take the case that he has engaged to supply more than he has of either (x yards). Then he has to go out and buy $x - l$ and $x - l'$ yards of the two sorts at q and q' shillings a yard. This costs him $q\,(x - l)$ and $q'\,(x - l')$ shillings, and he then sells his x yards of both sorts for $p\,x$ and $p'\,x$ shillings respectively. If the second be greater than the first he receives

$$p\,x - q\,(x - l) \text{ and } p'\,x - q'\,(x - l')$$

shillings for the two, and

$$p\,x - q\,(x - l) = p'\,x - q'\,(x - l')$$

by the problem: but if the second be less than the first, he loses

$$q\,(x - l) - p\,x \text{ and } q'\,(x - l') - p'\,x$$

shillings by the two, and

$$q\,(x - l) - p\,x = q'\,(x - l') - p'\,x$$

by the problem. Both these last equations give

$$x = \frac{q\,l - q'\,l'}{p' - q' - (p - q)}$$

the same as in the first case.

Let $l > l'$, and suppose he can furnish the stipulated quantity of the first stock, but not of the second, that is, x is less than l, and greater than l'. He has then by the problem to dispose of his remainder $l - x$ and to make up the deficiency $x - l'$, at q and q' shillings a yard. Consequently, he receives from the first $p\,x + q\,(l - x)$, and from the second he receives $p'\,x$ and has to pay $q'\,(x - l')$ which leaves him $p'\,x - q'\,(x - l')$. The first must be greater than the second, for by the problem he has the same (receipts or deficits) from both, and from the first he clearly receives; therefore he does so from the second. And the equation is

$$p\,x + q\,(l - x) = p'\,x - q'\,(x - l')$$

giving the same value of x as before.

Let us take another possible variety of the same question. He does not engage to *furnish*, but to *take*, the same number of yards of both, at p and p' shillings a yard. He then concludes such a bargain as rids him of his whole stock at q and q' shillings a yard, and finds his receipts or deficits the same from both. It is plain that he then first buys x yards of both for $p\,x$ and $p'\,x$ shillings, and sells his $l + x$ yards of the first, and $l' + x$ yards of the second at q and q' shillings a yard. If he gain by this, the equation is

$$q\,(l + x) - p\,x = q'\,(l' + x) - p'\,x;$$

if he lose, it is

$$p\,x - q\,(l + x) = p'\,x - q'\,(l' + x)$$

and both give

* For a complete explanation of this point the student may refer to the first chapter of the Treatise on the Differential Calculus, which contains nothing more than a student might here read.

$$x = \frac{q' l' - q l}{p' - q' - (p - q)} \text{ not as } \frac{q l - q' l}{p' - q' - (p - q)} \text{ before}$$

but these latter only differ in the sign of the numerator, that is, only differ in sign.

We shall now return to the general problem, and the case where $p' - q' = p - q$. Say that the lengths of the two pieces are 20 and 40 yards, the first and second prices of the first 10 and 8 shillings, those of the second 6 and 4 shillings, or

$$l = 20 \quad l' = 40 \quad p = 10 \quad p' = 8 \quad q = 6 \quad q' = 4$$

$$p' - q' = p - q = 4$$

If we suppose x yards (more than either 20 or 40) to be first bought, the equation is

$$10x - 6(x - 20) = 8x - 4(x - 40)$$

$$\text{or,} \quad 4x + 120 = 4x + 160$$

which cannot be, for the second side always exceeds the first by 40 (shillings). But this excess of 40 may evidently be made as small a part of the transaction as we please, by supposing x sufficiently great, and if two quantities be called nearly equal which have a very small difference when compared with their own magnitude, (which is the usual meaning of nearly equal,) then (Study of Mathematics, p. 41) this problem is said to be solved when x is infinite; that is, may be as nearly solved as we please by taking x sufficiently great.

If in the general problem we suppose $q\,l = q'\,l'$, and also $p' - q' = p - q$, the value of x takes the form $\frac{0}{0}$, and the equation becomes of the form $A\,x + B = A\,x + B$. *Any* value may be given to x. If in the preceding instance we suppose l' to be 30 instead of 40, all the rest remaining the same, the equation becomes

$$4x + 120 = 4x + 120$$

which is true of all values of x, or any quantity cut off or added to the stocks mentioned in the problem, satisfies the equation. The following are other cases of this problem:—

	l	l'	p	p'	q	q'	x	Receipts
I.	50	25	6	6	8	9	− 175	750
II.	20	20	3	9	10	6	8	144
III.	70	8	3	8	10	5	66	238
IV.	64	36	1	9	9	16	0	576

The student must remember that the answers to this problem will not always be whole numbers, but that these cases have been so contrived, in order to avoid fractions, and render the point we are now considering the only difficulty.

I. The negative sign of the answer shows a diametrically opposite meaning to that which was supposed in the equation. We supposed that the owner began by engaging to *furnish* a quantity of each; the answer shows that the problem cannot be solved in that way, but must be solved by supposing him engaged to *buy* 175 yards of each, both at 6 shillings; or, if it seems more clear, the problem proposed is not possible, but the corresponding problem which supposes him to begin by buying *is* possible, and the quantity so bought must be 175 yards; this is an outlay of 175 × 6, or 1050 for each; but the stocks of 175 + 50 and 175 + 25, or 225 and 200 then in hand, sold at 8 and 9 shillings give 1800 and 1800 shillings, so that the receipts for each are 750 shillings.

II. Is altogether within the limits of the first supposition.

III. Shows that he will have to buy 58 yards of the second to make up the 66 yards which the answer to the problem shows he is engaged to furnish. This costs 58 × 5, or 290 shillings, and the whole 66 at 8 shillings yields 528, giving, as the balance of receipt, 238. Of the first stock he sells 66 yards at 3 shillings, yielding 198 shillings, and the remaining 4 at 10 shillings, yielding 238 shillings in all.

IV. Shows that he must not cut off any of either piece in the first bargain, for selling the remainders (which in this case are the wholes) at 9 and 16 shillings, he just gets the same from both.

Let the student reconsider this question again and again, taking additional examples, and explaining them in the preceding manner. In every subsequent question (particularly in geo-

metry) he must always be on the watch to explain the three forms, namely, negative quantity, $\frac{A}{0}$, and $\frac{0}{0}$.

I. When the value of an unknown quantity appears to be negative, look back at the problem, and consider the negative quantity as unmeaning and unexplained, until it has been shown that the problem requires the unknown quantity to be of the kind diametrically opposite to that which it was at first supposed to be.

2. Consider the form $\frac{A}{0}$ as unmeaning until it is shown that the greater the value given to the unknown quantity, the more nearly is a solution produced to the problem; then, and not before, use the abbreviated form of speech that the unknown quantity is infinite.

3. When $\frac{0}{0}$ *is the result of an equation of the first degree*, let it be clearly ascertained that any value of the unknown quantity is a solution of the problem. What it means as to an equation of the second degree, we shall afterwards explain.

Section 18. *On Equations which are required to be solved in whole Numbers.*

This subject, though of very little practical use, will tend to impress on the mind of the student some considerations connected with whole numbers which will make him more expert in common arithmetic.

The following theorems and definitions are necessary :—

1. A *prime* number is one which admits of no divisors, except 1 and itself: such as 7, 29, 31.

2. All italic letters in this section signify *whole* numbers, and Greek letters whole numbers or fractions.

3. All numbers divisible by a are contained in the formula $a\,b$. Thus, all numbers divisible by 6 are contained in the set $6 \times 1, 6 \times 2, 6 \times 3, 6 \times 4$, &c.

All numbers which divided by a leave a remainder c, are contained in $a\,b + c$. Thus, all the numbers which divided by 13 leave a remainder 4, are contained in the set $13 \times 1 + 4, 13 \times 2 + 4, 13 \times 3 + 4$, &c.

5. All numbers are either prime numbers, or are made by multiplying prime numbers together. 20 is 2.2.5 64 is 2.2.2.2.2.2, 2420 is 2.2.5.11.11, 2310 is 2.3.5.7.11, and so on. That is, every whole number may be represented by

$$a^m \times b^n \times c^q \times \ldots\ldots$$

where $a, b, c, \ldots$ are prime numbers, and m, n, q, &c. are whole numbers, prime or not.

6. No number can be *resolved into prime factors* (the process of the last) in two different ways. Thus, if $a\,b\,c = d\,e\,f$, it is impossible that all the six, $a\,b$, &c., can be prime.

7. If a divide b, the prime factors of a are all among the prime factors of b. Thus, 360 is $2^3.3^2.5$, and all the divisors of 360 are represented by a case of one or other of the sets

$$2^m.3^n.5,\ 2^m.5,\ 3^n.5,\ 2^m.3^n,\ 2^m,\ 3^n,\ 5$$

where m is not $>$ 3 n is not $>$ 2.

8. If a number represented by its prime factors, be $a^m\,b^n\,c^p$, the number of divisors does not at all depend upon a, b, and c, but entirely upon m, n, and p. Thus, 600 being $2^3.5^2.3^1$ has the same number of divisors as 360, or $2^3.3^2 5^1$. The number of divisors (unity and the number itself included) is $(m + 1)(n + 1)(p + 1)$. Thus, 360 and 600 both have $(3 + 1)(2 + 1)(1 + 1)$ or 24 divisors. We exhibit those of 360.

$1, 3, 3^2, 2, 2.3, 2.3^2, 2^2, 2^2.3, 2^2.3^2, 2^3, 2^3.3, 2^3.3^2, 5, 5.3, 5.3^2, 5.2, 5.2.3, 5.2.3^2, 5.2^2, 5.2^2.3, 5.2^2.3^2, 5.2^3, 5.2^3.3, 5.2^3.3^2$.

Give the reason why the number of divisors is here $(3 + 1)(2 + 1)(1 + 1)$, and try to give a general demonstration of the theorem.

It is required to solve the equation $11\,x + 7\,y = 108$ in whole numbers, or to find all the values of x and y? Or it is required to divide 108 into two numbers, one a multiple of 11, the other of 7, in as many different ways as possible. The process is as follows:— Since y is a whole number, or $\frac{1}{7}(108 - 11\,x)$, or $15 - x + \frac{1}{7}(3 - 4\,x)$, of which $15 - x$ is a whole number; it follows that $\frac{1}{7}(3 - 4\,x)$ is a whole number. Let it be t; then $3 - 4\,x = 7\,t$, or $x = -t + \frac{1}{4}(3 - 3\,t)$. This algebraic fraction is to be a whole number, let it be t', then $3 - 3\,t = 4\,t'$, or $t = 1 - t' - \frac{1}{3}t'$, so that $\frac{1}{3}t'$ must be a whole number. Let t'' be this whole number; then $t' = 3\,t''$ $t = 1 - 3\,t'' - t'' = 1 - 4\,t''$; $x = -1 + 4\,t'' + \frac{1}{4}(3 - 3 + 12\,t'') = 7\,t'' - 1$

$$y = 15 - 7\,t'' + 1 + \tfrac{1}{7}\,(3 - 28\,t'' + 4) = 17 - 11\,t''$$

$$11\,x + 7\,y = 77\,t'' - 11 + 119 - 77\,t'' = 108$$

This result being independent of t'', it would seem that we have thus an infinite number of answers. And so we have if we consider algebraical answers, that is, positive or negative whole numbers; but if we restrict ourselves to arithmetical whole numbers, that is, to *positive* algebraical answers, we must so assume t'' that $7\,t''$ is greater than 1, and $11\,t''$ less than 17. The only value of t'' which satisfies both conditions is 1, which gives $x = 6$, $y = 6$, or $11.6 + 7.6 = 108$, which is the only arithmetical answer.

As these questions are entirely for exercise in numbers, we shall give no rule, but only a method. It is required to find a set of numbers, which being divided by 4, 5, and 6, give remainders 1, 2, and 3. Let us consider the two first conditions: let x and y be the quotients (rejecting remainders) of this number when divided by 4 and 5; hence, since the remainders are to be 1 and 2, we must have $4\,x + 1$, and $5\,y + 2$, both equal to the required number. Consequently

$$4\,x + 1 = 5\,y + 2, \text{ or } 4\,x - 5\,y = 1$$

must be solved in whole numbers. We might find this as follows: we want to find a multiple of four which exceeds a multiple of five by 1. A case is evidently found where $x = 4$, $y = 3$; add to these any multiples of 5 and 4, such as $5\,t$ and $4\,t$, and we have

$$x = 4 + 5t \qquad y = 3 + 4\,t$$

$$4\,(4 + 5\,t) - 5\,(3 + 4\,t) = 1.$$

But the preceding, though it shows that these forms satisfy the conditions, does not show that they are the *only* forms which do so. To show this, proceed as before: we have $x = y + \frac{1}{4}(y + 1)$, therefore $y + 1$ must be of the form $4\,t$ or y of the form $4\,t - 1$. The preceding gave $4\,t + 3$, which does not appear at first to be the same; but it is so in reality: for though $4\,t - 1$ is not the same as $4\,t + 3$, for any one value of t, yet it is plain that 1 less than *one* multiple of 4 is the same as 3 more than *another*. And the value of t may be any whole number. Taking $y = 4\,t + 3$, then the number $5\,y + 2$ is $20\,t + 17$, which formula contains all such numbers as, being divided by 4 and 5, will leave remainders 1 and 2. But this divided by 6 is to leave a remainder 3, by the last clause of the problem. Divide by 6 algebraically, giving $3\,t + 2 + \frac{1}{6}(2\,t + 5)$, but $3\,t + 2$ is a whole number, therefore the remainder 3 must come from $2\,t + 5$, which must therefore be of the form $6\,t' + 3$. But $2\,t + 5 = 6\,t' + 3$ gives $t = 3\,t' - 1$, and $20\,t + 17 = 60\,t' - 3$, which is the form required. For example, let $t' = 1$, then 57 satisfies the conditions; let $t' = 2$, then 117 also satisfies them; and so on. The student may make abundance of examples for himself, the test of correctness being obvious.

Theorem. Show that $a\,x + b\,y = c$ cannot be solved in whole numbers if either two of the three, a, b, and c, have a common measure which the third has not.

We shall now suppose it required to solve the equation

$$x^2 + y^2 = z^2$$

in whole numbers.

Always in such a case endeavour to write the equation so that both sides may be reducible to a pair of factors. In the present instance write

$$x^2 = z^2 - y^2,$$

or $\qquad x.x = (z - y)(z + y)$

1. Suppose neither of the three to be given. Assume

$$z + y = v\,x, \text{ then } z - y = \frac{x}{v}$$

$$z = \frac{x}{2}\left(v + \frac{1}{v}\right) \quad y = \frac{x}{2}\left(v - \frac{1}{v}\right)$$

Assume* any *even* number for x, and let $2\,v$ be any divisor of x. Or assume x any odd number, and v any divisor of it. For example, let $x = 9$, $v = 3$, then $z = 15$, $y = 12$, and $9^2 + 12^2 = 81 + 144 = 225 = 15^2$. This also contains the case where x is given, for it must then be assumed as given.

2. If z be given, the preceding equations give

$$x = \frac{2\,v\,z}{v^2 + 1} \quad y = \frac{v^2 - 1}{v^2 + 1}\,z.$$

Write $\frac{m}{n}$ for v, which gives $(m > n)$

$$x = \frac{2\,m\,n\,z}{m^2 + n^2} \quad y = \frac{m^2 - n^2}{m^2 + n^2}\,z$$

* Any proof that may be wanting is left for the student.

Prove 1. that if m and n be whole numbers with a common measure, the preceding can be reduced, and other numbers, which have no common measure, substituted for them without altering x or y. 2. That in the latter case $m\,n$ and $m^2 + n^2$ cannot have a common measure.

Hence, prove that when x and y are whole numbers, the problem is impossible except when z or one of its factors is the sum of two unequal squares. Thus, when $z = 5$ or $4 + 1$, let $m^2 = 4$ $n^2 = 1$, and $x = 4$ $y = 3$. But when $z = 11$, the problem is impossible. When $z = 10$, or $9 + 1$, then $m = 3$ $n = 1$ gives $x = 6$ $y = 8$. But 5, a factor of 10, is $4 + 1$, and $m = 2$ $n = 1$ gives $x = 8$ $y = 6$, which is only a reversal of the former.

Show the following, I. If m be greater than n, and both be integers, then

$$x = m^2 - n^2 \quad y = 2\,m\,n \quad z = m^2 + n^2$$

satisfies the preceding problem. II. The square of an even number is divisible by four, and that of an odd number must leave a remainder one. III. A prime number divided by 6, must have a remainder 1 or 5. IV. One more than a square may be a prime number, but one more than a cube cannot.

Problem. To find two numbers, of which the sum of the squares added to the product gives a square, or to solve

$$x^2 + x\,y + y^2 = z^2$$

show that, if x, y, and z, satisfy this equation, any of the same multiples of the three also satisfy it; and hence prove that any real fractional solution gives a solution in whole numbers.

Show that, in this problem, either of the following equations is a consequence of the other.

$$z + y = v\,(x + y) \quad z - y = \frac{x}{v}$$

From these deduce

$$y = \frac{v^2 - 1}{2 - v} \cdot \frac{x}{v} \quad z = \frac{v^2 - v + 1}{2 - v}\,\frac{x}{v}$$

Show that v must be greater than 1, and less than 2, or that if $v = \frac{m + n}{m}$, n must be less than m. Substitute this value of v, and thus obtain the results

$$y = \frac{2\,m\,n + n^2}{m^2 - n^2}\,x \quad z = \frac{m^2 + m\,n + n^2}{m^2 - n^2}\,x$$

And from a preceding part show that $x = m^2 - n^2$, $y = 2\,m\,n + n^2$ $z = m^2 + m\,n + n^2$. For instance, let $m = 3$, $n = 2$, then $x = 5$, $y = 16$, $z = 19$. Verify this, and find other instances.

Show that if n be greater than m, then

$$x = n^2 - m^2 \quad y = 2\,m\,n + n^2 \quad z = m^2 + m\,n + n^2$$

satisfy

$$x^2 - x\,y + y^2 = z^2$$

1. By the theory of the negative sign applied to the preceding. 2. By an independent process similar to the above.

Apply a similar process to

$$a\,x^2 \pm b\,x\,y + y^2 = z^2, \quad a\,x^2 \pm b\,x\,y + c\,y^2 = c\,z^2$$

a, b, and c, being given whole numbers. And, finally, apply the same process to

$$a\,x^2 + b\,x\,y + c^2\,y^2 = e^2\,z^2$$

Problem. Required two numbers, of which the sum of the squares shall be a given number of times the sum.

Let m be the given number of times, so that

$$x^2 + y^2 = m\,(x + y)$$

Assume $x = \frac{p}{q}\,y$, which gives by substitution

$$y = \frac{m\,q\,(p + q)}{p^2 + q^2} \quad x = \frac{m\,p\,(p + q)}{p^2 + q^2}$$

If possible, take p and q so that $m = p^2 + q^2$, or $m = c\,(p^2 + q^2)$, where c is a whole number. Then

$$y = c\,q\,(p + q) \quad x = c\,p\,(p + q)$$

for instance, if $m = 25 = 5\ (4 + 1)$ $x = 15$, $y = 30$.

Problem. Required *triangular* numbers which are also squares. A triangular number is any number which results from making x a whole number in $\frac{1}{2}\,x\,(x + 1)$. Show that this must give a whole number.

We have $\frac{1}{2}\,x\,(x + 1) = y^2$

Assume $x = \frac{m}{n}\,y$ then $x + 1 = 2\,\frac{n}{m}\,y$

or $y = \frac{m\,n}{m^2 - 2\,n^2}$, $x = \frac{2\,n^2}{m^2 - 2\,n^2}$

If we can now get m and n two whole numbers such that $m^2 - 2n^2 = 1$, we have a solution in $y = mn$, $x = 2n^2$. Let one set of values of $m^2 - 2n^2 = 1$ be found, say p and q, we shall show how by this one to find another.

Assume $m + 1 = \frac{p}{q} n$

whence $m - 1 = 2 \frac{q}{p} n$

giving $n = \frac{2pq}{p^2 - 2q^2}$ $m = \frac{p^2 + 2q^2}{p^2 - 2q^2}$

let p and q be the instances which satisfy the preceding: hence it follows that $p^2 - 2q^2 = 1$, and $n = 2pq$, $m = p^2 + 2q^2$, which give $m^2 - 2n^2 = p^4 - 2p^2q^2 + 4q^4 = (p^2 - 2q^2)^2 = 1$: and $x = 8p^2q^2$ $y = 2pq(p^2 + 2q^2)$

For instance, if $p = 3$ $q = 2$, and $p^2 - 2q^2 = 1$, which gives from the first method $x = 8$ $y = 6$, and we have $\frac{1}{2}.8.9 = 6 \times 6$: and from the second $x = 288$ $y = 204$, and $\frac{1}{2}$ $288.289 = 204.204$.

Observe, that we are not sure of thus getting all solutions; for it is not necessary that $m^2 - 2n^2$ should be 1, it is sufficient that it divide mn and $2n^2$.

Section 19. *Permutations and Combinations.*

Let $[n,p]$ be the abbreviation of the product of all numbers between n and p, both inclusive. Thus

$[4, 9]$ means $4 \times 5 \times 6 \times 7 \times 8 \times 9$

Let Γn stand for the product of all numbers from 1 up to n *exclusive*: thus

$\Gamma 8$ means $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

Now show the following:

$$[n, n + m] = \frac{\Gamma(n + m + 1)}{\Gamma n}$$

$$\frac{[n + m, n]}{[m + 1, 1]} = \frac{\Gamma(n + m + 1)}{\Gamma n \times \Gamma(m + 2)}$$

Show that $\frac{[m, m + p]}{[n, n + q]}$ must be a whole number, if p be greater than q, and $m + p$ greater than $n + q$. Take instances, and show this proposition: for example,

$$\frac{6.7.8.9.10.11}{2.3.4.5.6}$$

and from the instances endeavour to collect a general proof.

If there be n counters marked C_1 C_2 C_3 C_4 .... C_n, and if p be less than n, the number of different ways in which p counters can be drawn, one after the other, counting every two orders, however slightly they differ, as different, is $[n, n - p + 1]$. These are *permutations* of p out of n. But the number of different ways in which p can be taken out of n at once, is $[n, n - p + 1]$ divided by $[1, n]$. These are *combinations* or *selections* of p out of n.

Question. How many different hands can be held at the game of whist, or how many combinations are there of 13 out of 52?

Answer.

$\frac{[52, 40]}{[1, 13]}$ or 635,013,559,600

How many different choices of 5 may be made out of twelve persons?

Answer.

$\frac{12.11.10.9.8}{1.\ 2.\ 3.4.5}$ or $\frac{1.11.1.9.8}{1.\ 1.1.1.1}$ or 792

this method of abbreviation must be learned by practice: the 3.4 in the denominator is equivalent to the 12 in the numerator, and the 2.5 in the denominator to the 10 in the numerator.

In how many orders may four be drawn out of 20?

Ans. 20.19.18.17, or 116280.

Question. There are four boxes, containing a, b, c, and d balls. In how many ways may 4 balls be drawn, one from each?

Ans. From every ball in the first arises b methods of drawing from the first two; there are c times as many ways of drawing from the first three; for every way of drawing from the first two may be followed by any of the c ways of drawing from the third. Therefore abc is the number of possible drawings from the first three; and by similar reasoning, $abcd$ is the number of drawings from all four.

In the preceding question, how many ways are there of drawing three from three of the four boxes?

Ans. The first thing to be done is the selection of the three boxes to be drawn from: this may be done in $(4.3.2) \div (1.2.3)$ or four different ways (an abbreviation of this sort might be used: in so many ways as three can be taken, in so many ways can one be left; that is, the number required must be four.)

Calling A B C and D the four boxes, the selections may be

A B C, A B D, A C D, or B C D,

and the drawings from each set may be done in $a\,b\,c$, $a\,b\,d$, $a\,c\,d$, or $b\,c\,d$ ways; whence the total number is

$$a\,b\,c + a\,b\,d + a\,c\,d + b\,c\,d.$$

There are 3 boxes, with four, five, and six counters. In how many ways may 4 balls be drawn, not taking more than 2 from either? Firstly, consider how many ways 4 can be composed out of three numbers not greater than 2 (0 being included); this gives only $0 + 2 + 2$ and $1 + 1 + 2$.

This gives as follows, since there are three different ways of varying the cases:—

Balls are taken out of A (4)	B (5)	C (6)	Number of ways of taking them.
0	2	2	$\frac{5.4}{2} \times \frac{6.5}{2}$ or 150
2	0	2	$\frac{4.3}{2} \times \frac{6.5}{2}$ or 90
2	2	0	$\frac{4.3}{2} \times \frac{5.4}{2}$ or 60
1	1	2	$4 \times 5 \times \frac{6.5}{2}$ or 300
1	2	1	$4 \times \frac{5.4}{2} \times 6$ or 240
2	1	1	$\frac{4.3}{2} \times 5 \times 6$ or 180
			In all 1020

If there be n counters, all of a different mark, the total number of different orders in which they can be arranged is $[n, 1]$. What case is this of a preceding theorem? If there be n_1 counters marked C_1, n_2 marked C_2, and n_3 marked C_3, then the total number of different arrangements is

$$\frac{[n_1 + n_2 + n_3, 1]}{[n_1, 1]\,[n_2, 1]\,[n_3, 1]}.$$

Deduce this from what precedes.

SECTION 20. *On Expressions of the first and second Degrees.*

The following points contain the summary of the theory of expressions of the first and second degrees, which is one of the most important parts of algebra, and without which the mere solution of equations is of little use.

First degree. Let $m\,x + n$ be the expression of the first degree with respect to x. Let R be its root, or the value of x which makes $m\,x + n = 0$; then

1. R the root is $-\frac{n}{m}$.

2. $m\,x + n = m\,(x - \text{R})$ for all values of x.

3. When x is *greater* than R, $m\,x + n$ and m have the *same* signs; when x is *less* than R, $m\,x + n$ and m have different signs.

Example. To verify these rules in the case of $2\,x - 7$ and $-\frac{3}{2}\,x - \frac{1}{5}$.

In $2\,x - 7$ $\quad \text{R} = \frac{7}{2}$, $m = 2$ $n = -7$

$$2\,x - 7 = 2\left(x - \frac{2}{7}\right) = 2\,(x - \text{R})$$

Let x be greater than $\frac{7}{2}$ say $= 5$: then $2\,x - 7 = 3$ and is of the same sign as 2. Let x be less than $\frac{7}{2}$, say * -1: then $2\,x - 7 = -9$, and is of a different sign from 2. Again, let x be positive, but less than $\frac{7}{2}$, say $= 3$, then $2\,x - 7 = -1$, and differs in sign from 2.

* Study of Mathematics, page 49.

$$\text{In } -\frac{3}{2}x - \frac{1}{5},\ R = -\frac{2}{15},\ m = -\frac{3}{2}\ n = -\frac{1}{5}$$

$$-\frac{3}{2}x - \frac{1}{5} = -\frac{3}{2}\left(x + \frac{2}{15}\right) = -\frac{3}{2}\left(x - \left(-\frac{2}{15}\right)\right)$$

Let x be greater than $-\frac{2}{15}$.

1. Let $x = -\frac{1}{15}$, $-\frac{3}{2}x - \frac{1}{5} = -\frac{1}{10}$, $m = -\frac{3}{2}$
2. Let $x = 0$ .... $= -\frac{1}{5}$ .....
3. Let $x = 1$ .... $= -\frac{17}{10}$ .....

Let x be less than $-\frac{2}{15}$.

1. Let $x = -\frac{3}{15}$ $-\frac{3}{2}x - \frac{1}{5} = +\frac{1}{10}$, $m = -\frac{3}{2}$
2. Let $x = -\frac{1}{6}$ .... $= \frac{1}{20}$ .....

Verify these assertions upon other expressions, such as $x + 1$, $\frac{1}{2}x - 3$, &c.

Show that the root of $m x + n + m' x + n'$ must lie between the roots of $m x + n$ and $m' x + n'$, by several instances, and by algebraical reasoning. Show also that the root of the first is greater than, equal to, or less than, the average of the roots of the second and third, according as $n m'^2 + n' m^2$ is greater than, equal to, or less than $m m' (n + n')$ if $m m' (m + m')$ be positive: or according as $n m'^2 + n' m^2$ is less than, equal to, or greater than, $m m' (n + n')$, if $m m' (m + m')$ be negative.

Second degree. Let $a x^2 + b x + c$ be the given expression of the second degree, where a, b, and c may be severally positive, negative, or nothing: and let m n and r be the absolute numbers in a b and c, without signs; so that if a and b be positive, and c negative, we mean that a is $+ m$, b is $+ n$, and c is $- r$.

In making a summary of the properties of the expression above given which are most useful, we distinguish 1, those which are common to all cases; 2, those which distinguish the eight following cases from each other:

$$\left.\begin{matrix} m x^2 + n x + r \\ - n x^2 - n x - r \end{matrix}\right\} \qquad \left.\begin{matrix} m x^2 + n x - r \\ - m x^2 - n x + r \end{matrix}\right\}$$

$$\left.\begin{matrix} m x^2 - n x + r \\ - m x^2 + n x - r \end{matrix}\right\} \qquad \left.\begin{matrix} m x^2 - n x - r \\ - m x^2 + n x + r \end{matrix}\right\}$$

General properties. 1. If $b^2 - 4 a c$ be positive, there are two different roots: if $b^2 - 4 a c = 0$, these two roots are both the same: if $b^2 - 4 a c$ be negative, there are no possible roots whatsoever.

2. The roots of the expression $a x^2 + b x + c$, when they exist (call them R_1 and R_2) are

$$R_1 = \frac{-b + \sqrt{b^2 - 4 a c}}{2 a} \qquad R_2 = \frac{-b - \sqrt{b^2 - 4 a c}}{2 a}$$

either of these substituted for x makes

$$a x^2 + b x + c = 0.$$

3. Between the roots and the co-efficients the following equations exist :—

$$R_1 + R_2 = -\frac{b}{a} \qquad R_1 R_2 = \frac{c}{a}.$$

4. The following equation is true for every value of x (when there are roots).

$$a x^2 + b x + c = a (x - R_1)(x - R_2).$$

5. When the two roots are equal, that is, when $b^2 - 4ac = 0$; then

$$R_1 = R_2 = -\frac{b}{2a}. \quad ax^2 + bx + c = a(x - R_1)^2$$

and $ax^2 + bx + c$ is then a perfect square with respect to x, giving

$$\sqrt{ax^2 + bx + c} = \pm\sqrt{a}\left(x + \frac{b}{2a}\right)$$

6. The following equation is always true, and shows (see 1, preceding), that $(ax^2 + bx + c)a$ is the sum of two positive quantities when $b^2 - 4ac$ is negative.

$$ax^2 + bx + c = \frac{(2ax + b)^2 + 4ac - b^2}{4a}.$$

7. $ax^2 + bx + c$ and a *never* differ in sign, *except* when the two roots of the former are *possible* and *different*, and x is taken so as to lie between them.

8. When there are no roots, the least *numerical* value of $ax^2 + bx + c$ happens when

$$2ax + b = 0, \quad \text{or } x = -\frac{b}{2a}$$

$$\text{and is } c - \frac{b^2}{4a}.$$

The preceding properties occur so continually in all applications of algebra, and particularly in geometry, that the student should know them perfectly.

Example 1. What are the properties of

$$-2x^2 + 4x + 3$$

$$a = -2 \qquad b = +4 \qquad c = +3$$

1. $b^2 - 4ac = 40$; there are two different roots.

2. These roots are

$$R_1 = \frac{-4 + \sqrt{40}}{-4} = 1 - \frac{1}{2}\sqrt{10} \quad \text{(neg.)}$$

$$R_2 = \frac{-4 - \sqrt{40}}{-4} = 1 + \frac{1}{2}\sqrt{10} \quad \text{(pos.)}$$

3. $$R_1 + R_2 = 2 \qquad R_1 R_2 = -\frac{3}{2}$$

4. $$-2x^2 + 4x + 3 = -2(x - R_1)(x - R_2)$$
$$= -2(x - 1 + \tfrac{1}{2}\sqrt{10})(x - 1 - \tfrac{1}{2}\sqrt{10})$$

5. Does not apply to this case.

6. $$-2x^2 + 4x + 3 = \frac{(-4x + 4)^2 - 40}{-8}$$
$$= -2(x - 1)^2 + 5.$$

7. $-2x^2 + 4x + 3$ is negative for every value of x, except those which lie between

$1 - \frac{1}{2}\sqrt{10}$ or $-1{\cdot}5811389$ nearly

$1 + \frac{1}{2}\sqrt{10}$ or $2{\cdot}5811389$ „

in which cases it is positive. Show that it is positive when $x = 1$, $1{\cdot}5$, or $2{\cdot}5$.

8. Does not apply.

Example 2. $3x^2 - 12x + 12$.

$a = 3 \qquad b = -12 \qquad c = 12.$

1. $b^2 - 4ac = 0$; there are two equal roots.

2. These roots are both $= 2$.

5. $3x^2 - 12x + 12 = 3(x - 2)^2$

7. The expression is always positive, except only when $x = 2$ (0 is neither + nor −).

Example 3. $2x^2 + x + 1$

$a = 2 \qquad b = 1 \qquad c = 1$

1. $b^2 - 4ac = -7$; there are no roots.

6. $2x^2 + x + 1 = \frac{(4x+1)^2 + 7}{8}$

7. This expression is always positive.

8. Its least value is $\frac{7}{8}$, which is when $x = -\frac{1}{4}$.

When the co-efficients are fractional, reduce the whole to a common denominator; for instance, reduce

$$\frac{1}{2}x^2 + \frac{2}{3}x - \frac{3}{4} \text{ to } \frac{6x^2 + 8x - 9}{12}$$

then from the properties of the numerator, those of the fraction may be obtained.

Particular properties. The pairs bracketted together in page 81, present no distinction whatever except difference of sign. Each one is the other of its pair with all the signs changed.

1. When a and c have the same sign, the roots *may* be impossible, and we have

$b^2 - 4ac$ is less than b^2

$\sqrt{b^2 - 4ac}$ „ „ „ b, numerically.

2. When a and c have different signs, the roots cannot be impossible, and we have

$b^2 - 4ac$ is greater than b^2

$\sqrt{b^2 - 4ac}$ „ „ b, numerically.

3. In $\left.\begin{array}{l} +mx^2 + nx + n \\ \text{and } -mx^2 - nx - r \end{array}\right\} \begin{array}{l} + + + \\ - - - \end{array}$

the particular distinction is, that the roots, *if any*, are both negative.

4. In $\left.\begin{array}{l} +mx^2 - nx + r \\ \text{and } -mx^2 + nx - r \end{array}\right\} \begin{array}{l} + - + \\ - + - \end{array}$

the roots, *if any*, are both positive.

5. In $\left.\begin{array}{l} +mx^2 + nx - r \\ \text{and } -mx^2 - nx + r \end{array}\right\} \begin{array}{l} + + - \\ - - + \end{array}$

There must be two roots of different signs, the numerically greater root being negative.

6. In $\left.\begin{array}{l} +mx^2 - nx - r \\ \text{and } -mx^2 + nx + r \end{array}\right\} \begin{array}{l} + - - \\ - + + \end{array}$

There must be two roots of different signs, the numerically greater being positive.

These may all be contained in one rule, as follows: — there may be as many positive roots as there are changes of sign from term to term of the expression, and as many negative roots as there are continuations of sign; and according as the first term and the second present change or continuation, the greater root, numerically, is positive or negative.

Show that $$\sqrt{b^2 - 4ac} - b = \frac{-4ac}{\sqrt{b^2 - 4ac} + b}$$

and from hence that

$$R_1 = \frac{-2c}{b + \sqrt{b^2 - 4ac}} \qquad R_2 = \frac{-2c}{b - \sqrt{b^2 - 4ac}}$$

and also that if a be supposed to vary, becoming smaller and smaller,

R_1 continually approaches to $-\frac{c}{b}$

R_2 increases without limit.

Why cannot this be made clear *in both cases*, when the first forms are used? Also show from the preceding that when a is very small,

$$\sqrt{b^2 - 4ac} = b - \frac{2ac}{b} \text{ nearly.}$$

If the roots of $ax^2 + bx + c$ be R_1 and R_2, those of $cx^2 + bx + a$ are $\frac{1}{R_1}$ and $\frac{1}{R_2}$.

Show this by actually forming the roots.

Show that a change of sign in b changes only the sign of the roots, and not their numerical magnitude.

Example. A number $2p$ is divided into two parts, whose product is q^2. What are those parts?

Ans. The roots of the expression $x^2 - 2px + q^2$, or the solutions of the equation

$$x^2 - 2px + q^2 = 0$$

namely, $p + \sqrt{p^2 - q^2}$ and $p - \sqrt{p^2 - q^2}$

Prove from this that it is impossible the product of the two parts into which any number is divided should exceed the square of the half: for instance, that no two numbers or fractions which added together make 10, can have a product exceeding 25.

Example. What are the solutions of

$$(x - a)(x - b) = c?$$

Ans. $\dfrac{a + b \pm \sqrt{(a - b)^2 + 4c}}{2}$.

What are the *conditions of possibility* of this equation? Of the two following, which is possible?

$$(x - 4)(x - 1) = 12 \quad (x - 1)(x - 2) = -100.$$

What do the roots become when $c = 0$? How does this appear from the equation itself?

Example. When is the following expression possible?

$$\sqrt{(b^2 - 4ac)x^2 + (2bd - 4ae)x + d^2 - 4af} \dots . \quad (1)$$

The roots of the expression where square root is shown are

$$\frac{2ae - bd \pm \sqrt{4a\{(cd^2 + ae^2 - bde) + (b^2 - 4ac)f\}}}{b^2 - 4ac}.$$

For what values of x will the following equation allow of being solved by possible values of y?

$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

Show by solving the equation on the supposition that y is to be found, that this question reduces itself to the preceding.

Show that the roots of $ax^2 + 2bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - ac}}{a}.$$

As this form often occurs it should be remembered. Compare it with that for the roots of $ax^2 + bx + c = 0$, and explain the difference.

What are the roots of

$$(1 + a)x^2 - 2(1 + 2a)x + 1 + 3a?$$

Ans. 1 and $\dfrac{1 + 3a}{1 + a}$.

For what values of a is the second root negative? Find this out from the expression, without looking at the root; and from the root without looking at the expression.

Ans. When a lies between $-\dfrac{1}{3}$ and -1.

Question. If the difference of the two roots of $ax^2 + bx + c = 0$ be D, what are the roots, and what equation must exist between a, b, c, and D?

Ans.

$$R_1 = \frac{-b + aD}{2a} \quad R_2 = \frac{-b - aD}{2a}$$

$$b^2 - a^2D^2 = 4ac.$$

Prove this in two different ways.

Question. There are two numbers whose sum is p, and the sum of their squares is q: what are the numbers?

Ans.

$$\frac{p + \sqrt{2q - p^2}}{2} \quad \text{and} \quad \frac{p - \sqrt{2q - p^2}}{2}$$

Show that $x + \dfrac{1}{x}$ cannot be less than 2, or that

$$x + \frac{1}{x} = 2a \quad (a < 1)$$

has no possible roots.

If the sum of two numbers be p, and the sum of their cubes q, then those two numbers are the roots of the expression

$$x^2 - px + \frac{p^3 - q}{3p}.$$

If the roots of $ax^2 + bx + c$ be R_1 and R_2, what is the expression whose roots are $R_1 + k$ and $R_2 + k$?

Ans. $ax^2 + (b - 2ak)x + ak^2 - bk + c$.

What is the expression where roots are $m R_1$ and $m R_2$?.
Ans. $a x^2 + m b x + m^2 c$.

What number is that which exceeds its square root by a? Let v be the square root, and v^2 the number; then

$$v = \frac{1 + \sqrt{1 + 4a}}{2} \quad \text{or} \quad \frac{1 - \sqrt{1 + 4a}}{2}$$

$$v^2 = \frac{1 + 2a + \sqrt{1 + 4a}}{2} \quad \text{or} \quad \frac{1 + 2a - \sqrt{1 + 4a}}{2}$$

verify and explain both cases. Show that v and v^2 are rational when a is a number or fraction added to its square, and that only one square root of v^2 will satisfy the condition. Find also the number, which added to its square root is equal to a, and show that the square root of v^2 which does not satisfy one question, satisfies the other.

What are the roots of $a x^4 + b x^2 + c$?

$$Ans. \pm \sqrt{\frac{-b + \sqrt{b^2 - 4ac}}{2a}} \quad \text{and} \pm \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

Point out the cases in which two roots only are possible, and in which all are possible or none possible. Show that there cannot be one or three possible, and the others or other impossible.

What are the roots of $a x^{2n} + b x^n + c$?

$$Ans. \sqrt[n]{\frac{-b + \sqrt{b^2 - 4ac}}{2a}} \quad \text{and} \quad \sqrt[n]{\frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

which may be taken with either sign if n be even. We do not here enter on the impossible roots.

What two numbers are there whose sum is m times their difference, and whose product is n times their sum?

If x and y be the numbers, then we have the equations

$$x + y = m(x - y) \qquad x y = n(x + y)$$

From the first, $y = \dfrac{m-1}{m+1}x$,

which substituted in the second, gives

$x = \dfrac{2mn}{m-1}$ and thence $y = \dfrac{2mn}{m+1}$.

What numbers are there, the sum of whose squares is m times their product, and the difference of whose squares is n times their product. Examine the equations and prove that these two conditions are contradictory unless $m^2 - n^2 = 4$. Thence prove that they are never true when m is a whole number, unless $m = 2$ $n = 0$, and never true when n is a whole number. Show, as in page 77, that when

$$m = k + \frac{1}{k} \qquad n = k - \frac{1}{k},$$

any values of x and y which satisfy the first of the equations also satisfy the second.

Solve the equations

$$x^2 + y^2 = m(x + y) \qquad x + y = n(x - y).$$

From the second, $\qquad y = \dfrac{n-1}{n+1}x,$

which, substituted in the first, gives, after reduction,

$$2(n^2 + 1)x^2 = 2mn(n + 1)x.$$

Ans. Either $x = 0$ $\qquad y = 0$;

$$\text{or } x = mn\frac{n+1}{n^2+1} \qquad y = mn\frac{n-1}{n^2+1}.$$

Solve the equations

$$x^2 - y^2 = mxy \ldots . \text{(A)} \qquad x - y = nxy \ldots . \text{(B)}.$$

First deduce $x + y = \dfrac{m}{n}$ . . . . (C).

Substitute y obtained from this in both of the first equations, and show that they produce different results, namely,

$$n^2 (2 + m) x^2 + m n (2 - m) x - m^2 = 0 \quad . \; . \; . \; . \; (D)$$

$$n x^2 + (2 - m) n x - m = 0 \quad . \; . \; . \; . \; (E)$$

the roots of either of these are values of x, which with $y = \frac{m}{n} - x$, solve the equation; namely,

$$(D) \qquad x = \frac{- m (2 - m) \pm m \sqrt{12 + m^2}}{2 n (2 + m)}$$

$$(E) \qquad x = \frac{- (2 - m) n \pm \sqrt{n^2 (2 - m)^2 + 4 m n}}{2 n}.$$

The truth is, that by the artifice of division, which produced the equation (C), we have obtained equations of the *second* degree; whereas, had we simply substituted in (A) the value of y from (B), namely,

$$y = \frac{x}{n x + 1},$$

we should, after reduction, have obtained an equation of the *fourth* degree, which may have four possible roots.

We must here remind the student that there is a degree of connexion between algebraical equations, more than is actually and logically contained in the forms of speech by which they are connected. Let us take the following assertions:

(A) All right angles are equal.
(B) P and Q are right angles.
(C) P and Q are equal angles.

If (A) and (B) be both true, (C) must be true; but if (A) and (C) be both true, it does not necessarily follow that (B) must be true, as will easily be seen. But take for (A) (B) and (C) three algebraical equations, of which (C) necessarily follows, or is true, when (A) and (B) are true; for instance,

$$(A) \qquad x + y = 7$$
$$(B) \qquad x - y = 5$$
$$(C) \qquad x^2 - y^2 = 35,$$

of which (C) must be true when (A) and (B) are true. It follows that when (A) and (C) are true, either (B) is true, or (B) is one of *two* equations *in this case* (it might be three, four, &c., in problems of a higher degree) one of which must be true. Assume equations (A) and (C) or

$$x + y = 7 \qquad x^2 - y^2 = 35.$$

If we divide the second by the first, we have $x - y = 5$, which it appears at first must follow absolutely; and this is true, finite numbers only being considered. But we have (*Study of Mathematics*, p. 41) to consider the possibility of an *infinite* solution, or of this problem being one particular case of a general problem, which admits of more solutions than one, in general, but of which solutions one or other increases without limit, as the general problem is made to approach to the particular case in question. We have seen that a problem of the second degree must generally have two solutions, but this seems to have only one; for since equals divided by equals must give equals, it follows that *if* $x + y = 7$, and $x^2 - y^2 = 35$, we *must* have $x - y = 5$, and $x + y = 7$, together with $x - y = 5$, give $x = 6$ $y = 1$, and nothing else whatsoever. The question is, what is become of the second solution which the general problem $a x + b y = m$ $\;p x^2 - q y^2 = n$ will be found to have? To solve this question, we must first see whether we really have only one solution. Instead of dividing the second equation by the first, substitute in the second the value of y from the first, which gives

$$x^2 - (7 - x)^2 = 35,$$

an equation of the second degree at first sight, and therefore with two roots or none. But on looking further, we see that this equation is no more of the second degree, than $x = 1$ is of the hundredth degree (being $x + x^{100} = 1 + x^{100}$), as the terms of the second degree destroy each other, and leave

$$14 x - 49 = 35 \qquad x = 6.$$

But let us now alter the problem by proposing

$$x + y = 7 \qquad (1 + k) x^2 - y^2 = 35.$$

We here present a problem which may be made as near as we please to the former, by taking k sufficiently small, and which absolutely becomes the former, when $k = 0$. Now substitute as before, and we have

$$(1 + k)\,x^2 - (7 - x)^2 = 35, \quad \text{or}$$
$$k\,x^2 + 14\,x = 84$$
$$x = \frac{-14 \pm \sqrt{196 + 336\,k}}{2\,k}$$
$$= \frac{-7 + \sqrt{49 + 84\,k}}{k} \quad \text{or} \quad \frac{-7 - \sqrt{49 + 84\,k}}{k}$$
$$(\text{p. } 83) \quad = \frac{84}{\sqrt{49 + 84\,k} + 7} \quad \text{or} \ - \frac{84}{\sqrt{49 + 84\,k} - 7}$$

If we now suppose k to diminish continually, the first root approaches continually to

$$\frac{84}{\sqrt{49} + 7} \quad \text{or } 6,$$

while the second is always negative, and has a denominator which diminishes without limit; that is, the root increases numerically without limit. When k is small, the first expression for the second root shows that it is

$$\frac{-7 - \sqrt{49}}{k}, \quad \text{or} - \frac{14}{k} \text{ nearly;}$$

so that if k were $\frac{1}{1000}$, the root would be nearly -14000.

Let us suppose k to be very small, and let the great and negative value of x be taken. Then $x + y = 7$ shows that y or $7 - x$ is a little greater and positive. We now ask, what is the substitute for the equation $x - y = 5$, which is necessarily true in the first problem? Is it nearly true in the second problem? To investigate this, take the second equation in the form

$$(x^2 - y^2) + k\,x^2 = 35,$$

divide the three terms by the equals $x + y$, 7, and 7, which gives

$$x - y + \frac{k\,x^2}{7} = 5.$$

This equation will be true, as might be proved by actual substitution: but it will not be true, because $x - y = 5$ nearly, but because $x - y$ is a negative quantity, and $\frac{1}{7}\,k\,x^2$ is a positive quantity greater by 5. And though k may be small, x is so large that $k\,x^2$ is considerable. With respect to $x = 6$, which will give $y = 1$ very nearly, which is one solution of the second problem when k is small, we may say that

$$x - y + \frac{1}{7}\,k\,x^2 = 5$$

means $x - y = 5$ very nearly, because the term rejected is only about 36 sevenths of k, and is small: but we may not call the latter equation nearly true of the larger root.

Question. If $a\,x^2 + b\,x + c$, and $a'\,x^2 + b'\,x + c'$ have a root in common, what equation must exist between a, b, c, a', b', c'?

Let P be the common root, and let Q be the other root of the first, and Q′ the other root of the second. Then we have

$$\text{P} + \text{Q} = -\frac{b}{a} \ldots (1) \qquad \text{P}\,\text{Q} = \frac{c}{a} \ldots (3)$$

$$\text{P} + \text{Q}' = -\frac{b'}{a'} \ldots (2) \qquad \text{P}\,\text{Q}' = \frac{c'}{a'} \ldots (4)$$

From these equations deduce the following:

$$\text{Q}' - \text{Q} = \frac{b\,a' - b'\,a}{a\,a'} \qquad \frac{\text{Q}'}{\text{Q}} = \frac{a\,c'}{a'\,c},$$

from which deduce $\quad \text{Q} = \dfrac{c}{a}\,\dfrac{b\,a' - b'\,a}{a\,c' - a'\,c}.$

Now, from (1) and (3) deduce

$$a\,\text{Q}^2 + b\,\text{Q} + c = 0,$$

and thence (from the preceding)

$$c\,(b\,a' - b'\,a)^2 + b\,(b\,a' - b'\,a)\,(a\,c' - a'\,c) + a\,(a\,c' - a'\,c)^2 = 0,$$

which show to be the same as

$$(a\,c' - a'\,c)^2 = (c\,b' - c'\,b)\,(b\,a' - b'\,a) \quad \ldots \ldots (5).$$

Now deduce the same as follows:—since P is a root of both, we have

$$a\,P^2 + b\,P + c = 0$$
$$a'\,P^2 + b'\,P + c' = 0$$

Hence, deduce

$$(a\,c' - a'\,c)\,P^2 - (c\,b' - c'\,b)\,P = 0$$

and also

$$(b\,a' - b'\,a)\,P - (a\,c' - a'\,c) = 0,$$

and from the two last deduce (5).

Question. Given the values of a and c; required the method of finding such values of b as will make the roots of $a\,x^2 + b\,x + c$ rational. Show, from the method explained in page 77, that $b^2 - 4\,a\,c$ is a square when

$$b = a\,m + \frac{c}{m},$$

m being any number or fraction whatsoever. Show that the roots in this case are

$$-\,m \qquad \text{and} - \frac{c}{a\,m}.$$

Question. In how many different ways can whole and positive rational roots be given to the expression

$$3\,x^2 + b\,x + 36?$$

Show, from the preceding, that m must be either -1, -2, -3, -4, -6, or -12, and that the values of b are -39, -24, -21, -21, -24 and -39. Explain the recurrence of the values of b.

Miscellaneous questions. Deduce from the general equation, the roots of $a\,x^2 + c = 0$, and of $a\,x^2 + b\,x = 0$. On what does it depend whether the roots of the first are possible or not? Show that when b is very small compared with a and c, the roots (if any, on what does this depend?) are one positive and the other negative, but numerically very nearly equal. Show that when c is very small compared with a and b, that one of the roots must be very small, but not the other, necessarily. What are the necessary conditions that both the roots must be very small? If c and a be equal, what relation must exist between the roots? Show that if the sum of the squares of the roots be unity, we must have $b^2 - 2\,a\,c = a^2$. In what relation do the roots of $a\,x^2 + m\,b\,x + m^2\,c$ stand to those of $a\,x^2 + b\,x + c$. Show from the method already given, that if $a\,x^2 + b\,x + c$ and $a'\,x^2 + b'\,x + c'$ have a common root,

$$(a + p\,a')\,x^2 + (b + p\,b')\,x + c + p\,c'$$

has the same for all values of p.

Section 21. *On Equations of the first degree, with more unknown quantities than one.*

As few cases of these will occur in the future reading of the student, which present any very complicated operations, we shall here only describe the best method of proceeding when only one of the unknown quantities (three in number) is wanted.

Suppose the equations to be

$$2\,x + 3\,y + 4\,z = 20$$
$$3\,x - 2\,y - 2\,z = 4$$
$$4\,x - \;\;y + \;\;z = 12.$$

Suppose the value of z is required. Multiply the second equation by p, and the third by q, then add the three together, and suppose p and q to be such that

$$2 + 3\,p + 4\,q = 0$$
$$3 - 2\,p - \;\;q = 0.$$

Show that we must then have

$$z = \frac{20 + 4\,p + 12\,q}{4 - 2\,p + q}$$

that from the preceding equations,

$$p = \frac{14}{5}\;q = -\frac{13}{5},\ \text{whence } z = 0.$$

Again, suppose the value of x is wanted in

$$a\,x + b\,y - c\,z = 1$$
$$b\,x + c\,y - a\,z = 1$$
$$c\,x + a\,y - b\,z = 1.$$

Show that if

$$b + c\,p + a\,q = 0$$
$$c + a\,p + b\,q = 0$$

then $$x = \frac{1 + p + q}{a + b\,p + c\,q}$$

and that

$$p = \frac{b^2 - a\,c}{a^2 - b\,c} \qquad q = \frac{c^2 - a\,b}{a^2 - b\,c}$$

$$z = \frac{a^2 + b^2 + c^2 - a\,b - b\,c - c\,a}{a^3 + b^3 + c^3 - 3\,a\,b\,c}$$

Determine the values of x and y by a similar method.

Section 22. *On Exponents.*

Explain the following expressions:

$$x^4,\quad x^{-4},\quad x^{\frac{1}{4}},\quad x^{-\frac{1}{4}},\quad x^{\frac{2}{3}},\quad x^{-\frac{2}{3}},\quad x^{\frac{a}{b}}$$

$$\left(x^{\frac{m}{n}}\right)^{\frac{a}{b}},\qquad \left\{(x+1)^{\frac{3}{2}}\right\}^{\frac{2}{3}}$$

$$x^m \times x^n = x^{m+n},\qquad x^m \times x^{-n} = x^{m-n}$$

$$x^m \div x^n = x^{m-n},\qquad x^m \div x^{-n} = x^{m+n}$$

$$x^{-m} \times x^{-n} = x^{-m-n} = x^{-(m+n)}$$

$$x^{-m} \div x^{-n} = x^{n-m} = x^{-(m-n)}$$

$$\left(x^m\right)^n = x^{mn},\quad \left(x^{-m}\right)^n = x^{-mn},\quad \left(x^{-m}\right)^{-n} = x^{mn}$$

$$x^2 \times x^3 \times x^{-4} = x \qquad x \times x^{-2} \div x^{-4} = x^3$$

What is the value of p in the following equations?

$$x^p \times x^2 = x^4 \times x^{-p},\qquad x^{-p} \times x^{-3p} = x^{12}.$$

Reduce the following expressions to fractional exponents and find the results:

$$\sqrt{x} \times \sqrt[3]{x} = \sqrt[6]{x^5} \qquad \sqrt[a]{x^b} \times \sqrt[b]{x^a} = \sqrt[ab]{x^{a^2+b^2}}$$

$$\sqrt{x\sqrt{x}} = \sqrt[4]{x^3} \qquad \sqrt[3]{x^2\sqrt[4]{x}} = \sqrt[4]{x^3}$$

$$\left(a^4 \times \left(a^3 \times a^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{3}} = a^{\frac{23}{18}} \qquad \sqrt[p]{x\sqrt[q]{x}} = \sqrt[pq]{x^{q+1}}$$

The qth root of the pth power of x having been taken, the reciprocal of the result is raised to the rth power, and its sth root is taken: this result having been first multiplied by the pth root of the qth power of x, is squared, and the result being then multiplied by the cube of x, the fourth root of the whole is taken. What is the algebraical method of stating this process, and what power of x is the result?

Expression, $\left\{x^3\left(x^{\frac{q}{p}}\left(x^{-\frac{p}{q}}\right)^{\frac{r}{s}}\right)^2\right\}^{\frac{1}{4}}$

Result, $x^{\frac{3pqs+2q^2s-2p^2r}{4pqs}}$

or, $\dfrac{x^{\frac{3}{4}} \times x^{\frac{q}{2p}}}{x^{\frac{pr}{2qs}}}$

$$\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2 = a + 2\,a^{\frac{1}{2}}\,b^{\frac{1}{2}} + b$$

$$\left(a^{\frac{1}{3}} + b^{\frac{1}{3}}\right)^3 = a + 3\,a^{\frac{2}{3}}\,b^{\frac{1}{3}} + 3\,a^{\frac{1}{3}}\,b^{\frac{2}{3}} + b$$

$$\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = a - b$$

$$\left(a^{\frac{m}{n}} + b^{-\frac{m}{n}}\right)^2 = a^{\frac{2m}{n}} + 2\,a^{\frac{m}{n}}\,b^{-\frac{m}{n}} + b^{-\frac{2m}{n}}$$

$$a + b = a^{\frac{1}{2}}\left(a^{\frac{1}{2}} + a^{-\frac{1}{2}} b\right) = a^{\frac{1}{3}}\left(a^{\frac{2}{3}} + a^{-\frac{1}{3}} b\right)$$

$$= a^{-\frac{3}{5}}\left(a^{\frac{8}{5}} + a^{\frac{3}{5}} b\right) = a\, b^{-1}\left(b + a^{-1} b^{2}\right) = a^{-1} b^{-1}\left(a^{2} b + b^{2} a\right)$$

$$= a^{\frac{m}{n}}\left(a^{\frac{n-m}{n}} + a^{-\frac{m}{n}} b\right) = a^{m} b^{n} c^{r}\left(a^{-(m-1)} b^{-n} c^{-r} + a^{-m} b^{-(n-1)} c^{-r}\right)$$

$$a^{\frac{1}{2}} - b^{\frac{1}{2}} = a\left(a^{-\frac{1}{2}} - a^{-1} b^{\frac{1}{2}}\right) = a^{\frac{1}{4}}\left(a^{\frac{1}{4}} - a^{-\frac{1}{4}} b^{\frac{1}{2}}\right)$$

$$a^{m} : a^{n} :: a^{p} : a^{p+n-m} \qquad a^{\frac{3}{2}} : a^{\frac{1}{4}} :: a^{\frac{2}{3}} : a^{-\frac{7}{12}}$$

Multiply together the two series

$$a + a' x + a'' x^{2} + a''' x^{3} + a^{\text{iv}} x^{4} + \ldots\ldots$$

and $\quad b + b' x^{-1} + b'' x^{-2} + b''' x^{-3} + b^{\text{iv}} x^{-4} + \ldots$

and give the terms of the product which involve

$$x^{3}, \quad x^{-3}, \quad x^{n}, \text{ and } x^{-n}.$$

Answers.

$$(b a''' + b' a^{\text{iv}} + b'' a^{\text{v}} + \ldots\ldots) x^{3} \qquad (b\, a^{(n)} + b' a^{(n+1)} + \ldots\ldots) x^{n}$$

$$(a b''' + a' b^{\text{iv}} + a'' b^{\text{v}} + \ldots\ldots) x^{-3} \qquad (a\, b^{(n)} + a' b^{(n+1)} + \ldots\ldots) x^{-n}$$

$$a - b = \left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)$$

$$= \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}} b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{1}{4}} - b^{\frac{1}{4}}\right)\left(a^{\frac{3}{4}} + a^{\frac{1}{2}} b^{\frac{1}{4}} + a^{\frac{1}{4}} b^{\frac{1}{2}} + b^{\frac{3}{4}}\right)$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{a^{\frac{1}{n}}\, b^{\frac{n-1}{n}}}{b} = \frac{a}{a^{\frac{n-1}{n}}\, b^{\frac{1}{n}}} = \frac{a^{\frac{mn+1}{n}}\, b^{\frac{pn+n-1}{n}}}{a^{m}\, b^{p+1}}$$

$$\sqrt[4]{\left(\frac{a\sqrt{b}}{\sqrt[3]{a b}}\right)} = a^{\frac{1}{6}}\, b^{\frac{1}{24}} \qquad \sqrt[n]{\left(\frac{a^{m}\sqrt[p]{b}}{\sqrt[q]{a b}}\right)} = a^{\frac{mq-1}{qn}}\, b^{\frac{q-p}{pqn}}.$$

Section 23. *Miscellaneous Questions.*

How much time elapses between two consecutive conjunctions of the minute hand and hour hand of a watch?

Ans. 1 hour, 5 minutes, 27 seconds, and $\frac{3}{11}$ of a second.

If one hand revolved in a hours, and the other in b hours, what time would elapse between two conjunctions?

Ans. If b be greater than a, $\frac{a b}{b - a}$ hours.

In the last question, how long will it be before the quicker hand has gained $\frac{p}{q}$ of a whole revolution upon the other?

Ans. $\frac{p}{q}\,\frac{a b}{b - a}$ hours.

In $\frac{a + b x}{1 + x}$ determine a and b so that if in the expression the expression itself be substituted for x, the result will be $= x$.

Ans. a may have any value, provided $b = -1$, or the expression must be $\frac{a - x}{1 + x}$.

Show that in the series

$$1 + n x + n \frac{n-1}{2} x^2 + n \frac{n-1}{2} \frac{n-2}{3} x^3 + \ldots\ldots$$

the same result is produced by writing $n + 1$ instead of n, as would be produced by multiplying the series by $1 + x$. Write the values of the preceding series when $n = 1$ $n = 2$ $n = 3$ $n = 4$, and show that they are the same as $1 + x$, $(1 + x)^2$, $(1 + x)^3$, and $(1 + x)^4$.

Multiply together the two following series:

$$1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \text{ \&c.}$$

$$1 + y + \frac{y^2}{2} + \frac{y^3}{2.3} + \frac{y^4}{2.3.4} + \text{ \&c.}$$

and show that the product is the expression obtained by writing $x + y$ instead of x in the first.

Multiply together the series

$$a + a' x + a'' x^2 + a''' x^3 + a^{\text{iv}} x^4 + \ldots.$$

$$\text{and } b + b' x + b'' x^2 + b''' x^3 + b^{\text{iv}} x^4 + \ldots.$$

What is the term of the product involving x^n?

Ans.

$$\left(a^{(n)} \; b + a^{(n-1)} b' + a^{(n-2)} b'' + \ldots\ldots + a' b^{(n-1)} + a \, b^{(n)} \right) x^n$$

In the preceding question, let the second series be

$$1 + x + x^2 + x^3 + x^4 + \ldots\ldots$$

or let $\quad b = 1 \quad b' = 1 \quad b'' = 1$ &c.

Find from the result an easy method of multiplying any series by the last mentioned, and make use of it to find the first five terms of the fifth power of

$$1 + x + x^2 + x^3 + \text{\&c.}$$

Ans. $\quad 1 + 5 x + 15 x^2 + 35 x^3 + 70 x^4.$

Show that in the product of the two series

$$a + a' x + a'' x^2 + a''' x^3 + a^{\text{iv}} x^4 + \ldots.$$

$$a - a' x + a'' x^2 - a''' x^3 + a^{\text{iv}} x^4 - \ldots. :$$

there can be no terms involving odd powers of x.

Show that the following are all equal to each other and to $\dfrac{1}{1-x}$

$$1 + \frac{x}{1-x} \qquad 1 + x + \frac{x^2}{1-x} \qquad 1 + x + x^2 + \frac{x^3}{1-x} \quad \text{\&c.}$$

Three men, A, B and C, could complete a work as follows: A and B in c days, B and C in a days, A and C in b days. In what time could each complete it, and in what time could they all do it together?

Ans. A B and C could severally do it in

$$\frac{2\,a\,b\,c}{a\,b + a\,c - b\,c} \qquad \frac{2\,a\,b\,c}{a\,b + b\,c - a\,c} \qquad \frac{2\,a\,b\,c}{b\,c + a\,c - a\,b} \quad \text{days,}$$

and all three together in $\dfrac{2\,a\,b\,c}{a\,b + b\,c + c\,a}$ days.

Explain the case in which $a = 1$, $b = 4$, and $c = 6$, and also that in which $a = 1$, $b = 2$, $c = 2$.

Every whole number is either one of the series of powers of 2 contained in 1, 2, 4, 8, 16, &c., or may be made by adding together terms of this series without repeating any term twice. And every whole number is either one of the series

of powers of 3 contained in 1, 3, 9, 27, 81, &c., or may be made by addition and subtraction of terms of this series without using any one twice.

Prove the following formulæ:

$$\frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}$$

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4};$$

and having proved these, deduce from them the following theorems: 1. That the sum of all whole numbers up to n is $\frac{1}{2}n(n+1)$. 2. That the sum of the squares of all whole numbers up to n^2 is $\frac{1}{6}n(n+1)(2n+1)$. 3. That the sum of the cubes of all whole numbers up to n^3, is the square of the sum of all whole numbers up to n.

From what immediately precedes, prove that the sum of n terms of the series

$$a, \quad a+b \quad a+2b \ \ldots . \ \text{is } na + n\frac{n-1}{2}b$$

that the sum of n terms of

$$a^2 \quad (a+b)^2 \quad (a+2b)^2 \ldots . \ \text{is } na^2 + n(n-1)ab + \frac{1}{6}n(n-1)(2n-1)b^2$$

and that the sum of n terms of a^3, $(a+b)^3$ &c., is

$$na^3 + \frac{3}{2}n(n-1)a^2b + \frac{1}{2}n(n-1)(2n-1)ab^2 + \frac{1}{4}n^2(n-1)^2b^3$$

What is the inverse operation to adding one nth part of the whole, and subtracting one nth part of the whole?

Answers. The subtraction of the $(n+1)$th part, and the addition of the $(n-1)$th part. (The principal difficulty is in the correct understanding of the words of the question.)

There is a number to which I add its fourth part; from the sum I take 3, and to the difference I add its fifth part. The result is 10. What was the number?

Ans. $9\frac{1}{15}$.

There is a number to which a is added, and the result is divided by b. To the quotient a' is added, and the result divided by b'. To the quotient a'' is added, and the result divided by b''. The result of the last process is found to be h. What was the number?

Ans. $hb''b'b - a''b'b - a'b - a$.

In the preceding, let the addition be changed into a subtraction, and the division into multiplication. What is the number?

Ans.

$$x = a + \frac{a'}{b} + \frac{a''}{bb'} + \frac{h}{bb'b''}.$$

Multiply the expression $\sqrt{a} + \sqrt{b} + \sqrt{c}$ by $\sqrt{a} + \sqrt{b} - \sqrt{c}$, give the product the form $P + \sqrt{Q}$, and multiply by $P - \sqrt{Q}$. What is the result?

Ans. $a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$.

How many different cases are there of $\pm a \pm b$, and what is the product of all?

Ans. $\left(a^2 - b^2\right)^2$.

How many different cases are there of $\pm a \pm b \pm c$, and what is the product of all?

Ans. $\left(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2\right)^2$.

Show that one of the three, $a - b$, $b - c$, $c - a$, must be negative, and one must be positive; and that $a^2 + b^2 + c^2$ always exceeds $ab + bc + ca$.

If m be a given number, then x can always be taken so great that $(x+m)^2$ shall exceed x^2 as much as we please; and at the same time by as small a fraction of x as we please.

Show that the number of different ways (counting different orders as different ways) in which p numbers, no one of which exceeds m, can be put together so as to make q, must be the co-efficient of x^q in the development of

$$\left(x + x^2 + x^3 + \ldots\ldots + x^{m-1} + x^m\right)^p.$$

What are the roots of the following equation

$$a\,(p\,x^2 + q\,x + r)^2 + b\,(p\,x^2 + q\,x + r) + c = 0\,?$$

Ans. The four cases of

$$\frac{-\,q\,\sqrt{a} \pm \sqrt{a\,q^2 - 4\,a\,p\,r - 2\,b\,p \pm 2\,p\,\sqrt{b^2 - 4\,a\,c}}}{2\,\sqrt{a}.\,p}.$$

Show that the sum of these four roots is $-\,\dfrac{2\,q}{p}$.

Prove the following formula by verification:

$$\sqrt{2}\,\sqrt{a \pm b\,\sqrt{c}} \;=\; \sqrt{a + \sqrt{a^2 - b^2\,c}} \;\pm\; \sqrt{a - \sqrt{a^2 - b^2\,c}}.$$

For what numbers or fractions is $x^2 - c\,y^2$ a square?

Ans. m, n, and p, being any whole numbers or fractions, let

$$x = p\,(c\,m^2 + n^2) \qquad y = 2\,p\,m\,n.$$

How must m, n, and p, be taken, so that c being a fraction, $x^2 - c\,y^2$ may be a square whole number?

Assuming the following notation

$$V_0 = 1\,,\quad 2\,V_1 = x + \frac{1}{x},\quad 2\,V_2 = x^2 + \frac{1}{x^2},\quad 2\,V_3 = x^3 + \frac{1}{x^3},\ \&c.$$

show that $V_{n+1} + V_{n-1} = 2\,V_n\,V_1$.

Let p be a given whole number, and show that the following equation is satisfied by one value of n and m, and by one only;

$$m\,2^{\,n+1} + 2^n - 1 = 2\,p + 1.$$

Form a series of terms beginning with 1, and such that each term exceeds the preceding by the cube of the units figure in the sum of all the preceding.

Ans. 1, 2, 29, 37, 766, 891, &c.

Show that the following equation, $x^3 = a\,x + b$ is verified by

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}}$$

and show that the product of the two terms in the value of x just given is $\dfrac{a}{3}$.

Find a value of P from the equations $P = Q + x^n$ $P = 1 + Q\,x$, and show how this may be applied to deduce the following equation:

$$\frac{1 - x^n}{1 - x} = 1 + x + x^2 + x^3 + \ldots\ldots + x^{n-1}$$

from which deduce the following:

$$\frac{y^n - x^n}{y - x} = y^{n-1} + y^{n-2}\,x + y^{n-3}\,x^2 + \ldots\ldots + x^{n-1}.$$

Detect the mistake in the following process:

Let $a = b$; then $a^2 = a\,b$, or $a^2 - a\,b = 0$, and $a^2 = b^2$ or $a^2 - b^2 = 0$, thence $a^2 - a\,b = a^2 - b^2$, or $a\,(a - b) = (a + b)\ (a - b)$, or $a = a + b$. But $b = a$, then $a = a + a = 2\,a$.

If a and b be very nearly equal, then

$$\frac{\sqrt{a} - \sqrt{b}}{a - b} = \frac{1}{2\,\sqrt{a}} \text{ very nearly.}$$

In the equations $a\,x + b\,y = a\,b$, $x^2 + y^2 = c^2$, what relation must exist between a, b, and c, in order that the resulting values of x may be equal?

Ans. $c = \dfrac{a\,b}{\sqrt{a^2 + b^2}}$.

Two circles may cut each other in two points; two straight lines in one point, and a straight line and circle in two points. How many different points

of intersection may there be where there are 12 circles and 10 straight lines?

Ans. 427.

What is the answer to the preceding, when there are m circles, and n straight lines?

Ans. $(m+n)^2 - m - n\dfrac{n+1}{2}$.

Prove that the preceding expression is always positive when m and n are not less than 1.

There are three species of curves, marked A, B, and C. Two of the sort A may cut each other in a points, and two of the sorts B and C in b and c points. Again, A may cut B in c' points, B may cut C in a' points, and C may cut A in b' points. There are m, n, and p curves of the three sorts: how many points of intersection may there be in the figure?

Ans.

$$\frac{1}{2}\left(am^2 + bn^2 + cp^2 + 2a'np + 2b'pm + 2c'mn - am - bn - cp\right)$$

Verify the following equations:

$$1 = (x+1) - x$$
$$1.2 = (x+2)^2 - 2(x+1)^2 + x^2$$
$$1.2.3 = (x+3)^3 - 3(x+2)^3 + 3(x+1)^3 - x^3$$
$$1.2.3.4 = (x+4)^4 - 4(x+3)^4 + 6(x+2)^4 - 4(x+1)^4 + x^4$$

And also the following,

$$0 = (x+2) - 2(x+1) + x$$
$$0 = (x+3)^2 - 3(x+2)^2 + 3(x+1)^2 - x^2$$
$$0 = (x+4)^2 - 4(x+3)^2 + 6(x+2)^2 - 4(x+1)^2 + x^2$$
$$0 = (x+4)^3 - 4(x+3)^3 + 6(x+2)^3 - 4(x+1)^3 + x^3.$$

And also the following:

$$x^2 = x + 2x\frac{x-1}{2}$$

$$x^3 = x + 6x\frac{x-1}{2} + 6x\frac{x-1}{2}\,\frac{x-2}{3}$$

$$x^4 = x + 14x\frac{x-1}{2} + 36x\frac{x-1}{2}\,\frac{x-2}{3} + 24x\frac{x-1}{2}\,\frac{x-2}{3}\,\frac{x-3}{4}.$$

If there be a series of terms $a + a'x + a''x^2 +$ &c., of which the coefficients a, a', a'' &c. follow this law, namely, that each one, after the second, is the sum of the two preceding, then if V represent the sum of the series *ad infinitum*, we must have

$$V = \frac{a + (a'-a)x}{1 - x - x^2};$$

and if V_n represent the sum of the series as far as $a^{(n)}x^n$ inclusive, we must have

$$V_n = \frac{a + (a'-a)x - a^{(n+1)}x^{n+1} - a^{(n)}x^{n+2}}{1 - x - x^2}$$

Show that $a - b + c - e + \ldots$ must be less than a, and greater than $a - b$, if a, b, c, &c., be a series of decreasing positive quantities.

Reduce the binomial theorem

$$(1+x)^n = 1 + nx + n\frac{n-1}{2}x^2 + n\frac{n-1}{2}\,\frac{n-2}{3}x^3 + \ldots$$

to the following form:

$$(1+ax)^{\frac{1}{a}} = 1 + x + \frac{x^2}{2}(1-a) + \frac{x^3}{2.3}(1-a)(1-2a) + \ldots$$

What expressions are those, which, substituted instead of x in the following,

1. $\frac{1-x}{1+x}$ 2. $a x + x^2$ 3. $\frac{x - x^2}{1+x}$ 4. $a(x+m) - m$

will reduce them severally to x.

Answers.

1. $\frac{1-x}{1+x}$ 2. $-\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + x}$ 3. $\frac{1 - x \pm \sqrt{1 - 6x + x^2}}{}$

4. $\frac{1}{a}(x+m) - m$.

What expression must be substituted for x in $\frac{a + b x + c x^2}{b + x}$ in order that it may become $b + x$?

Ans. $\frac{x \pm \sqrt{x^2 + 4bcx - 4c(a - b^2)}}{2}$

If the series $a + a' x + a'' x^2 + \ldots$ be reduced to the form $a(1 + px + pqx^2 + pqrx^3 + \ldots)$, what are p, q, r, &c.?

If the expression $x^2 + xy + y^2$ change from 2 to 10 when x changes from 1 to 2, what are the changes of y?

What is the least number or fraction by which 7 more than a square number or fraction can exceed 5 times the number or fraction itself?

Ans. $\frac{3}{4}$.

Find the sum of the squares of the roots of $x^2 - (1+a)x + \frac{1 + a + a^2}{2}$, without finding the roots. *Ans.* a.

Verify the last by finding and squaring the roots. What is the expression which has for its roots the squares of the roots of $a x^2 + bx + c$?

Ans. $a^2 x + (2ac - b^2) x + c^2$.

Divide the number a into two such parts, that the first shall be the square of the second.

Ans. $\frac{2a + 1 - \sqrt{1 + 4a}}{2}$ and $\frac{\sqrt{1 + 4a} - 1}{2}$.

Show that if a be a whole square number, the answer of the last must be irrational, and also that the answer cannot be rational unless a be of the form $b(b+1)$ where b is rational.

Divide the number a into two parts, the product of which shall be a square.

Ans. $\frac{m^2 a}{m^2 + n^2}$ and $\frac{n^2 a}{m^2 + n^2}$

where m and n are any whole numbers. Show that these parts cannot be whole numbers unless a or one of its factors be the sum of two square numbers.

Divide the number a into two such parts that the sum of their squares shall be a square.

Ans. $\frac{(n^2 - m^2)a}{n^2 + 2mn - m^2}$ and $\frac{2mna}{n^2 + 2mn - m^2}$

where m and n are any numbers, n being the greater. Show that these parts may be made whole numbers when a is a whole number, itself or one of its factors being the difference between one square number and the double of another.

Solve the equations

$$\frac{1}{x} + \frac{1}{y} = a \qquad \frac{1}{y} + \frac{1}{z} = b \qquad \frac{1}{z} + \frac{1}{x} = c\,;$$

and explain the solution when $a + b = c$.

There are n numbers, the sum of all but the first is a_1, of all but the second, a_2, &c. &c., and of all but the last, a_n. What are the numbers?
Ans. The first is

$$\frac{1}{n-1}(a_1 + a_2 + a_3 + \ldots + a_n) - a_1 \qquad \text{the second is}$$

$$\frac{1}{n-1}(a_1 + a_2 + a_3 + \ldots + a_n) - a_2 \qquad \text{and so on.}$$

If there be three whole numbers, the product of every two of which is a square, then the numbers themselves must be squares.

Required the least number, which, divided by 4, 6, or 9, leaves a remainder 3, and by 15, a remainder 12. *Ans.* 147.

Of the four numbers, $x \quad xy \quad xy^2 \quad xy^3$, in continued proportion, the sum of the first and last is b, of the second and third a. Required x and y.

$$\textit{Ans.}\quad y = \frac{a + b \pm \sqrt{b^2 + 2ab - 3a^2}}{2a}$$

$$x = \frac{4a^2}{(a + b \pm \sqrt{b^2 + 2ab - 3a^2})\,(3a + b \pm \sqrt{b^2 + 2ab - 3a^2})}$$

The upper or lower sign being used throughout. Explain the two solutions, show that of the two values of y, each is unity divided by the other. Show that the preceding results are rational when, m and n being any whole numbers,

$$b = \frac{3n^2 + m^2}{n(n-m)}\,\frac{a}{2}.$$

In x, xy, xy^2, xy^3, let the sum of the first and third be p, and that of the second and fourth q. Required x and y.

$$\textit{Ans.}\quad y = \frac{q}{p} \qquad x = \frac{p^3}{p^2 + q^2}.$$

$$\text{Let } x = \frac{a}{b+p} \qquad p = \frac{c}{d+q} \qquad q = \frac{e}{f+r} \qquad r = \frac{g}{h}.$$

Required the value of x, so that p, q, and r shall not appear in it.

$$\textit{Ans.}\quad \cfrac{a}{b + \cfrac{c}{d + \cfrac{e}{f + \cfrac{g}{h}}}} = \frac{(dfh + dg + eh)\,a}{bdfh + bdg + beh + cfh + cg}$$

Explain this in the case where $dfh + dg + eh = 0$.

Show that the following equations may all be satisfied by a value of x which is less than unity.

$$(a + h)^2 = a^2 + 2(a + x).h$$
$$(a + h)^3 = a^3 + 3(a + x)^2.h$$
$$(a + h)^3 = a^3 + 3a^2h + 3(a + x)h^2$$
$$(a + h)^4 = a^4 + 3(a + x)^3.h$$
$$\sqrt{a + h} = \sqrt{a} + \frac{1}{2}\frac{h}{\sqrt{a + x}}$$
$$\sqrt{(a + h)^2 + 1} = \sqrt{a^2 + 1} + \frac{a + x}{\sqrt{(a + x)^2 + 1}}\,h.$$

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$$\begin{array}{r}
4)\ 1. \\
12)\overline{11.25} \\
20)\ \overline{5.9375} \\
\overline{8.296875}
\end{array}$$

$P = £73.\ 15s. = 73.75$

$r = .05,$

$\therefore Pr = 3.6875,$

$$\begin{array}{r}
3.6875)8.296875(2.25 \\
\underline{7\ 3750} \\
92187 \\
\underline{73750} \\
184375 \\
184375
\end{array}$$

And the answer therefore is 2.25 years, or 2 years and a quarter.

300. M being the amount of £P in n years, it is evident that P is the present value of £M due in n years. And equation (2) of last article gives us

$$P = \frac{M}{1 + nr}$$

which is, under a different form, the rule given in art. [298] for finding the present value. Similarly if D be the discount.

$$D = M - P = M - \frac{M}{1 + nr}$$
$$= \frac{Mnr}{1 + nr}.$$

It is unnecessary to give examples of these expressions, as they are of the same nature as those of the last article.

301. In almost all money transactions, it is usual, when a deduction is made by way of discount in consequence of immediate payment, to calculate the interest of the sum to be paid, instead of the discount as above given. This gives an advantage to the person so paying, inasmuch as he deducts the interest of the sum to be paid instead of the interest of its present value. But the person receiving is willing to forfeit the difference for being freed from all doubts and uncertainty.

In the same way interest is substituted for discount in the general method of calculating equations of payments.

A owes B £P_1 due at the end of n_1 years, and £P_2, due at the end of n_2 years from the present *time;* at what time must he pay B the sum of £P_1 and P_2, that neither party may gain or lose?

Let n be the number of years required. Then $(n - n_1)$ years is the extra time during which A has the use of P_1, and he is therefore benefited by the interest of £P_1 for that time, or by $P_1(n - n_1)r$. But he pays £P_2 $(n_2 - n)$ years before it is due, and is a loser, therefore, by the discount of P_2 for that time, or by

$$\frac{P_2(n_2 - n)r}{1 + (n_2 - n)r}.$$

Now, in order that he may neither gain nor lose, he must be as much a loser by paying P_2 before it is due, as he is a gainer by paying P_1 after it is due. Equating therefore his gain and loss, and dividing by r we have

$$P_1(n - n_1) = \frac{P_2(n_2 - n)}{1 + (n_2 - n)r},$$

which by reduction becomes a quadratic equation. But in the ordinary method of treating this subject A is considered a loser not by the discount, but by the interest on P_2 for $(n_2 - n)$ years, or by $P_2(n_2 - n)r$. In that case, we have, proceeding as before,

$$P_1(n - n_1) = P_2(n_2 - n);$$

the solution of which gives

$$n = \frac{P_1 n_1 + P_2 n_2}{P_1 + P_2}.$$

Similarly, if n be the equated time for the payment of any number of debts, P_1, P_2, P_3, &c., due at the several times n_1, n_2, n_3, &c., we should, by the same process, arrive at the equation

$$n = \frac{P_1 n_1 + P_2 n_2 + P_3 n_3 + \&c.}{P_1 + P_2 + P_3 + \&c.};$$

which expression is tantamount to the rule usually given: *Add together the products of each debt multiplied by the time when it is due, and divide by the sum of the debts.* Here, as before, the substitution of interest for discount is to the advantage of the debtor. The rule is so simple that it is unnecessary to illustrate it by examples.

302. As soon as a sum of money is payable, it matters little whether it be due under the name of principal or interest; the use of it would be of equal value to its owner. It would, therefore, appear to be equitable that it should be charged with interest in one case as well as the other; in other words, that a debt forborne should be charged with compound interest. It is, however, a singular fact that the laws of

this country never consider it so charged. Thus, supposing a person to have unjustly withheld for any number of years, the annual payment of a certain sum of money by way of interest; he would be simply compelled to pay the annual sum multiplied by the number of years. Considering interest payable yearly, the following rule for finding the amount of a sum of money at compound interest requires no explanation. *Multiply the principal sum by the rate per cent., divide by a hundred, and add the principal sum.* This gives the amount due at the end of the first year. *Multiply again by the rate per cent., divide by a hundred, and add the amount due at the end of the first year.* The result is the amount due at the end of the second year. *The same operation is to be performed as many times as the number of years for which the interest is to be calculated.*

The difference between the amount so found and the original sum is the compound interest. When there is any fraction of a year, it is usual to add the simple interest for this portion of the amount due at the end of the preceding year.

What is the amount of £76. 15*s*. in 2¾ years at 3 per cent. compound interest? We shall reduce the quantities to their respective decimals, and perform the operation as recommended in art.[167], neglecting all decimals beyond the third.

```
        £.
        76.75
            3
   100)230.25
        2.303  interest for the first year.
       76.75
       79.053  amount due at end of do.
            3
  100)237.159
        2.372  interest for the second year.
       79.053
       81.425  amount due at end of do.
```

We must find the simple interest of this for three quarters of a year.

Multiplying by the rate per cent. we have

244.275, and dividing by 100
2.443

This is the interest for a whole year. To find that for 3 quarters of a year, multiply by ¾, or by .75.

```
           2.443
            .75
           1710
           122
           1.832
adding    81.425   amount due at end }
                   of second year.   }
          83.257   which becomes by
                   reduction.
```

	£.	s.	d.
	83	5	1½
Deducting original sum	76	15	
Compound interest is	6	10	1½.

303. The observations made in the beginning of art. [302] apply to the calculations of discount as well as those of interest. Correctly, then, the present value of a sum of money due in a given time, is such a sum as would at a given rate of compound interest amount to the given sum in that time. The same rule applies here as to the finding the present value at simple interest; with this difference, that the amount of £100. must be calculated at compound interest.

What is the discount on £173. 5*s*. due in 2 years at 5 per cent. compound interest?

	£.	s.
The amount of £100. in 1 year is	105	0
. . . . 2 years .	110	5

or, reducing to the decimal of a pound, 110.25. Reducing also £173. 5*s*. to the decimal of a pound, and proceeding as in art. [167],

```
          173.25
             100
110.25)17325(157.142
        11025
         63000
         55125
          78750
          77175
           1575
           1103
            472
            441
             31
             22
              9
```

The present value is £157.142, which becomes by reduction £157. 2*s*. 5½*d*., which subtracted from £173. 5*s*. gives for discount £16. 2*s*. 6¾*d*.

304. We shall retain for the algebraical formulæ for compound interest the notation adopted in art. [292], supposing, moreover, R to represent the amount of £1. in one year, which is evidently the same as $1 + r$.

Now, since £1. amounts to R in the first year, by a simple proportion R must amount to R^2 in the second year, and therefore £1. in two years amounts to R^2. It is thence clear, that in n years £1. amounts to R^n. Hence we have

$$M = PR^n \quad \ldots\ldots\ldots \quad (1.)$$

$$I = PR^n - P = P(R^n - 1) \quad (2.)$$

It is hardly necessary to observe, that these expressions present, under a slight variety of forms, the rules given in art. [302]. The following example will be sufficient.

What is the amount of £5. in 3 years at 5 per cent. compound interest?

Here $R = 1.05$, and $n = 3$.

Multiplying as in art. [167], retaining, however, 4 decimal places, as we shall multiply by 5, art. [167],

$$\begin{array}{r l}
1.05 & \\
\underline{1.05} & \\
105 & \\
\underline{525} & \\
1.1025 & \\
\underline{1.05} & \\
1{,}1025 & \\
\underline{551} & \\
1.1576 & = R \\
\underline{5} & = P \\
5.788 & \\
\underline{20} & \\
15.76 & \\
\underline{12} & \\
9.12 &
\end{array}$$

The amount is therefore £5. 15*s*. 9*d*.

This method of calculating compound interest for any number of years is exceedingly tedious, and in practice we must have recourse to logarithms.

305. The interest having been supposed payable at the end of each year, it would seem impossible to calculate *compound* interest for a less period than a year, or to assign any but integral and positive values for n in the equation (1) of the last article. And, indeed, the manner in which the equation was obtained would appear to confine us to such values. But a little further consideration will convince us that, a proper signification being attached to the quantity which M represents, the expression is also true for fractional and negative values. We have before observed one or two instances of the extension which algebraical notation gives to the terms of a question, and the continuity which it implies in the quantities it is employed on. The following remarks will still further illustrate this fact, while the applicability of equation (1) to all values of n, presents another instance of the law of continuity.

By the method of calculating compound interest adopted in art. [295], after finding the interest for one year, we added it to the principal, considered the two together as a new principal, found the interest for a year, added again, and so on. But the algebraical mode of treating the subject, considering R as the amount of £1. in one year, mentions no particular time at which the interest is to be added to the principal, to be from that time itself considered as principal, and charged with interest; on the contrary, the generality of this language forbids that any particular time should be selected in preference to another. The conversion, therefore, of interest into principal must be considered to proceed continuously. The amount, therefore, in a year only fixes the rate of increase, and we may calculate compound interest for a less period than a year with as much propriety as for a greater.

What is the amount of £P at compound interest in 6 months?

R being the amount of £1. in one year, let x be the amount of £1. in 6 months. It is evident, then, that 1, x, and R, are continued proportionals, and, therefore,

$$x^2 = R,$$

$$\therefore x = R^{\frac{1}{2}}.$$

And the amount of P in 6 months is $PR^{\frac{1}{2}}$, the expression we should have derived from equation (1), by making n equal to $\frac{1}{2}$.

Again, what is the amount of £P at

compound interest in the $\frac{1}{m^{th}}$ part of a year.

Let x be the amount of £1. in this time. Then it is clear, from the last example, that x^2 is the amount in $\left(\frac{2}{m}\right)$th parts of a year, and so on; so that $1, x, x^2 \ldots\ldots R$

is a series of continued proportionals of $m + 1$ terms, and

$$\therefore R = 1 \times x^m,$$

$$\therefore x = R^{\frac{1}{m}}.$$

And the amount of P is $R^{\frac{1}{m}}$, the expression we should have obtained by putting $n = \frac{1}{m}$ in equation (1).

Similarly, to find the amount in 2 years, and the $\frac{1}{m^{th}}$ part of a year, we have the amount at the end of 2 years equal to $P\,R^2$. Let this equal P_1. The amount of this in the $\frac{1}{m^{th}}$ part of a year $= P_1 R^{\frac{1}{m}} = P\,R^2 R^{\frac{1}{m}} = P\,R^{\frac{2m+1}{m}}$.

Now M is the value in n years of £P due at the present time, and in the same way as if n refers to a succeeding period M is the amount of £P, so if it refers to a preceding one P is the amount of £M in n years,

or in the latter case $P = M \, . \, R^n$,

and $\therefore M = \frac{P}{R^n} = P \, . \, R^{-n}$.

From all this we conclude that the equation (1) of the last article originally obtained for integral is also true for fractional and negative values of n. It will have occurred to the reader that the method formerly adopted of calculating the compound interest for $2\frac{3}{4}$ years, where, after finding the amount at the end of 2 years, we took the simple interest of this amount for $2\frac{3}{4}$, was incorrect, according to the principles last laid down. We shall apply these to two examples.

Required the amount of £153. 10s. in 1 year and a half at 21 per cent. compound interest.

R here = 1.21.

And reducing to decimals, the amount is $(153.5) \times (1.21)^{\frac{3}{2}}$.

$$\text{But } (1.21)^{\frac{3}{2}} = (1.21) \times (1.21)^{\frac{1}{2}}$$
$$= (1.21) \times (1.1)$$
$$= 1.331$$

```
 153.5
 1.331
 -----
 153.5
 46 05
  4 605
   154
 -------
 204.309
```

which becomes, by reduction, £204.6s.2d., which is the answer.

What is the amount of £6. in $2\frac{1}{2}$ years at 3 per cent. compound interest?

Here R = 1.03

$$n = 2\frac{1}{2} = \frac{5}{2}$$

$$\therefore R^n = (1.03)^{\frac{5}{2}} = (1 + .03)^{\frac{5}{2}}.$$

Now the following equation, art. [280], is true as far as it goes, that is, there is no term included in the &c. which is not multiplied by a higher power of x than the second,

$$(1+x)^m = 1 + m\,x + m\frac{m-1}{2}x^2 + \text{\&c.}$$

We shall suppose $x = .03$ and $m = \frac{5}{2}$, and since we only require a result accurate to 4 places of decimals, art. [167], we may neglect every power beyond the second of .03,

for $(.03)^3 = .000027$.

We have, then,

$$1 = 1$$

$$m\,x = \frac{5}{2} \times .03$$
$$= \frac{.15}{2}$$
$$= .075$$

$$m\frac{m-1}{2}x^2 = \frac{5}{2} \times \frac{3}{4} \times .0009$$
$$= \frac{15}{8} \times .0009$$
$$= \frac{1}{8} \times .0135$$
$$= .0017,$$

and, adding, $R^n = 1.0767$
$P = \quad 6$
$\therefore P R^n = 6.4602,$

which, by reduction, becomes £6.9s.2¼d.

306. The equation $M = P R^n$ contains 4 different quantities, from knowing 3 of which we can find the other. It was observed in art. [299], that in calculating the compound interest for any number of years we were obliged to have recourse to logarithms. We are also unable to find the value of n, when that is the unknown quantity, without similar assistance. Thus taking the logarithms of both sides of the above equation, and referring to our section on logarithms,

$$\begin{aligned}\text{Log. } M &= \log. P . R^n \\ &= \log. P + \log. R^n \\ &= \log. P + n \log. R \ldots (1)\end{aligned}$$

$$\text{And} \therefore n = \frac{\log. M - \log P}{\log. R} \ldots (2)$$

Thus to find the amount of £15. 10s. in 10 years at 5 per cent. compound interest, we should have to multiply 1.05 by itself 10 times by the ordinary process. Referring to equation (1), and observing that £15. 10s. = £15. 5, we have

$$\begin{aligned}\text{Log. } P &= 1.1903317 \\ n \text{ Log. } R &= .211893 \\ \therefore \text{Log. } M &= 1.4022247\end{aligned}$$

And $\therefore$ M = 25.248, which, by reduction, becomes £25. 4s. 11½d.

Of Annuities.

307. From the greater complication of the subject we shall dispense with giving arithmetical rules for calculating annuities, and proceed at once to the algebraical method. We thus find the amount of an annuity forborne any number of years, supposing it charged with simple interest.

Let the annuity be represented by A, and supposed to be payable at the end of every year. Retaining in other respects the same notation as in art. [299], and observing that the interest *for* each year is charged on the sum of the annual payments due at the end of the preceding year, we have

Due *for* the 1st year . A
,, 2nd . . . $A + A r$
,, 3rd . . . $A + 2 A r$
,, n^{th} . . . $A + (n-1) A r$,

and the sum of these or $n A + (1 + 2 + \&c. + (n-1)) A r$ gives the whole amount. The coefficient of $A r$ is an arithmetic series of $n - 1$ terms, whose common difference is 1, and, therefore, the sum of it, art. [143], is

$$\left(2 + (n-2)\right)\frac{n-1}{2}, \text{ or } n.\frac{n-1}{2},$$

$$\therefore M = n A + n . \frac{n-1}{2} r A.$$

A promised to pay B £10. at the end of every year, but neglected to do so. What was due to B at the end of the 20th year, simple interest being charged at 5 per cent.?

Here $A = 10, r = .05, n = 20$, and

$$\therefore n \frac{n-1}{2} = 190,$$

and $\qquad r A = .5.$

Hence $\quad n . \dfrac{n-1}{2} . r A = 95,$

$\therefore M = 200 + 95$, and the amount due is £295.

Estimated at simple interest, the present value of an annuity is found as follows.

The present value of A, art. [298], to be paid at the end of 1 year, is equal to $\dfrac{A}{1 + r}$, to be paid at the end of 2 years is equal to $\dfrac{A}{1 + 2 r}$, and so on, and to be paid at the end of n years $= \dfrac{1}{1 + n r}$.

Now the sum of these present values is the present value of A to be paid at the end of 1, 2, and n years, or of an annuity of £A to continue n years. Thus we have P representing the present value, P =

$$A\left(\frac{1}{1+r} + \frac{1}{1+2r} + \&c. + \frac{1}{1+n r}\right)(1).$$

If we had supposed the first annual payment to be made at the end of the m^{th} instead of the 1st year, and then to continue n years, we should, by a like process, obtain for the present value

$$A\left\{\frac{1}{1 + m r} + \frac{1}{1+(m+1)r} + \&c. + \frac{1}{1 + (m + n) r}\right\}$$

What is the present value of an annuity of £5., to continue 3 years, at $1\frac{1}{2}$ per cent. simple interest?

Here $n = 3$,

$$\therefore P = 5 \times \left\{\frac{1}{1+r} + \frac{1}{1+2r} + \frac{1}{1+3r}\right\}.$$

Now $r = .015$ and r^2 and the higher powers of r are so small as not to have any influence in the result, and may, therefore, be neglected. We have then, art. [284],

$$\frac{1}{1+r} = 1 - r + r^2$$

$$\frac{1}{1+2r} = 1 - 2r + 4r^2$$

$$\frac{1}{1+3r} = 1 - 3r + 9r^2,$$

and their sum $= 3 - 6r + 14r^2$, which, putting for r its value, and performing the operations indicated, is equal to 2.9132, and multiplying by 5, $P = 14.566$, which, by reduction, becomes £14. 11*s*. $3\frac{3}{4}$*d*.

308. Some writers have defined the present value, estimated at simple interest, of an annuity to continue any number of years, to be that sum the amount of which would, in the given number of years, be equal to the amount of the annuity. But the sum thus obtained is not the present value of the annuity, but of the amount of the annuity after the given number of years. This amount, by art. [307], is $nA + n \cdot \frac{n-1}{2} rA$ and P' being the present value,

$$P'(1 + nr) = nA + n\frac{n-1}{2} rA,$$

or $$P' = \frac{nA + n \cdot \frac{n-1}{2} rA}{1 + nr},$$

which differs from P the present value of the *annuity*, found in art. [307], as would be shown by substituting any number greater than unity for n in the values of P and P′. The meaning we give to the expression present value would naturally lead us to expect the two quantities, P and P′, to be equal. Their inequality is the strongest proof of the inadequacy of a mode of calculation, like that of simple interest, which, as it were, sets a mark upon any sums of money that may have accrued by way of interest, and forbids their future accumulation. The reason of their inequality is easily explained. Suppose p to be the present value of £m due in one year. Then $p = \frac{m}{1+r}$, and let us suppose m to be unpaid for a second year and charged with interest; it amounts to $m(1+r)$. But p in two years amounts to $p(1+2r)$, or to $\frac{m(1+2r)}{1+r}$, which is different from the amount of m, and the reason is, because pr, the interest on p for the first year, is not charged with interest for the second year; and, therefore, in one case m was charged with interest and in the other only p. Therefore p, which is the present value of m, is not the present value of the amount of m after any number of years. The application of this to each payment of the annuity is manifest.

The present value of an annuity to continue for ever is found by making n infinite in the expressions for P and P′. The first becomes equal to

$$A\left(\frac{1}{1+r} + \frac{1}{1+2r} + \&c.\right),$$

the series being continued *ad infinitum*; and the latter becomes itself infinite, which is an additional proof of the inapplicability to practice of the principles upon which it rests.

309. R being, as before, the amount of £1. in one year, we thus find the amount of an annuity, forborne any number of years, at compound interest. Observing that the *whole* sum due at the end of each successive year is one of the annual payments, together with the amount in one year of the sum due at the end of the preceding year, we have due at the end of

1st year, A

2nd . $A + AR$

3rd . . $A + AR + AR^2$

n^{th} . . $A + AR + \&c. + AR^{n-1}$,

or the amount in n years is $A(1 + R + \&c. + R^{n-1})$. Now the quantity within the brackets is a geometric series of n terms, commencing with unity and having R for a common ratio, and therefore the sum of it [art. 151] is $\frac{R^n - 1}{R - 1}$,

and $\therefore M = A\dfrac{R^n - 1}{R - 1} \dots$ (1).

To find the present value estimated at compound interest we have, art. [299], the present value of A to be paid at the end of 1 year is equal to $\dfrac{A}{R}$, to be paid at the end of 2 years is equal to $\dfrac{A}{R^2}$, and to be paid at the end of n years is equal to $\dfrac{A}{R^n}$, so that the sum of these present values, which is the present value of the annuity to continue n years, is

$$\frac{A}{R} + \frac{A}{R^2} + \&c. + \frac{A}{R^n},$$

or, P being the present value,

$$P = \frac{A}{R}\left(1 + \frac{1}{R} + \&c. + \frac{1}{R^{n-1}}\right) \dots (2).$$

The quantity within the brackets is a geometric series of n terms, whose common ratio is $\dfrac{1}{R}$ and first term unity, and its sum, art. [151], is $\dfrac{1 - \dfrac{1}{R^n}}{1 - \dfrac{1}{R}}$,

and $\therefore P = A \,.\, \dfrac{\left(1 - \dfrac{1}{R^n}\right)}{R - 1}$.

The present value of the amount M of the annuity in n years is $\dfrac{M}{R^n}$, or (putting for M its value in the present article) it becomes $A \times \dfrac{1 - \dfrac{1}{R^n}}{R - 1}$, which is the same as the present value of the annuity. The reader remembers that the two were different when calculated by simple interest.

Had the first payment been made at the end of the m^{th} instead of the first year, and continued n years afterwards, we should have had by the same process the present value represented by

$$\frac{A}{R^m} + \frac{A}{R^{m+1}} + \&c. + \frac{A}{R^{m+n-1}},$$

or by

$$\frac{A}{R^m}\left\{1 + \frac{1}{R} + \&c. + \frac{1}{R^{n-1}}\right\},$$

that is summing the geometric series, art. [151], by

$$\frac{A}{R^m} \times \frac{1 - \dfrac{1}{R^n}}{1 - \dfrac{1}{R}},$$

or

$$\frac{A}{R^{m-1}} \times \frac{1 - \dfrac{1}{R^n}}{R - 1} \dots (3).$$

If the annuity be supposed to continue for ever, the series in equation (2) will go on *ad infinitum*, and P being the present value, we shall have

$$P = \frac{A}{R}\left(1 + \frac{1}{R} + \frac{1}{R^2} + \&c.\ ad\ infinitum\right),$$

or, art. [153],

$$P = \frac{A}{R} \times \frac{1}{1 - \dfrac{1}{R}} = \frac{A}{R - 1} \dots (4);$$

and similarly if the first payment be made after the m^{th} year, and then continue for ever, the present value is

$$\frac{A}{R^{m-1}} \times \frac{1}{R - 1}.$$

The rate of interest is $3\frac{2}{5}$ per cent., what sum of money is equivalent to an income of £3. a year?

$3\frac{2}{5}$ is equal to 3.4 per cent., and, therefore, $R = 1.034 \;\therefore R - 1 = .034$.

By equation (4) of this article

$$P = \frac{A}{R - 1} = \frac{3}{.034}, \text{ or } \frac{3000}{34},$$

which, by division, is equal to $88\frac{4}{17}$, or $88\frac{1}{4}$ nearly. The answer then is £$88\frac{1}{4}$.

Indeterminate Equations.

310. In our sections on simple and quadratic equations, art. [108], &c. and art. [204], &c., we considered those cases only where there was the same number of independent equations as of unknown quantities. The method of solution adopted in this case was, when there were several unknown quantities, by combining the equations in any manner to obtain an equation involving only one unknown quantity, see art. [119], and

art. [213]. We now propose to examine those cases where we have a less number of equations than of unknown quantities. Imperfect as our general means of solution were in the former case, except where our equations were all of the first degree, we shall find our powers in this case confined within still narrower limits.

We begin, as before, with equations of the first degree, and first with the most simple case of one equation involving two unknown quantities, which may be represented generally by

$$a x + b y = c.$$

We observe, in the first place, that if the values of x and y which we are seeking for may be of any kind, positive or negative, whole numbers or fractions, there are an infinite number of such values which satisfy this equation, see art. [119]. For we have only to substitute *any* value for one of the unknown quantities, y for instance, and then solve the equation with respect to the other, x, that is, considering x as the only unknown quantity, and the assumed value for y, and the value found for x, present us with a solution of the equation. It is for this reason, that equations of this kind are called indeterminate, because they do not fix, or determine, the values of the unknown quantities. All systems of equations, where there are more unknown quantities than independent equations, are indeterminate in the same sense of the word; for, retaining as many unknown quantities as there are equations, we may give the rest any values we please, and thus arrive at an infinite number of solutions. But returning to the equation $a x + b y = c$, suppose that we were seeking only such values of x and y as being whole numbers, or being whole numbers and positive, satisfied this equation, the above method would not be of any service, for although we might assume a whole number for y, yet the value of x obtained in the above manner, and which with the assumed value for y would form a solution of the equation, would, in all probability, be a fraction.

In the second place, the equation $a x + b y = c$, being cleared of fractions, and in its lowest terms, in order that any integral values of x and y may be capable of satisfying this equation, it is necessary that a and b be prime to each other. For, supposing for a moment this not to be the case, so that a and b having a common divisor, suppose r, may be written under the form $e r$ and $f r$, where e, f, and r are whole numbers, substituting their values of a and b in the equation $a x + b y = c$, it becomes $e r x + f r y = c$, or, dividing by r, $e x + f y = \frac{c}{r}$. But the equation having been previously in its lowest terms, $\frac{c}{r}$ must be a fraction, and it is in that case evident, that the equation $e x + f y = \frac{c}{r}$, cannot be satisfied by integral values of x and y.

We shall now show how we may find the whole numbers which satisfy the equation $a x + b y = c$, observing that questions which give rise to equations of this kind, usually require integral values of the unknown quantities. It will be best to commence by taking a numerical example of this equation, and to consider the more general case afterwards.

311. It is required to find the integral values of x and y in the equation $5 x + 7 y = 81$. We observe here, that the conditions above alluded to are satisfied, the equation being in its lowest terms, and 5 and 7 being prime to each other.

Transforming to the right-hand side of the equation, the term having the largest coefficient, we have

$$5 x = 81 - 7 y.$$

Dividing by 5,

$$x = \frac{81 - 7 y}{5}.$$

Dividing out as much as possible,

$$x = 16 + \frac{1}{5} - y - \frac{2 y}{5}$$

$$= 16 - y - \frac{2 y - 1}{5} \dots (1)$$

Our having transformed the term with the larger coefficient, has, we see, enabled us to simplify the expression by dividing out by the smaller coefficient. Now any value of y being substituted in this equation, that value of y, together with the value of x derived from the same equation, form, in the general sense of the word, a solution of the equation. But we are only seeking integral values of x and y; we shall, therefore, now find such a value of y as, being itself a whole number, will, when substituted in equation (1), make the

value of x derived therefrom a whole number.

Now, in order that x may be a whole number, it is necessary that $\frac{2y-1}{5}$ be a whole number.

Let $$\frac{2y-1}{5} = v,$$

v being any whole number,

$$\therefore 2y - 1 = 5v.$$

This is an equation of the same kind as the original one, but its terms are simpler, and necessarily so, from the operation of dividing out, before made use of. In order to find the values of y and v we proceed as before. We thus obtain

$$y = \frac{5v+1}{2} \dots . (2)$$

$$= 2v + \frac{v+1}{2}.$$

Now y being a whole number, $\frac{v+1}{2}$ must be a whole number.

Let $$\frac{v+1}{2} = w,$$

w being any whole number.

We obtain from this,

$$v = 2w - 1 \dots . (3).$$

From all these operations we see that, w being any whole number, the value of v derived from equation (3) is integral, as also the value of y from equation (2), and that of x from equation (1). The equation (3) is of the same nature as equation (1), but the coefficient of v being unity we may assume any value for w, and are certain of an integral value for v. Although the coefficient of v might have been some whole number greater than unity, yet by pursuing the same process we are sure of arriving ultimately at an equation of this form, the operation of dividing out constantly diminishing the coefficients of the quantities y, v, &c.

Substituting the value of v derived from equation (3), in equation (2) we have

$$y = 4w - 2 + w,$$

or $$y = 5w - 2 \dots\dots . (4).$$

And again, substituting this value of y in equation (1),

$$x = 16 - 5w + 2 - 2w + 1,$$

or

$$x = 19 - 7w \dots\dots\dots\dots . (5).$$

The corresponding values of x and y obtained by giving all integral values to w in the equations (4) and (5), present so many solutions of the equation. The quantity w is called an indeterminate quantity, or, more shortly, an indeterminate.

Supposing then w equal to 1, we have

$$x = 12,$$
$$y = 3.$$

Supposing w equal to 2,

$$x = 5,$$
$$y = 8.$$

And giving to w all succeeding integral values from 1 upwards, we obtain the following corresponding values of x and y.

12 and	3,
5	8,
− 2	13,
− 9	18,
&c.	&c.

We shall presently return to the law which the several values of x and y follow.

312. In solving equations of this kind we always endeavour to arrive at values of x and y expressed in terms of some indeterminate w, which is susceptible of all integral values, as in equations (4) and (5). The reader will find by trial, that unless the coefficients of x and y are prime to each other, the attainment of this result is impracticable, art. [308]. It is frequently much shortened by the employment of various artifices for which no general rule can be given. They can be only learnt by observation and practice.

Thus taking the equation

$$11x - 17y = 5.$$

Proceeding as before,

$$11x = 17y + 5,$$

and $$x = \frac{17y+5}{11}$$

$$= y + \frac{6y+5}{11}.$$

If we continued to proceed as before, we should put $\frac{6y+5}{11} = v$, but observing, that the difference between 6, the coefficient of y, and the denominator 11, is equal to the other term 5, we put the above equation under another form, namely,

$$x = y + \frac{11y - 5y + 5}{11},$$

$$x = 2y - \frac{5y - 5}{11},$$

$$= 2y - \frac{5(y-1)}{11}.$$

Now 5 being prime to 11, $\frac{y-1}{11}$ must be a whole number.

Let $\frac{y-1}{11} = w$,

$\therefore y - 1 = 11w$,

and $y = 11w + 1$.

Also $x = 2y - 5w$,

$= 22w + 2 - 5w$,

or $x = 17w + 2$.

So that the several integral values of x and y, obtained by giving to w all integral values from 0 upwards, are

2 and 1,

19 ... 12,

36 ... 23,

&c. &c.

To find a number which, when divided by 6, gives a remainder 5, and when divided by 7, a remainder 3. Let us suppose 6 to be contained x times in the number in question, with a remainder 5, then the number is $6x + 5$.

And similarly, supposing 7 to be contained y times, with a remainder 3, the number is $7y + 3$. Equating the two values of the number, we have the following equation:

$$6x + 5 = 7y + 3,$$

or $6x = 7y - 2$,

$$\therefore x = \frac{7y - 2}{6},$$

$$= y + \frac{y-2}{6}.$$

Let $\frac{y-2}{6} = w$,

then $y = 6w + 2$,

and $x = 6w + 2 + w$,

$= 7w + 2$.

The several pairs of values of x and y are

2 and 2,

9 8,

16 14,

25 20,

&c. &c.

And the several values of $6x + 5$, or $7y + 3$, or of the number required, are

17, 59, 101, &c.

numbers which will be found upon trial to satisfy the proposed condition.

A person has only crowns and 3 shilling pieces in his pocket, and wishes to pay a bill of £2. 16*s*. How many must he give of each?

Let x = the number of 3 shilling pieces, and y = ... crowns, then the sum expressed in shillings is $3x + 5y$, and therefore we have by the question,

$$3x + 5y = 56,$$

$$\therefore 3x = 56 - 5y,$$

$$x = \frac{56 - 5y}{3},$$

or $x = 18 - y - \frac{2y - 2}{3}$,

$$= 18 - y - \frac{2(y-1)}{3}.$$

Let $\frac{y-1}{3} = w$,

then $y - 1 = 3w$,

and $y = 3w + 1$;

also $x = 18 - (3w + 1) - 2w$,

$= 17 - 5w$.

And the several pairs of values for x and y are

17 and 1,

12 4,

7 7,

&c. &c.

− 3 and 13,

&c. &c.

So that 17 pieces of 3 shillings and 1 crown, 12 pieces of 3 shillings and 4 crowns, &c., make up the sum required. The negative values for x signify that so many pieces of 3 shillings must be given back. Thus the sum may be made up by giving 13 crowns, 3 pieces of 3 shillings being returned. Suppose it had been required to pay the same sum in crowns and half sovereigns. A moment's consideration shows this to be impossible, and forming the equation we shall find that the coefficients of x and y admit a divisor 5, to which 56, the number on the other side of the equation, is prime, art. [310].

313. If we observe the several values of either of the unknown quantities in any of the above examples, we shall find that they form an arithmetical progres-

sion, the common difference of which is the coefficient of the other quantity. For instance, in the first example, $5x + 7y = 81$, the several values of x were

$$12, \quad 5, \quad -2, \quad -9, \ \&c.$$

and those of y

$$3, \quad 8, \quad 13, \quad 18, \ \&c.$$

In the former we observe each term is less than the preceding by 7, the coefficient of y, and in the latter, each term is greater by 5, the coefficient of x.

In order to show that this is necessarily the case, we will consider the general equation $ax + by = c$. This equation is in its lowest terms, and a and b are prime to each other. Suppose we have found x' and y', two integral values of x and y, which satisfy this equation, we have

$$ax' + by' = c \ldots . (1).$$

Now let $x' + m$, and $y' + n$, be two other integral values of x and y, so that

$$a(x' + m) + b(y' + n) = c \ldots (2).$$

We propose to find what relations must subsist between m and n.

Substracting equation (1), term by term, from equation (2), we have

$$am + bn = 0,$$

or

$$am = -bn,$$

therefore

$$m = -\frac{bn}{a} \ldots . (3).$$

Now observing that b and a are prime to each other, in order that m may be a whole number it is necessary that n should be a multiple of a; and n must also be a whole number. Let then $n = aw$, w being capable of all integral values, positive as well as negative. Equation (3) gives us $m = bw$. Hence x' and y' being any two values of n and y, we have all others represented by

$$x' - bw,$$

$$y' + aw,$$

w being capable of all integral values.

From this result we learn, *first*, that the several integral values of x and y, which satisfy the equation $ax + by = c$, must necessarily be of the form above indicated.

Secondly. That a and b being both positive, while the value of y is increased by the coefficient of x, that of x is diminished by the coefficient of y, and *vice versâ*; but if a or b be negative, then their corresponding values are both increased or both diminished by the coefficient of the other.

Thirdly. That the indeterminate w being capable of negative as well as positive values, the arithmetic series at the beginning of the present article giving the several values of x and y, may be continued indefinitely to the left as well as the right.

It is clear from what has preceded, that having obtained any two integral values of x and y, which satisfy the equation, we can immediately (from the equations $x = x' - bw$, $y = y' + aw$) find an infinite number of such values. The main object then is the finding with rapidity two such values. A property of a *continued fraction* has been made use of for this purpose, and we shall briefly explain the manner in treating of that subject.

314. When we have one equation of the first degree involving more than two unknown quantities, the method of proceeding is very similar. We shall simply go through the operations, their explanation being the same as that given in the case of two unknown quantities.

$$4x + 9y + 10z = 103,$$

transforming

$$4x = 103 - 9y - 10z,$$

and

$$x = \frac{103 - 9y - 10z}{4},$$

$$= 25 - 2y - 2z - \frac{y + 2z - 3}{4} \ldots (1).$$

Let

$$\frac{y + 2z - 3}{4} = w,$$

$$\therefore y + 2z - 3 = 4w,$$

and

$$y = 4w + 3 - 2z \ldots . (2).$$

Substituting this value of y in equation (1),

$$x = 25 - 8w - 6 + 4z - 2z - w,$$

or

$$x = 19 - 9w + 2z \ldots . (3).$$

We have thus the values of x and y expressed in terms of z, one of the unknown quantities, and an indeterminate w, but the quantity z is so involved that giving to it any integral value, the corresponding values of x and y are also integral. Suppose z equal to 0, and giving to w sucessively the values 0, 1, 2, &c., we find the corresponding values of x and y to be

19 and 3,
10 ... 7,
1 ... 11,
&c. &c.

Next suppose $z = 1$, the corresponding values of x and y are

21 and 1,
12 5,
3 9,

and so on.

If it be necessary that the values of all the unknown quantities be positive, we must not give to z a greater value than 9. In that case x and y are each equal to unity.

A similar method may be adopted whatever be the number of unknown quantities, and we shall ultimately arrive at two values of x and y, expressed in terms of the other unknown quantities, and one indeterminate, or we shall have the values of several of the unknown quantities expressed in terms of the others, and of several indeterminates.

315. Still considering only simple equations, suppose that we have several equations involving, however, a greater number of unknown quantities, as, for example, two equations involving three.

$$14\,x + 11\,y + 9\,z = 360 \;\ldots\ldots\; (1),$$
$$x + y + z = 30 \;\ldots\ldots\; (2).$$

Our object is to find such *integral* values of x, y, and z as satisfy at the same time both these equations. The process adopted is analogous to that in art. [119], where we had two equations between two unknown quantities, and to the reasoning of that article the reader is referred for an explanation of what follows.

Multiplying equation (2) by 14, the coefficient of x in equation (1), we have

$$14\,x + 14\,y + 14\,z = 420.$$

Subtracting equation (1), term by term, from this, $14\,x$ disappears, and we have

$$3\,y + 5\,z = 60 \;\ldots\ldots\; (3).$$

From this equation we find the values of y and z in terms of an indeterminate w, which values are as follows,

$$y = 5 + 5\,w \;\ldots\ldots\; (4),$$
$$z = 9 - 3\,w \;\ldots\ldots\; (5).$$

Subtracting these values of y and z in equation (2), and reducing, we obtain

$$x = 16 - 2\,w \;\ldots\ldots\; (6).$$

Making w then successively equal to 0, 1, 2, &c., we obtain the following corresponding values of x, y, and z:

16, 5 and 9,
14, 10 ... 6,
12, 15 ... 3,
10, 20 ... 0,
&c. &c.

In the above equation the coefficients of the unknown quantities in one equation being equal to unity, the process was very simple. The following example will show more fully the method of solving equations of this kind.

$$2\,x + 5\,y + 3\,z = 108 \;\ldots\ldots\; (1),$$
$$3\,x - 2\,y + 7\,z = 95 \;\ldots\ldots\; (2).$$

Multiplying equation (1) by 3, the coefficient of x in equation (2), and equation (2) by 2, the coefficient of x in equation (1), we obtain

$$6\,x + 15\,y + 9\,z = 324,$$
$$6\,x - 4\,y + 14\,z = 190,$$

and, subtracting,

$$19\,y - 5\,z = 134 \;\ldots\ldots\; (3).$$

The values of y and z obtained from this equation in terms of an indeterminate t, are

$$y = 11 + 5\,t \;\ldots\ldots\; (4),$$
$$z = 15 + 19\,t \;\ldots\ldots\; (5).$$

Substituting these values of y and z in equation 2, we have

$$3\,x - 22 - 10\,t + 105 + 133\,t = 95,$$

and, transposing,

$$3\,x + 123\,t = 12.$$

Here we have again an equation between two unknown quantities, to which we should apply the method for solution of equations of this nature, and thus have values of x and t in terms of another indeterminate w. The substitution of the value of t in the equations (4) and (5), would give us the values of y and z in terms of w. We should thus have the values of x, y, and z in terms of the same indeterminate w. But observing the above equation, we see that all its terms are divisible by 3, and, dividing, we obtain

$$x + 41\,t = 4,$$

which gives us at once

$$x = 4 - 41\,t \;\ldots\ldots\; (6).$$

Thus equations (4), (5), and (6) give us at once the values of the three unknown quantities in terms of a quantity t, to which we may give any integral value. It will frequently happen that equations of this kind do not admit a solution in whole numbers, and that will be indicated by one of the equations which we arrive at between two unknown quantities, having its coefficients divisible by a number, of which the other sum is not a multiple. See art. [310].

316. By combining any two simple equations, whatever be the number of

unknown quantities comprised in them, in the way in which equations (1) and (2), in the last example, were combined, we may always arrive at an equation involving one unknown quantity less than those two equations. See art. [119]. When we do this, we are said to *eliminate* that unknown quantity. Thus equation (3), in the two last examples, arose from the elimination of x from the equations (1) and (2). Suppose then that we have p equations involving any greater number of unknown quantities. Combining the first of these equations with the second, third, and every other successively, and eliminating by each combination the same unknown quantity, we arrive at $p - 1$ equations, involving one unknown quantity less than the p original equations. Thus combining the first of the $p - 1$ equations with every other successively, we get rid of another unknown quantity, and arrive at $p - 2$ equations. Proceeding similarly, we shall at last arrive at *one* equation, involving a certain number of the unknown quantities, which we may solve by one of the methods which we have given, and find values of the unknown quantities comprised in it in terms of one or more indeterminates. This will lead us, by the continuation of a similar process, to the values of all the unknown quantities. It might at first appear, that, after combining the first of our p equations with each of the other, we might also combine the second with the third, and so on, and thus, eliminating the same unknown quantity as before, obtain more independent equations. But that, this is not the case may be proved as follows. Having transposed all the terms of each of the equations to the left-hand side, art. [109], we may write them $A = 0$, $B = 0$, $C = 0$, &c. Suppose a, b, c, &c. to be the coefficients of x, the quantity which we propose to eliminate, in the equations $A = 0$, $B = 0$, $C = 0$, &c. Then in order to get rid of x from A and B, we multiply the first by b, and the second by a, and subtract. We thus obtain

$$A b - B a = 0 \text{ .... } (1).$$

Similarly, combining $A = 0$ with $C = 0$, we obtain

$$A c - C a = 0 \text{ .... } (2).$$

Now combining $B = 0$ with $C = 0$, the equation we should arrive at, independent of x, is

$$B c - C b = 0 \text{ .... } (3);$$

and this equation is not independent of the equations (1) and (2), but derivable from them, as follows. Multiply equation (1) by c, and equation (2) by b, and subtract, we obtain

$$B a c - C a b = 0,$$

or, dividing by a,

$$B c - C b = 0,$$

which is the same as equation (3). The same may be proved of any combinations of the equations. We may hence conclude, that, when we have any number of simple equations less than the number of unknown quantities, we cannot, by combining them in any manner, arrive at the same number of independent equations as unknown quantities.

317. When indeterminate equations are above the first degree, their solution is one of considerable difficulty. If in one equation with two unknown quantities even the square of either of them occurs, its discussion would require a separate treatise. We will take one example, where the only term above the first degree is the product of the two unknown quantities, the solution of which will give some notion of the method to be generally adopted.

Required two numbers whose product, added to three times the first, and five times the second, is equal to 48.

x and y being the two numbers, the question gives us the following equation:

$$x y + 3 x + 5 y = 48.$$

Transposing the term not involving x, and collecting the coefficients of x,

$$x (y + 3) = 48 - 5 y,$$

and dividing by the coefficient of x,

$$x = \frac{48 - 5 y}{y + 3}.$$

We may get rid of y from the numerator by writing this

$$x = \frac{63 - 5 (y + 3)}{y + 3},$$

or $$x = \frac{63}{y + 3} - 5.$$

Now, in order that x may be a whole number, 63 must be a multiple of $y + 3$, and the several numbers of which 63 is a multiple are 3, 7, 9, 21. Hence $y + 3$ must be equal to one of these numbers. In the first case y would be equal to 0, but if we take 7, we have $y = 4$, and

$$x = \frac{63}{y+3} - 5$$
$$= 9 - 5 = 4,$$

or the two numbers are 4 and 4, as will be found correct by trial. Again, taking 9, we have $y = 6$, and

$$x = \frac{63}{9} - 5 = 2,$$

and we shall find the numbers 2 and 6 answer the proposed condition. The other value of $y + 3$, viz. 21, would make x negative.

On Continued Fractions.

318. When we have any fraction, as for instance $\frac{61}{13}$, dividing out as far as we can, the fractional part will become a proper fraction, and we have a mixed fraction $4\frac{9}{13}$, or, representing it algebraically,

$$4 + \frac{9}{13}.$$

Now this may be written

$$4 + \frac{1}{\frac{13}{9}}.$$

We may now proceed further with the division, for

$$\frac{13}{9} = 1 + \frac{4}{9}.$$

Writing this in the denominator we have

$$4 + \frac{1}{1 + \frac{4}{9}}.$$

Again, $$\frac{4}{9} = \frac{1}{\frac{9}{4}} = \frac{1}{2 + \frac{1}{4}}.$$

And putting this value of $\frac{4}{9}$ in the previous expression, we have the fraction $\frac{61}{13}$ represented by the following expression:

$$4 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4}}}.$$

A fraction represented in this way, where we observe the numerators of the several fractions which enter into it are all equal to unity, is called a *continued fraction*. We shall here be able to enter into a few only, and those the most simple and obvious, of the properties which it possesses.

In the first place it enables us to express in simpler terms the value of any given fraction to any degree of accuracy which the question may require. Thus taking the above expression representing the fraction $\frac{61}{13}$, and neglecting the fractional part altogether, we have for our first approximation 4, which differs from the accurate value of the fraction by $\frac{9}{13}$. Again, taking only the first term of the denominator, and neglecting the rest, our second approximation will be $4 + \frac{1}{1}$, or 5, and this differs from the accurate value of the fraction by $\frac{4}{13}$. Observe that this approximation is greater than the fraction itself, while the former one was less. Proceeding one step further, and representing the fraction by

$$4 + \cfrac{1}{1 + \cfrac{1}{2}},$$

our third approximation is $4 + \frac{2}{3}$, or $\frac{14}{3}$, which, like the first approximation, is less than the accurate value of the fraction, and differs from it by $\frac{1}{39}$, approaching nearer to it than either of the former values. We shall presently treat the subject more generally, and shall then see that it necessarily results from the form of the expression that the successive approximations, derived in the above manner, err alternately by excess and deficiency, and approach nearer and nearer to the accurate value of the fraction.

As an example of this, we will examine the length of the tropical year, or of the interval between the sun's leaving

and returning to the same equinox.* The most accurate calculations have proved this interval (upon which the seasons mainly depend) to be equal to 365,242264 days. Representing this fractionally, we have

$$365 + \frac{242264}{1000000},$$

or, reducing the fractional part to its lowest terms,

$$365 + \frac{30283}{125000},$$

or

$$365 + \cfrac{1}{\cfrac{125000}{30283}}.$$

And, dividing as before, we obtain

$$365 + \cfrac{1}{4 + \cfrac{3868}{30283}}.$$

This, by a further reduction, becomes

$$365 + \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{3207}{3868}}}.$$

Not proceeding any further with the division, we see that the three first approximations to the given fraction are 365, $365\frac{1}{4}$, $365\frac{7}{29}$. The first number gives us a very rude approximation to the length of the year, and one which, in the course of a few centuries, would completely invert the order of the seasons. The second answers to the Julian Calendar, by which one day was intercalated every four years; and the third differs, by a very small quantity, from the length of the day which is the basis of the Gregorian Calendar, and which is now acted upon in almost all the countries of Europe. According to this we intercalate a single day every four years, but omit three of these intercalations in four centuries. This makes us intercalate ninety-seven days in four centuries, and gives therefore for the length of each year $365\frac{97}{400}$ days.

319. But the utility of continued fractions in giving us approximations to the value of any fraction in simpler terms than the fraction itself, is not to be estimated very highly. Their principal utility consists in our frequently being able to express the value of an unknown quantity under this form only, or under no other so easily. On this account every thing connected with them becomes important. For instance, when we have the quantity sought for, *only* under this form, it is convenient to know, in stopping at any term of the continued fraction, how far our approximation differs from the value of the whole of it, or to know some limit to the error, so as to estimate the degree of accuracy. See art. [292]. We shall presently show how this limit may be assigned.

In art. [254] we showed how, from an equation of the form $a^x = b$, we might find the value of x by means of a table of logarithms. We may also express the value of x derived from such an equation in the form of a continued fraction. Then take the equation

$$2^x = 3.$$

Now, observing that

$$2^1 = 2,$$

and

$$2^2 = 4,$$

it is evident that in the above equation x must be greater than 1, and less than 2.

Suppose, then,

$$x = 1 + \frac{1}{x'},$$

where x' is greater than 1, (since x is less than 2.)

We have from the original equation

$$2^{1+\frac{1}{x'}} = 3,$$

or

$$2 \times 2^{\frac{1}{x'}} = 3,$$

or

$$2^{\frac{1}{x'}} = \frac{3}{2};$$

or, raising both sides of the equation to the x'^{th} power,

$$2 = \left(\frac{3}{2}\right)^{x'} \ldots\ldots (1).$$

Now $\left(\frac{3}{2}\right)^1 = \frac{3}{2}$, which is less than 2;

but $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$, which is greater than 2, and therefore x' must be greater than 1, and less than 2.

* In reality, the earth moves round the sun, but it aids conception sometimes to consider the earth at rest, and the sun moving round it. The sun is said to be in the *equinox* in those two positions where the plane of the earth's equator being produced passes through him. At these times the duration of day is equal to that of night all over the earth.

Suppose, then,

$$x' = 1 + \frac{1}{x''}.$$

Substituting this in equation (2),

$$2 = \left(\frac{3}{2}\right)^{1+\frac{1}{x''}},$$

or

$$2 = \frac{3}{2} \times \left(\frac{3}{2}\right)^{\frac{1}{x''}},$$

or

$$\frac{4}{3} = \left(\frac{3}{2}\right)^{\frac{1}{x''}};$$

which gives us

$$\left(\frac{4}{3}\right)^{x''} = \frac{3}{2} \ldots . (2).$$

Now

$$\left(\frac{4}{3}\right)^1 \text{ is less than } \frac{3}{2}$$

and

$$\left(\frac{4}{3}\right)^2, \text{ or } \frac{16}{9}, \text{ is greater than } \frac{3}{2};$$

whence we infer, as before, that x'' is greater than 1, and less than 2. Let, then,

$$x'' = 1 + \frac{1}{x'''}.$$

Equation (2) gives us

$$\left(\frac{4}{3}\right)^{1+\frac{1}{x'''}} = \frac{3}{2},$$

or

$$\frac{4}{3} \times \left(\frac{4}{3}\right)^{\frac{1}{x'''}} = \frac{3}{2},$$

or

$$\left(\frac{4}{3}\right)^{\frac{1}{x'''}} = \frac{9}{8},$$

which gives us

$$\frac{4}{3} = \left(\frac{9}{8}\right)^{x'''}.$$

We should here find that x''' was greater than 2, and less than 3, and might again proceed in the same manner as before.

The result of all this is, that

$$x = 1 + \frac{1}{x'},$$

or substituting for x'

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{x''}},$$

or substituting for x''

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x'''}}},$$

or

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \&c.}}},$$

since x''' is greater than 2.

The several approximations to the value of x derived from this expression are

$$1,\ 2,\ \frac{3}{2},\ \frac{8}{5},\ \&c.$$

320. In order to consider this subject generally, and exhibit the relations which the successive approximations bear to each other, and their respective degrees of accuracy, we will take the general form of a continued fraction, or

$$y + \cfrac{1}{y_1 + \cfrac{1}{y_2 + \cfrac{1}{y_3 + \&c.}}},$$

where y, y_1, y_2, y_3, &c. are whole numbers. The first approximation to the value of this fraction is y,

the second is $y + \dfrac{1}{y_1}$,

the third is $y + \cfrac{1}{y_1 + \cfrac{1}{y_2}}$,

and so on. With regard to their several approximations, we immediately observe that the *first* y is too *small*, there being some quantity, $\dfrac{1}{y_1 + \&c.}$, to be added to it; that the *second* $y + \dfrac{1}{y_1}$ is too *great*, the denominator of the fractional part not being y_1 but some quantity, $y_1 + \dfrac{1}{y_2 + \&c.}$, greater than y_1; the *third*, $y + \cfrac{1}{y_1 + \cfrac{1}{y_2}}$, is too *small* for $y_1 + \dfrac{1}{y_2}$, the denominator of the fractional part is too great for the

same reason that $y+\frac{1}{y_1}$ was too great in the second approximation, and so on. It is unnecessary to proceed further to prove the following principle. *The successive approximations to the value of a continued fraction are alternately too great and too small, the even ones too great, and the odd ones too small.*

321. In order to exhibit the connections of the several approximations with each other, let the *first* be represented by $\frac{p}{q}$, the *second* by $\frac{p_1}{q_1}$, the *third* by $\frac{p_2}{q_2}$, and so on. The several fractions $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, &c. are called converging fractions, or more shortly convergents; $\frac{p_1}{q_1}$ the first, $\frac{p_2}{q_2}$ the second, &c. and their several numerators and denominators are intended to represent the numerators and denominators of the several approximations, when reduced to the form of simple fractions by the process made use of in art. [318]. The n^{th} approximation then is the $(n-1)^{th}$ converging fraction; $\frac{p}{q}$ is not called a converging fraction, because, being equal to y, it is in fact not a fraction; q is of necessity equal to unity, but it is put under this form for the sake of symmetry.

The following equations need no explanation,

$$\frac{p}{q}=\frac{y}{1},$$

$$\frac{p_1}{q_1}=y+\frac{1}{y_1},$$

reducing this becomes

$$\frac{p_1}{q_1}=\frac{y\,y_1+1}{y_1},$$

so that

$$\left.\begin{aligned}p_1&=y\,y_1+1\\ q_1&=y_1\end{aligned}\right\}\ \ldots\ldots\ldots\ (1),$$

$\frac{p_2}{q_2}$ is found by writing $y_1+\frac{1}{y_2}$ for y_1 i the expression for $\frac{p_1}{q_1}$, so that it is equal to

$$\frac{y\left(y_1+\frac{1}{y_2}\right)+1}{y_1+\frac{1}{y_2}},$$

or

$$\frac{y\,y_1\,y_2+y+y_2}{y_1\,y_2+1},$$

which may be written

$$\frac{(y\,y_1+1)\,y_2+y}{y_1\,y_2+1}.$$

Now observing equations (1), and also that $y=p$ and $q=1$, we obtain by substitution in the above value of $\frac{p_2}{q_2}$

$$\frac{p_2}{q_2}=\frac{p_1\,y_2+p}{q_1\,y_2+q},$$

so that

$$\left.\begin{aligned}p_2&=p_1\,y_2+p\\ q_2&=q_1\,y_2+q\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ (2).$$

Again $\frac{p_3}{q_3}$ is found by substituting $y_2+\frac{1}{y_3}$ for y_2 in the value of $\frac{p_2}{q_2}$, and is therefore equal to

$$\frac{p_1\left(y_2+\frac{1}{y_3}\right)+p}{q_1\left(y_2+\frac{1}{y_3}\right)+q},$$

which may be written

$$\frac{(p_1\,y_2+p)\,y_3+p_1}{(q_1\,y_2+q)\,y_3+q_1};$$

and observing equations (2), and substituting for $p_1\,y_2+p$, and $q_1\,y_2+q$, we obtain

$$\frac{p_3}{q_3}=\frac{p_2\,y_3+p_1}{q_2\,y_3+q_1},$$

so that

$$\left.\begin{aligned}p_3&=p_2\,y_3+p_1\\ q_3&=q_2\,y_3+q_1\end{aligned}\right\}\ \ldots\ldots\ldots\ (3).$$

Proceeding similarly, it is evident from the way in which the quantities y_4 y_5, &c. enter into the several converging fractions, that we shall arrive at a similar result, and we have the following equation connecting each of the converging fractions with the two which precede it:

$$\frac{p_n}{q_n}=\frac{p_{n-1}\,y_n+p_{n-2}}{q_{n-1}\,y_n+q_{n-2}};$$

whence

$$\left.\begin{aligned}p_n&=p_{n-1}\,y_n+p_{n-2}\\ q_n&=q_{n-1}\,y_n+q_{n-2}\end{aligned}\right\}\ \ldots\ldots\ (n).$$

322. Multiplying the first of these equations by q_{n-1}, and the second by p_{n-1}, and subtracting, we obtain

$$p_n\times q_{n-1}-q_n\times p_{n-1}=p_{n-2}\,q_{n-1}-q_{n-2}\,p_{n-1},$$

or

$$p_n\times q_{n-1}-q_n\times p_{n-1}=-\left\{p_{n-1}\,q_{n-2}-q_{n-1}\,p_{n-2}\right\}\ \ldots\ldots\ (A).$$

This expression shows us *that the difference between the product of the numerator of any converging fraction, and the denominator of that which precedes it, and the product of the denominator of the same converging fraction, and the numerator of that which precedes it, is alternately positive and negative, and differs only in sign.*

Now going back to equations (1), and observing that $y = p$ and $q = 1$, we have

$$p_1 \times q - q_1 \times p = y y_1 + 1 - y y_1,$$

or $\quad p_1 \times q - q_1 \times p = 1.$

Again recurring to equations (2), we find

$$p_2 \times q_1 - q_2 \times p_1 = p \times q_1 - q \times p_1$$
$$= -(p_1 \times q - q_1 \times p)$$
$$= -1,$$

a result which might have been at once inferred from equation (A). The following equation immediately results from equation (A) and the remark which follows it:

$$p_n \times q_{n-1} - q_n \times p_{n-1} = \pm 1 \ \ldots . \text{(B)}.$$

And it is also evident from the values of $p_1 \times q - q_1 \times p$, and $p_2 \times q_1 - q_2 \times p_1$, that the positive sign must be taken when n is odd, and the negative one when it is even. We may hence derive the following property. *The difference between the product of the numerator of the n^{th} converging fraction, and the denominator of the $(n-1)^{th}$, and the product of the denominator of the n^{th}, and the numerator of the $(n-1)^{th}$, is equal to $+1$ when n is odd, and -1 when n is even.*

323. In treating of indeterminate equations, we observed, art. [311], that by means of a property of continued fractions we might quickly arrive at a solution in whole numbers of the equation $a x + b y = c$. We may thus find it. The fraction $\frac{a}{b}$ being reduced into the form of a continued fraction, let $\frac{P}{Q}$ represent the converging fraction previous to the last, which will be the complete fraction $\frac{a}{b}$. We have then by the last rule

$$a \times Q - b \times P = \pm 1.$$

In any particular case we should know whether $\frac{a}{b}$ was an even or an odd converging fraction, and therefore know which sign to take. We will suppose it to be an odd one, and according to the rule adopting the positive sign, and then multiplying by c, we have

$$a \times P \times c - b \times Q \times c = c \ldots (1.$$

Now observing the equation

$$a x + b y = c,$$

and comparing it with equation (1), we see that it is satisfied by the substitution of $P \times c$ for x, and $Q \times c$ for y. Hence $P \times c$, and $-Q \times c$, afford us a solution of the equation. By applying this to the equation we before examined, $5x + 7y = 81$, we should obtain for the values of x and y, 243, and -162.

Returning to equation (B),

$$p_n \times q_{n-1} - q_n \times p_{n-1} = \pm 1,$$

it is clear from this equation that every divisor of p_n and q_n must also divide 1, so that they admit no divisor greater than unity. This furnishes us with another property. *The numerator and denominator of each converging fraction are prime to each other.*

324. By writing $y_1 + \frac{1}{y_2}$ for y_1 in the expression for $\frac{p_1}{q_1}$, art. [319], we obtained the expression for $\frac{p_2}{q_2}$. But supposing that we had written

$$y_1 + \cfrac{1}{y_2 + \cfrac{1}{y_3 + \&c.}}$$

as far as the expression went, for y_1, in the same expression, we should evidently have obtained, upon reduction, the fraction which the continued fraction represents. Similarly, if in the expression for $\frac{p_n}{q_n}$, we put for y_n

$$y_n + \frac{1}{y_{n+1} + \&c.},$$

continuing the expression as far as it goes, we shall again obtain the original fraction. Representing then

$$y_n + \frac{1}{y_{n+1} + \&c.}$$

by Y, and the original fraction by F, and writing Y for y_n in the expression for $\frac{p_n}{q_n}$, in art. [319], we have

$$F = \frac{p_{n-1} Y + p_{n-2}}{q_{n-1} Y + q_{n-2}}.$$

This expression will lead us to the several errors we commit by representing the original fraction by its several converging fractions. For

$$F - \frac{p_{n-2}}{q_{n-2}} = \frac{p_{n-1}Y + p_{n-2}}{q_{n-1}Y + q_{n-2}} - \frac{p_{n-2}}{q_{n-2}},$$

$$= \frac{Y\{p_{n-1}q_{n-2} - q_{n-1}p_{n-2}\}}{(q_{n-1}Y + q_{n-2})\,q_{n-2}} \;\ldots\ldots\; (1).$$

And by a similar reduction we should obtain

$$F - \frac{p_{n-1}}{q_{n-1}} = \frac{-\{p_{n-1}q_{n-2} - q_{n-1}p_{n-2}\}}{q_{n-1}\{q_{n-1}Y + q_{n-2}\}} \;\ldots\ldots\; (2).$$

From these expressions we learn that the several converging fractions are alternately too great and too small; for if

$F - \frac{p_{n-2}}{q_{n-2}}$ is positive, $F - \frac{p_{n-1}}{q_{n-1}}$ is negative,

and *vice versâ*. This result we formerly arrived at from the way in which the successive approximations were derived from each other.

325. Now, in order to estimate the degree of accuracy of the several converging fractions, we observe in the first place that art. [320],

$$p_{n-1}q_{n-2} - q_{n-1}p_{n-2} = \pm 1.$$

Hence, from equation (2) in the last article,

$$F - \frac{p_{n-1}}{q_{n-1}} = \frac{\pm 1}{q_{n-1}(q_{n-1}Y + q_{n-2})};$$

now Y is greater than unity, and, therefore,

$F - \frac{p_{n-1}}{q_{n-1}}$ is less than $\frac{1}{q_{n-1}(q_{n-1} + q_{n-2})}$,

neglecting the sign, and consequently, *à fortiori*,

$F - \frac{p_{n-1}}{q_{n-1}}$ is less than $\frac{1}{(q_{n-1})^2}$.

The following property then is manifest. *The error committed by representing a fraction by any of its convergents is less than unity divided by the square of the denominator of that convergent.*

Thus the error committed by representing the length of the year, art. [316], by $365\frac{7}{29}$ days, was less than $\frac{1}{(29)^2}$ days, or less than the $\frac{1}{841}$ *th* part of a day. And similarly the error introduced by representing the value of x in the equation $2^x = 3$ by $\frac{8}{5}$ was less than $\frac{1}{25}$.

Once more, observing equations (1) and (2), and considering that Y is greater than unity, and q_{n-1} greater than q_{n-2}, (for $q_{n-1} = q_{n-2}y_{n-1} + q_{n-3}$, art. [319]) it will be manifest that $F - \frac{p_{n-1}}{q_{n-1}}$ is less than $F - \frac{p_{n-2}}{q_{n-2}}$, from which we derive the following property. *Each converging fraction approaches nearer to the value of the continued fraction than any of those which precede it.* It is for this reason that they are called *converging* fractions, or *convergents*.

Of the expansion of a^x and the formation of Logarithmic Tables.

326. We have already explained the purposes of logarithmic tables, art. [234], &c., and also the method of using them. We now proceed to show how they may be formed. It appeared from the nature of logarithms, art. [237], that comparatively few of them could be expressed in whole numbers, or terminating decimals, and we observed that this was not necessary, since they were sufficiently exact for all common purposes when carried to seven decimal digits. This naturally leads us to endeavour to express them in the shape of a series, and if we can arrive at one which is rapidly *convergent*, neglecting those terms which have no influence on the first seven decimal places, we shall form a table accurate to the degree required, art. [292]. A preliminary step in the arrival at this series is the expansion of a^x.

327. Before, however, we proceed to the direct consideration of the subjects placed at the head of this section, we must prove the following theorem.

If the equation

$$A + Bx + Cx^2 + \&c. = a + bx + cx^2 + \&c.$$

be true whatever value be given to x and A, B, C, &c. a, b, c, &c. being independent of x, we propose to prove that it is a necessary consequence of the above equation, that the coefficients of like powers of x in both sides of the equation are equal, that is, that $A = a$, $B = b$, $C = c$, &c. For since this equation is true, whatever value be given to x, it is true when $x = 0$, and the two expressions are reduced to their first terms, so that we have

$$A = a.$$

But A and a are independent of the value of x, and therefore being equal for one value of x, they are equal for all. We have then, striking out A and a from the two sides of the original equation, the following equation subsisting for all values of x,

$$B\,x + C\,x^2 + \&c. = b\,x + c\,x^2 + \&c.$$

Divide by x and the following equation is true for all values of x.

$$B + C\,x + \&c. = b + C\,x + \&c.$$

Hence for the same reason that A was equal to a, we have

$$B = b,$$

and so on for all the other coefficients.

This theorem is one of the strongest instruments in analytical reasoning, and we shall find it in the subsequent part of Algebra of frequent and extensive application.

328. We shall now consider the expansion of a^x, and endeavour to express it by a series ascending by integral powers of x. We have already seen, art. [281], that any power of a binomial for instance $(1 + y)^n$ may be represented by a series ascending by integral powers of y. We have in fact proved that

$$\left.\begin{aligned}(1 + y)^n = 1 + n\,y + n\,\frac{n-1}{2}\,y^2 + n\,\frac{n-1}{2}\,\frac{n-2}{3}\,y^3 + \&c.\\ + n\,.\frac{n-1}{2}\,\frac{n-2}{3}\,\ldots\,\frac{n-r+1}{r}\,y^r + \&c.\end{aligned}\right\}\ \text{(A)}.$$

This series goes on for ever when n is fractional or negative, but when it is a positive integer, contains only $n + 1$ terms. Now here the remark made in art. [287] is of importance to show that even if n be a positive integer, we introduce no error in supposing the series to go on to infinity. The method in which we shall alter the arrangement of the factors of the several terms in series (A) will show the necessity of retaining all the terms if we wish to arrive at a general result.

Returning to equation (A), and representing the exponent by x, and writing a for $1 + y$, so that $y = a - 1$, we have

$$\left.\begin{aligned}a^x = 1 + x\,(a-1) + x\,\frac{x-1}{2}\,(a-1)^2 + x\,\frac{x-1}{2}\,\frac{x-2}{3}\,(a-1)^3 + \&c.\\ + x\,\frac{x-1}{2}\,\frac{x-2}{3}\,\ldots\,\frac{x-(r-1)}{r}\,(a-1)^r + \&c.\end{aligned}\right\}\ \text{(B)}.$$

We here observe, in the first place, that only integral powers of x enter into the expression, and in the next, that all the terms after the first have x for a factor, so that 1 is the first term of the result arranged according to the ascending powers of x. In order to find the *second* term, or, which comes to the same thing, the coefficient of x, we proceed as follows. Observing the several terms of the series, we see that in the second the coefficient of x is $(a-1)$, in the third, $-\frac{1}{2}(a-1)^2$, in the fourth, $-\frac{1}{2} \times -\frac{2}{3}(a-1)^3$ or $+\frac{1}{3}(a-1)^3$, and so on for the remainder of the coefficients; or that in each term it is the product of the second terms of the numerators of all the factors (except x, which has no second term) which compose the coefficient, divided by the product of the denominators of all the factors. Now the second term of the numerator of each factor is always the same as the denominator of the preceding factor. A little consideration then will show, attention being paid to the sign, which is positive when there are an even number of factors besides x, and negative when there are an odd number, that the coefficient of x is

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \&c.,$$

this series going on to infinity. It is doubtless possible by a similar method to examine the coefficients of x^2, x^3, &c., but the operation would be tedious and almost impracticable, and would be different for each coefficient. The theorem demonstrated at the beginning of this section will enable us to obtain these several coefficients with great facility, and has the advantage of furnishing them all by the same process.

329. Resuming equation (B), and writing A for

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{\&c.}$$

and calling the several coefficients of x^2, x^3, &c. (the expressions for which we are about to investigate) M, N, &c. we have

$$a^x = 1 + A\,x + M\,x^2 + N\,x^3 + \text{\&c.} \quad \ldots (1).$$

This being true for all values of x, and A, M, N, &c. being independent of x, we may write z for x, and have the following equation, in which the values of A, M, N, &c. are the same as before,

$$a^z = 1 + A\,z + M\,z^2 + N\,z^3 + \text{\&c.} \quad \ldots (2).$$

Subtracting equation (2) term by term from equation (1), and combining the terms with like coefficients, we have

$$a^x - a^z = A(x-z) + M(x^2 - z^2) + N(x^3 - z^3) + \text{\&c.}$$

Now every term of the right-hand side of this equation is divisible by $x - z$, for every term has a multiplier of the form $x^r - z^r$, and $x^r - z^r$ being divided by $x - z$, gives

$$x^{r-1} + x^{r-2}.z + x^{r-3}.z^2 + \text{\&c.} + z^{r-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

a quotient in which there are r terms. The above equation then may be written, making $x - z$ a common factor of the several terms,

$$a^x - a^z = (x-z)\{A + M(x+z) + N(x^2 + x\,z + z^2) + \text{\&c.}\} \quad \ldots (4).$$

Again, $a^x - a^z = a^z(a^{x-z} - 1)$.

But by equation (1), putting $x - z$ for x,

$$a^{x-z} = 1 + A(x-z) + M(x-z)^2 + N(x-z)^3 + \text{\&c.}$$

Whence it appears, making $x - z$ a factor of the sum of all the terms which it multiplies, that

$$a^{x-z} - 1 = (x-z)\left\{\begin{matrix} A + M(x-z) \\ + N(x-z)^2 + \text{\&c.}\end{matrix}\right\},$$

so that

$$\begin{aligned} a^x - a^z &= a^z(a^{x-z} - 1) \\ &= a^z(x-z)\left\{\begin{matrix} A + M(x-z) \\ + N(x-z)^2 + \text{\&c.}\end{matrix}\right\}. \end{aligned}$$

Whence substituting this expression for $a^x - a^z$ in equation (4), and dividing by $x - z$, which is a factor on both sides of the resulting equation, we obtain

$$\begin{gathered} A + M(x+z) + N(x^2 + x\,z + z^2) + \text{\&c.} = \\ a^z(A + M(x-z) + N(x-z)^2 + \text{\&c.}) \end{gathered}$$

Now this equation subsisting for all values of x and z, we may suppose z equal to x, and we obtain, since all the terms after A on the right-hand side of the equation vanish,

$$A + M\,.\,2x + N\,.\,3x^2 + \text{\&c.} = a^x \times A;$$

and it is evident from equation (3), and the remark which follows it, that the coefficients on the left-hand side of this equation would go on following the same law. Putting then for a^x its value from equation (1), and multiplying each term by A, we have

$$\begin{gathered} A + 2\,M\,x + 3\,N\,x^2 + \text{\&c.} = \\ A + A^2 x + A\,M\,x^2 + \text{\&c.} \end{gathered}$$

Hence by the theorem proved at the beginning of this section, the coefficients of like powers of x must be equal; so that

$$\begin{gathered} 2\,M = A^2, \\ 3\,N = A \times M, \\ \text{\&c.} \quad \text{\&c} \end{gathered}$$

From the first we obtain

$$M = \frac{A^2}{2},$$

from the second

$$N = \frac{A\,M}{3}.$$

$$= \frac{A^3}{2 \cdot 3}$$

and the form of the two expressions shows that we should have for the coefficient of x^4, $\frac{A^4}{2.3.4}$, and so on for the other coefficients. Hence substituting the values of the several coefficients in equation (1),

$$a^x = 1 + Ax + \frac{A^2 x^2}{1.2} + \frac{A^3 . x^3}{1.2.3} + \&c. \ldots\ldots\ldots\ldots (C),$$

the series continuing to follow the same law. This is the expansion required.

330. We now propose to deduce from this expression a series for the logarithm of a number to any base. See art. [234]. The process, though at first view circuitous and indirect, requires only to be understood to appear simple.

In equation (C) any value may be given to a; and A, therefore, which, art. [326], is equal to

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.,$$

admits a corresponding variety of values. Let us suppose A equal to *unity*, and represent by e the value of a corresponding to this value of A. We have, by equation (C), A being equal to 1, and a to e,

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c. \text{ (D)},$$

and making x equal to 1,

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c.$$

Of this series we may take any number of terms. The summation of them is sufficiently easy, since each term, after the second, is derived from the preceding one by dividing successively by 2, 3, &c. The student can perform the operation, and neglecting the terms after the eleventh, which have no influence on the seven first places of decimals, he will find

$$e = 2.7182818.$$

This quantity then is known. The discovery of it does not at present appear to have brought us nearer our object, but we shall find it a necessary instrument in arriving at it. It is the base of a system of logarithms called the Napierean, from Napier, a celebrated mathematician of the seventeenth century, who invented logarithms and calculated them to this base. Logarithms calculated to this base are also sometimes, though without much propriety, called hyperbolic logarithms, from their entering into some of the properties of the hyperbola.

Resuming equation (C), suppose x equal to unity, the equation becomes

$$a = 1 + A + \frac{A^2}{1.2} + \&c.$$

But making x equal to A in equation (D) the series for e^A is the same as the above series for a. We thus have

$$a = e^A \ldots\ldots\ldots\ldots (1).$$

As we shall consider a as the base of the system for which we are forming our tables, we will put equation (1) under another form, and write n for a, and represent

$$(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.,$$

which is what A becomes, by p. This being done, equation (1) becomes

$$n = e^p \ldots\ldots\ldots\ldots (2).$$

From the definition of logarithms, this equation shows us that p is the logarithm of n to base e, or Nap. log. $n = p$. But

$$p = n - 1 - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.$$

Substituting, then,

$$\text{Nap. log. } n = n - 1 - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c. \ldots\ldots\ldots (3).$$

We have thus obtained a series for the logarithm of a number in the Napierean system. We thus proceed to find it in any other.

331. Taking the logarithms of both sides of equation (2) in the system required, for instance, in that whose base is a, we have

$$\log. n = \log. e^p$$
$$= p \log. e,$$

or $\log. n = \log. e \left\{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.\right\} \ldots . (4),$

since $p = (n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.$

We here appear to be no further advanced, since log. e being taken to base a, we have at present no means of finding it. Equation (1) will help us out of this difficulty.

The equation (1) of the last article $a = e^A$ shows, taking the logarithm of both sides to base a, that $1 = \log. e^A = A \log. e$, therefore

$$\frac{1}{A} = \log. e,$$

the logarithm being taken in the system whose base is a, and $\frac{1}{A}$ is known, since A is equal to

$$a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$$

But this, it is evident, from equation (3), is the logarithm of a in the Napierean system. Representing, then, for the sake of distinction, logarithms taken in the Napierean system by log.$'$, and those taken to base a by log., we have, putting for A its value, in the above equation

$$\frac{1}{\log.' a} = \log. e,$$

and substituting for log. e in equation (4),

$$\log. n = \frac{1}{\log.' a}\left\{n - 1 - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.\right\}.$$

The series within brackets in the above equation is the Napierean logarithm of n given in equation (3). The quantity $\frac{1}{\log.' a}$, by which the Napierean logarithm is multiplied, so as to produce the logarithm of the same number in the system calculated to base a, is called the modulus of that system, and is evidently the same for all values of n, depending only on the base of the system. It is usually represented by M; representing it so, the equation becomes

$$\log. N = M\left\{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.\right\} \ldots . (5).$$

(332.) In order that this equation, which affords the complete algebraical solution of the question, may be practically adequate to the computation of logarithms, we must alter its form, for if n be any number greater than 2, it is evident that the terms of the series instead of *converging*, become continually greater and greater. In fact, although we stated in the last article, that A could be found, being equal to

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.,$$

yet we should find this expression of little use in computing its numerical value. We proceed then to alter the form of the series in the above equation, and to the numerical computation of M.

Putting $n = 1 + n'$ in equation (5), and therefore $n - 1 = n'$, it becomes

$$\log. (1 + n') = M\left\{n' - \frac{1}{2}n'^2 + \frac{1}{3}n'^3 - \&c.\right\}.$$

Again putting $n = 1 - n'$ in equation (5), and therefore $n - 1 = -n'$, it becomes

$$\log. (1 - n') = M\left\{-n' - \frac{1}{2}n'^2 - \frac{1}{3}n'^3 - \&c.\right\}.$$

Subtracting this equation from the last, and observing that

$$\log. (1 + n') - \log. (1 - n') = \log. \frac{1 + n'}{1 - n'},$$

and that the series within the brackets go on following the same law, so that all the even powers of n' destroy each other, and all the terms involving the odd powers become doubled, we have

$$\log.\frac{1+n'}{1-n'} = 2\,\mathrm{M}\left\{n' + \frac{n'^3}{3} + \frac{n'^5}{5} + \&c.\right\}.$$

To reduce this to a more favourable form, let

$$\frac{1+n'}{1-n'} = \frac{b}{c},$$

the solution of which simple equation gives

$$n' = \frac{b-c}{b+c}.$$

Substituting for $\frac{1+n'}{1-n'}$ and n' in the above equation

$$\log.\frac{b}{c} = 2\,\mathrm{M}\left\{\frac{b-c}{b+c} + \frac{1}{3}\left(\frac{b-c}{b+c}\right)^3 + \frac{1}{5}\left(\frac{b-c}{b+c}\right)^5 + \&c.\right\} \ldots\ldots (6).$$

(333.) The series here is convergent, and rapidly so where the difference between b and c is very small. It is convenient to make b and c consecutive numbers, so that $b - c$ is equal to unity, and we can by means of the above series, observing that

$$\log. b = \log.\frac{b}{c} + \log. c,$$

find the logarithms of all numbers successively. Supposing, for a moment, that we have found the value of M for Brigg's system when the base is 10. It being equal to

.43429448,

so that 2 M is equal to

.86858896,

we have the following rule which is taken substantially from Dr. Hutton's mathematical tables. *Call* s *the sum of any number* (b) *whose logarithm is sought, and the number* (c) *next less by unity. Divide* .86858896 *by* s, *and reserve the quotient; divide the reserved quotient by the square of* s, *and reserve this quotient; divide this last quotient by the square of* s, *and again reserve this quotient, and thus proceed continually dividing the last quotient by the square of* s, *as long as division can be made. Then write these quotients under one another, the first uppermost, and divide them respectively by the uneven numbers* 1, 3, 5, *&c. Add all these last quotients together, then the sum will be the logarithm of* $\frac{b}{c}$, *as given by equation* (6).

And therefore to this logarithm, adding also the logarithm of c, *the next less number, the sum will be the required logarithm of* b, *the number required.*

The operations for finding the logarithms of 2 are indicated below. The next less number is 1, so that s = 3, and the square of s = 9.

3).86858896
9).28952965
9).03216996
9).00357444
9).0003 716
9).00004412
9).00000490
9).00000054
.00000006

Proceeding with the rule

1).28952965(.28952965
3).03216996(.01072332
5).00357444(.00071488
7).00039716(.00005673
9).00004412(.00000490
11).00000490(.00000044
13).00000054(.00000004

Adding the quotients

$$\log.\frac{2}{1} = .30102996$$
$$\text{Add } \log. 1 = .00000000$$
$$\log. 2 = .30102996$$

By a similar process we may find the logarithm of 3 and of all the higher numbers. But it is only necessary to find the prime numbers by this direct method, the others being easily found by composition and division.

Thus $\log. 4 = \log. 2^2$
$= 2 \log. 2,$
$\log. 6 = \log. (2 \times 3)$
$= \log. 2 + \log. 3,$

and so on.

$$\text{Log. } 5 = \log. \frac{10}{2}$$
$$= \log. 10 - \log. 2$$
$$= 1 - \log. 2,$$

&c. &c.

The series in equation (6) admits other transformations which are of service to the practical computer of logarithms by affording *verifications* of his result; it being evident that when we have arrived at the same result by several different processes, we may feel a more perfect assurance of its accuracy. But the discussion of them here would be misplaced, our object being only to *explain* the construction of tables.

(334.) The multiplier M may be found immediately from equation (6). It is equal to $\frac{1}{\log.' a}$, the accent indicating as before that the logarithms are taken in the Napierean system. This, as was before remarked, is the quantity by which we multiply the Napierean logarithm of a number in order to arrive at the logarithm to base a. Equation (6) then gives us, withdrawing M,

$$\log.' \frac{b}{c} = 2\left(\frac{b-c}{b+c} + \frac{1}{3}\left(\frac{b-c}{b+c}\right)^3 + \frac{1}{5}\left(\frac{b-c}{b+c}\right)^5 + \&c.\right) \ldots. (7).$$

Now for Brigg's system $a = 10$ In order, then, to find M, or $\frac{1}{\log.' a}$, we must first find log.' 10.

But
$$\log.' 10 = \log.' (2 \times 5)$$
$$= \log.' 2 + \log.' 5.$$

By equation (7) b being equal to 2, and c to 1,

$$\log.' 2 = 2\left(\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \&c.\right).$$

By a process similar to that used in the last article we shall find

$$\log.' 2 = .69314718.$$

From this we obtain log.' 4, or 2 log.' 2.

Again, putting 5 for b, and 4 for c in equation (7), we have

$$\log.' \frac{5}{4} = 2\left(\frac{1}{9} + \frac{1}{3} \cdot \frac{1}{9^3} + \frac{1}{5} \cdot \frac{1}{9^5} + \&c.\right).$$

which may be easily computed, being very convergent. Adding log.' 4, we obtain log'. 5. These operations being performed, in which the reader will find no difficulty, and log.' 2 being added to log.' 5, we shall obtain

$$\log.' 10 = 2.30258509,$$

and therefore

$$M = \frac{1}{\log.' 10}$$
$$= .43429448.$$

(335.) In art. [233] we promised to inquire into the real value of such quantities as $a^{\cdot 00000029}$, and to show that it differed from unity by a quantity which, within a certain degree of accuracy, might be neglected.

By art. [327] we have

$$a^x = 1 + Ax + \frac{A^2 x^2}{1.2} + \&c.,$$

where

$$A = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.,$$

which is the Napierean logarithm of a. It is in most cases a small number, and is not greater than 10 unless a be greater than 10000. Now x being equal to .00000029, the third term will not influence the twelve first places of decimals, and may, therefore, with those that follow it, be put out of the question. Supposing A to be less than 10, Ax will have no significant figure in the five first decimal places, and may therefore be neglected if the degree of accuracy required does not extend so far. In this case we may therefore consider $a^{\cdot 00000029}$ as equal to 1.

In art. [240] we remarked that the *common* logarithms of consecutive numbers of five or more digits were nearly in arithmetical progression. We may thus prove this to be the case:

$$\log. (c+1) - \log. c = \log. \frac{c+1}{c},$$

$$\log. (c+2) - \log. (c+1) = \log. \frac{c+2}{c+1}.$$

Calling the first D, and the second D′, and subtracting

$$D - D' = \log. \frac{c+1}{c} - \log. \frac{c+2}{c+1} = \log. \frac{\frac{c+1}{c}}{\frac{c+2}{c+1}},$$

$$= \log. \frac{(c+1)^2}{c(c+2)} = \log. \frac{c^2+2c+1}{c(c+2)}$$

$$= \log. \left\{1 + \frac{1}{c(c+2)}\right\},$$

or, expressing the logarithm in a series,

$$D - D' = M\left\{\frac{1}{c(c+2)} - \frac{1}{2}\,\frac{1}{c^2(c+2)^2} + \&c.\right\}.$$

Now c containing five digits, the denominator of the first term contains at least nine digits, and M being less than $\frac{1}{2}$, (as was shown in the last article,) $D - D'$ can have no significant figure in the eight first decimal places. We have therefore in the tables calculated to seven places of decimals,

$$D - D' = 0,$$

or $$D = D',$$

so that the differences between the logarithms of consecutive numbers are equal. From this property the method of finding the logarithms of numbers of six and seven digits in art. [242] directly flowed.

(336.) We have subjoined a small table of common logarithms, which will, for many purposes, supply the place of a whole volume of them. By means of this table the logarithms of all numbers, from *one* up to *ten thousand*, or of all numbers consisting of *four* digits, wherever the decimal point be placed, may be found to a certain number of decimal places. It will be seen, by reference to the table, that the logarithms *directly* given are those of all numbers only from 10 up to 999. But since the logarithms of 20 and 2, 30 and 3, &c., as well as those of all numbers consisting of the same digits, but varying in the position of the decimal point, differ only in their *characteristics*, art. [238], the manner of deducing from this table the logarithms of all numbers consisting of *one*, *two*, or *three* digits, is evident. These logarithms, it will be observed, are given as far as *four* decimal places only, while those tables, the construction of which we more fully explained in arts. [236], &c., were calculated as far as *seven*. For an explanation of the use of the *proportional parts* arranged to the right of the double line in each page, we must refer to arts. [240]...[243]. We are enabled, by means of them, to find from our table the logarithm of any number consisting of *four* digits, as in the articles last referred to, we derived the logarithms of numbers, consisting of 6 digits, from tables where the logarithms of numbers up to 99999 only, were tabulated. Thus, to find from our table the logarithm of 9863:—From the table we see that the logarithm of 986 is 2.9939 (2 being the characteristic), and, consequently, the logarithm of 9860 is 3.9939. Now the *proportional part* corresponding to 3, the last digit of the number whose logarithm we proposed to find, is 1, and, therefore, adding this, we have

$$\log. 9863 = 3.9940.$$

See arts. [240]...[243].

(337.) The table of antilogarithms contains the natural numbers corresponding to all logarithms of four places, arranged in the order of the logarithms. This table is an extract from Dodson's Antilogarithmic Canon. The first two figures of the given logarithm being found in the first column, and the third figure on the top of the page, the corresponding natural number is opposite to the former, and under the latter. The quantity to be added for the fourth logarithmic figure must be taken from

the columns headed proportional parts, in the same line with the number already found, and under the given fourth figure. The natural numbers thus obtained will generally be exact to a unit in the last place, except towards the end of the table; the number of decimal places will depend upon the *characteristic* of the logarithm. This table is chiefly intended to save time when many logarithms in succession are to be looked out.

PROPORTIONAL PARTS.

Nat. Nums.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	·0000	·0043	·0086	·0128	0170	0212	·0253	·0294	0334	·0374	4	8	12	17	21	25	29	33	37
11	·0414	·0453	·0492	·0531	·0569	0607	·0645	·0682	·0719	·0755	4	8	11	15	19	23	26	30	34
12	·0792	·0828	·0864	·0899	·0934	·0969	·1004	·1038	·1072	·1106	3	7	10	14	17	21	24	28	31
13	·1139	·1173	·1206	·1239	·1271	·1303	·1335	·1367	·1399	·1430	3	6	10	13	16	19	23	26	29
14	·1461	·1492	·1523	·1553	·1584	·1614	·1644	·1673	·1703	·1732	3	6	9	12	15	18	21	24	27
15	·1761	·1790	·1818	·1817	·1875	·1903	·1931	·1959	·1987	·2014	3	6	8	11	14	17	20	22	25
16	·2041	·2068	·2095	·2122	·2148	·2175	·2201	·2227	·2253	·2279	3	5	8	11	13	16	18	21	24
17	·2304	·2330	·2355	·2380	·2 05	·2430	·2455	·2480	·2504	2529	2	5	7	10	12	15	17	20	22
18	·2553	·2577	·2601	·2625	2618	2672	·2695	·2718	2742	·2765	2	5	7	9	12	14	16	19	21
19	·2788	·2810	·2833	·2856	·2878	·2900	·2923	·2945	·2967	2989	2	4	7	9	11	13	16	18	20
20	·3010	·3032	·3054	·3075	·3096	3118	·3139	·3160	·3181	·3201	2	4	6	8	11	13	15	17	19
21	·3222	·3243	·3263	·3284	·3304	·3324	·3345	·3365	·3385	·3404	2	4	6	8	10	12	14	16	18
22	·3424	·3444	·3464	·3483	3502	3522	·3541	·3560	·3579	·3598	2	4	6	8	10	12	14	15	17
23	·3617	·3636	·3655	·3674	·3692	·3711	·3729	·3747	·3766	·3784	2	4	6	7	9	11	13	15	17
24	·3802	·3820	·3838	·3856	·3874	·3892	·3909	·3927	·3945	·3962	2	4	5	7	9	11	12	14	16
25	·3979	·3997	·4014	·4031	·4048	·4065	·4082	·4099	·4116	·4133	2	3	5	7	9	10	12	14	15
26	·4150	·4166	·4183	·4200	·4216	·4232	·4242	·4265	4281	4298	2	3	5	7	8	10	11	13	15
27	·4314	·4330	·4346	·4362	·4378	4393	·4409	·4425	4440	·4456	2	3	5	6	8	9	11	13	14
28	·4472	·4487	·4502	·4518	·4533	4548	·4564	4579	·4524	·4609	2	3	5	6	8	2	11	12	14
29	·4624	4639	·4654	·4669	·4683	4698	·4713	4728	4742	·4757	1	3	4	6	7	9	10	12	13
30	·4771	·4786	·4800	·4814	4829	4843	·4857	·4871	·4886	·4900	1	3	4	6	7	9	10	11	13
31	·4914	·4928	·4942	·4955	4969	4983	·4997	·6011	·5024	5038	1	3	4	6	7	8	10	11	12
32	·5051	·5065	·5079	·5092	·5105	5119	5132	·5145	5159	·5172	1	3	4	5	7	8	9	11	12
33	·5185	·5198	·5211	·5224	·5237	·5250	·5263	5276	5289	·5302	1	3	4	5	6	8	9	10	12
34	·5315	·5328	·5340	·5353	5366	·5378	·5391	·5403	5416	5428	1	3	4	5	6	8	9	10	11
35	·5441	·5453	·5465	5478	·5490	·5502	·5514	·5527	5539	·5551	1	2	4	5	6	7	9	10	11
36	·5563	·5575	·5587	5599	5611	·5623	·5635	·5647	·5658	5670	1	2	4	5	6	7	8	10	11
37	·5682	·5694	·5705	·5717	5729	5740	·5752	·5763	5775	·5786	1	2	3	5	6	7	8	2	10
38	·5798	·5809	·5921	·5832	·5843	·5855	·5866	·5877	·5888	·5899	1	2	3	5	6	7	8	2	10
39	·5911	·5922	·5933	5944	5955	·5966	5977	·5988	5999	·6010	1	2	3	4	5	7	8	9	10
40	·6021	·6031	·6042	6053	·6064	·6075	·6085	·6096	6107	·6117	1	2	3	4	5	6	8	9	10
41	·6128	·6138	·6149	·6160	·6170	·6180	6191	·6201	6212	·6222	1	2	3	4	5	6	7	8	9
42	·6232	·6243	·6253	6263	·6274	·6284	·6294	6304	·6314	·6325	1	2	3	4	5	6	7	8	9
43	·6335	·6345	·6355	·6365	·6375	·6385	·6395	·6405	6415	·6425	1	2	3	4	5	6	7	8	9
44	·5435	·6144	·6454	·6464	·6474	·6484	·6493	·6503	·6513	6522	1	2	3	4	5	6	7	8	9
45	·6532	·6542	·6551	·6561	6571	·6580	·6590	6599	·6609	·6618	1	2	3	4	5	6	7	8	9
46	·6628	·6637	6646	·6656	6665	·6675	·6684	·6693	·5702	6712	1	2	3	4	5	6	7	7	8
47	·6721	·6730	·6739	·6749	·6758	·6767	·5776	6785	·6794	·6803	1	2	3	4	5	5	6	7	8
48	6812	·6821	6830	6839	·6848	·6857	·6866	·6875	·6884	·6893	1	2	3	4	4	5	6	7	8
42	·6902	·5911	·6920	·6928	·6937	·6946	·6955	6964	·6972	·6981	1	2	3	4	4	6	6	7	8
50	·6990	·6998	·7007	·7016	·7024	7033	·7042	·7050	·7059	·7067	1	2	3	3	4	5	5	7	8
51	·7076	·7084	·7093	·7101	·7110	·7118	·7126	7135	·7143	·7152	1	2	3	3	4	5	6	7	8
52	·7160	·7168	·7177	·7185	·7193	·7202	·7210	·7218	7226	·7235	1	2	2	3	4	5	6	7	7
53	·7243	·7251	·7259	·7267	7275	7284	7292	7300	7308	·7316	1	2	2	3	4	5	6	6	7
54	·7324	·7332	·7340	·7348	·7356	·7364	·7372	·7380	·7388	·7396	1	2	2	3	4	5	6	6	7

TABLE OF LOGARITHMS.

PROPORTIONAL PARTS.

Nat. Nums.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	·7404	·7412	·7419	·7427	·7435	7443	7451	·7459	·7466	·7474	1	2	2	3	4	5	5	6	7
56	·7482	·7490	·7497	·7505	·7513	7520	7528	·7536	·7543	·7551	1	2	2	3	4	5	5	6	7
57	·7559	·7566	·7574	·7582	·7589	7597	7604	·7612	·7619	·7627	1	2	2	3	4	5	5	6	7
58	·7634	·7642	·7649	·7657	7664	7672	7679	·7686	·7694	·7701	1	1	2	3	4	4	5	6	7
59	·7709	·7716	·7723	·7731	·7738	7745	·7752	·7760	·7767	·7774	1	1	2	3	4	4	5	6	7
60	·7782	·7789	·7796	·7803	7810	7818	·7825	·7832	·7839	·7846	1	1	2	3	4	4	5	6	6
61	·7853	·7860	·7868	·7875	·7882	7889	·7896	·7903	·7910	·7917	1	1	2	3	4	4	5	6	6
62	·7924	·7931	·7938	·7945	·7952	7959	·7966	·7973	·7980	·7987	1	1	2	3	3	4	5	6	6
63	·7993	·8000	·8007	·8014	·8021	8028	·8035	·8041	·8048	·8055	1	1	2	3	3	4	5	5	6
64	·8062	·8069	·8075	·8082	·8089	8096	·8102	·8109	·8116	·8122	1	1	2	3	3	4	5	5	6
65	·8129	·8136	·8142	·8149	·8156	8162	·8169	·8176	·8182	·8189	1	1	2	3	3	4	5	5	6
66	·8195	·8202	·8209	·8215	·8222	8228	·8235	·8241	·8248	·8254	1	1	2	3	3	4	5	5	6
67	·8261	·8267	·8274	·8280	·8287	8293	·8299	·8306	·8312	·8319	1	1	2	3	3	4	5	5	6
68	·8325	·8331	·8338	·8344	·8351	8357	·8363	·8370	·8376	·8382	1	1	2	3	3	4	4	5	6
69	·8388	·8395	·8401	·8407	·8414	·8420	·8426	·8432	·8439	·8445	1	1	2	2	3	4	4	5	6
70	·8451	·8457	·8463	·8470	·8476	8482	·8488	·8494	·8500	·8506	1	1	2	2	3	4	4	5	6
71	·8513	·8519	·8525	·8531	·8537	8543	·8549	·8555	·8561	·8567	1	1	2	2	3	4	4	5	5
72	·8573	·8579	·8585	·8591	·8597	8603	·8609	·8615	·8621	·8627	1	1	2	2	3	4	4	5	5
73	·8633	·8639	·8645	·8651	·8657	8663	·8669	·8675	·8681	·8686	1	1	2	2	3	4	4	5	5
74	·8692	·8698	·8704	·8710	·8716	8722	·8727	·8733	·8739	·8745	1	1	2	2	3	4	4	5	5
75	·8751	·8756	·8762	·8768	·8774	8779	·8785	·8791	·8797	·8802	1	1	2	2	3	3	4	5	5
76	·8808	·8814	·8820	·8825	·8831	8837	·8842	·8848	·8854	·8859	1	1	2	2	3	3	4	5	5
77	·8865	·8871	·8876	·8882	·8887	·8893	·8899	·8904	·8910	·8915	1	1	2	2	3	3	4	4	5
78	·8921	·8927	·8932	·8938	·8943	8949	·8954	·8960	·8965	·8971	1	1	2	2	3	3	4	4	5
79	·8976	·8982	·8987	·8993	·8998	9004	·9009	·9015	·9020	·9025	1	1	2	2	3	3	4	4	5
80	·9031	·9036	·9042	·9047	·9053	·9058	·9063	·9069	·9074	9079	1	1	2	2	3	3	4	4	5
81	·9085	·9090	·9096	·9101	·9106	9112	·9117	·9122	9128	·9133	1	1	2	2	3	3	4	4	5
82	·9138	·9143	·9149	·9154	·9159	·9165	·9170	·9175	·9180	9186	1	1	2	2	3	3	4	4	5
83	·9191	·9196	·9201	·9206	·9212	·9217	·9222	·9227	·9232	·9238	1	1	2	2	3	3	4	4	5
84	·9243	·9248	·9253	·9258	·9263	·9269	·9274	·9279	·9284	·9289	1	1	2	2	3	3	4	4	5
85	·9294	·9299	·9304	·9309	·9315	·9320	·9325	·9330	·9335	·9340	1	1	2	2	3	3	4	4	5
86	·9345	·9350	·9355	·9360	·9365	·9370	·9375	·9380	·9385	·9390	1	1	2	2	3	3	4	4	5
87	·9395	·9400	·9405	·9410	·9415	·9420	·9425	·9430	·9435	·9440	0	1	1	2	2	3	3	4	4
88	·9445	·9450	·9455	·9460	·9465	·9469	·9474	·9479	·9484	·9489	0	1	1	2	2	3	3	4	4
89	·9494	·9499	·9504	·9509	·9513	·9518	·9523	·9528	·9533	·9538	0	1	1	2	2	3	3	4	4
90	·9542	·9547	·9552	·9557	·9562	·9566	·9571	·9576	·9581	·9586	0	1	1	2	2	3	3	4	4
91	·9590	·9595	·9600	·9605	·9609	·9614	·9619	·9624	·9628	·9633	0	1	1	2	2	3	3	4	4
92	·9638	·9643	·9647	·9652	·9657	·9661	·9666	·9671	·9675	·9680	0	1	1	2	2	3	3	4	4
93	·9685	·9689	·9694	·9699	·9703	·9708	·9713	·9717	·9722	·9727	0	1	1	2	2	3	3	4	4
94	·9731	·9736	·9741	·9745	·9750	·9754	·9759	·9763	·9768	·9773	0	1	1	2	2	3	3	4	4
95	·9777	·9782	·9786	·9791	·9795	·9800	·9805	·9809	·9814	·9818	0	1	1	2	2	3	3	4	4
96	·9823	·9827	·9832	·9836	·9841	·9845	·9850	·9854	·9859	·9863	0	1	1	2	2	3	3	4	4
97	·9868	·9872	·9877	·9881	·9886	·9890	·9894	·9899	·9903	·9908	0	1	1	2	2	3	3	4	4
98	·9912	·9917	·9921	·9926	·9930	·9934	·9939	·9943	·9948	·9952	0	1	1	2	2	3	3	4	4
99	·9956	·9961	·9965	·9969	·9974	·9978	·9983	·9987	·9991	·9996	0	1	1	2	2	3	3	3	4

TABLE OF ANTILOGARITHMS.

Log.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											Proportional Parts.								
.00	1000	1002	1005	1007	1009	1012	1014	1015	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1145	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1154	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1321	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1425	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1526	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1649	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2045	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	5
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

TABLE OF ANTILOGARITHMS.

PROPORTIONAL PARTS.

Log.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

ERRATA IN SOME OF THE EDITIONS.

Page 86, column 1, line 38, for $(x+a)^5$ read $(x+a)^6$

Page 87, column 2, line 27, for x read x^n

Page 88, column 2, line 23 of note, for a read α

line 24 ditto, for $(1+z)^{m+a}$ read $(1+z)^{m+n}$

line 27 ditto, for $(1+z)^{m+a}$ read $(1+z)^{m+n}$

The same in lines 32 and 35

Page 90, column 1, line 2, for $(1+x)^{\frac{1}{2}}$ read $(1-x)^{\frac{1}{2}}$

line 6, for $(1+x)^{\frac{1}{2}}$ read $(1+x)^{-\frac{1}{2}}$

line 19, dele $-m$ at the end of the line

line 20, at beginning, for $\frac{m+1}{2}$ read $-m.\frac{m+1}{2}$.

column 2, line 5, for $\frac{\frac{m}{n}-1}{2}$ read $\frac{\frac{m}{n}-1}{2}$

line 7 from bottom, for $\frac{m(m-1)\ldots m-p+1}{1.2\ldots.p}$ read $\frac{m(m-1)\ldots(m-p+1)}{1.2\ldots.p}$

Page 91, column 1, line 26, for $\frac{y^2}{a^2}$ read $\frac{y}{a^2}$

line 27, for $\sqrt{1+\frac{y^2}{a^2}}$ read $\sqrt{1+\frac{y}{a^2}}$

Page 92, column 1, lines 4 and 5 should stand thus:

$$(1+x)^{\frac{1}{n}}=1+\frac{1}{n}\cdot x+\frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}\cdot x^2$$

$$+\frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}\cdot\frac{\frac{1}{n}-2}{3}\cdot x^3+\&c.$$

line 8 thus: $\sqrt[n]{\mathrm{N}}=a\left\{1+\frac{1}{n}\cdot\frac{y}{a^n}+\frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}\cdot\frac{y^2}{a^{2n}}\right.$

line 9 thus: $\left.+\frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}\cdot\frac{\frac{1}{n}-2}{3}\cdot\frac{y^3}{a^{3n}}+\&c.\right\}$

line 12, for $+\frac{1}{a}\cdot\frac{y}{a.^n}$ read $+\frac{1}{n}\cdot\frac{y}{a^n}$

line 13 thus: $\left.+\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{y^3}{a^{3n}}-\&c.\right\}$

column 2, line 40, for f read S

Page 93, column 2, line 5 of note, for $\frac{x^2}{a+x^2}$ read $\frac{x^2}{(a+x)^2}$

line 9 ditto, for $\left(\frac{1}{1-\frac{a-x}{a+x}}\right)^n$ read $\left(\frac{1}{1-\frac{a-x}{a+x}}\right)^n$

Page 94, column 1, line 2, dele comma at the end of the line.

line 5, for $(a+n)^m$ read $(a+x)^m$

Page 96, column 2, line 22, for [292] read [297].

Questo r..u.to è incompleto. duplicato del precedente (r..tt.10 N.F.

EXAMPLES OF THE PROCESSES

OF

ARITHMETIC AND ALGEBRA.

To prevent any misconception as to the use of this treatise, we state that it is intended only for those who study the principles of arithmetic and algebra, and the reasons of the rules laid down in those sciences. The plan we should recommend is the following:—Let the student repeat examples of each rule upon paper, choosing the most simple numbers which can be found, as well those given in this work as others, until he is capable of solving such instances mentally. Let him then proceed to the cases which contain more complicated numbers or expressions. This is by much the shortest way of proceeding, and eventually the easiest.

We presume a knowledge of the four fundamental operations of arithmetic in whole numbers, and shall therefore content ourselves with showing how examples may be formed which shall contain their own verification.

As soon as the pupil knows the processes of addition and subtraction, let him take a series of nnmbers, each of which contains one more figure than the preceding; say 154, 2879, 31673, 200104, and 7172618. Let him subtract each of these from the succeeding as follows:—

$$\begin{array}{rrrr} 2879 & 31673 & 200104 & 7172618 \\ \underline{154} & \underline{2879} & \underline{31673} & \underline{200104} \\ 2725 & 28794 & 168431 & 6972514 \end{array}$$

Let him then add all his results, together with the least number chosen. The result ought to be the greatest number.

$$\begin{array}{r} 6972514 \\ 168431 \\ 28794 \\ 2725 \\ \underline{154} \\ 7172618 \end{array}$$

As an exercise in multiplication, let two numbers be written down for the student, each of which he is to multiply by itself. For instance, 142 and 361.

$$\begin{array}{r} 361 \times 361 = 130321 \\ 142 \times 142 = \underline{\ \ 20164} \\ \text{Subtract } 110157 \end{array}$$

Let him then take the sum and difference of the two numbers first chosen, and multiply these together, which should give the same result as the preceding.

$$\begin{array}{rr} 361 & 361 \\ \underline{142} & \underline{142} \\ \text{add } 503 & 219 \text{ subtract.} \end{array}$$

$$503 \times 219 = 110157$$

For division, let the student multiply two numbers by themselves, and divide the difference of the results by the difference of the numbers; which should give their sum. But the division of any two numbers by one another may be made, and the result verified by multiplication as usual.

Section 1.—*Common Fractions.*

Operations containing fractions with very high numbers are of little practical use; decimal fractions being pre-

ferred. But as exercises of arithmetical accuracy we shall give, among the rest, a few cases of high numbers.

I.—*To reduce a fraction to its lowest terms.*

Definition. — A fraction is in its lowest terms, when there is no fraction equal to it which has a smaller numerator and denominator.

Principle employed.—The value of a fraction is not altered by dividing both its numerator and denominator, or multiplying both its numerator and denominator by the same number.

$$\frac{1}{2} \quad \frac{2}{4} \quad \frac{3}{6} \quad \frac{4}{8} \quad \frac{5}{10} \text{ \&c., \&c., are all equal.}$$

$$\frac{3}{7} \quad \frac{6}{14} \quad \frac{9}{21} \quad \frac{12}{28} \quad \frac{15}{35} \text{ \&c., \&c., are all equal.}$$

Rule. Divide both numerator and denominator by the greatest whole number which will divide them both without remainder.

Case 1. Where it is evident that a certain number will divide both numerator and denominator without remainder, and that the result is in its lowest terms.

$$\frac{11}{33}=\frac{1}{3} \qquad \frac{12}{48}=\frac{1}{4} \qquad \frac{21}{14}=\frac{3}{2} \qquad \frac{18}{99}=\frac{2}{11} \qquad \frac{27}{15}=\frac{9}{5}$$

Point out here by what numbers the numerators and denominators are divided.

Case 2. Where it is evident that the numerator and denominator are divisible by some number, but not evident that the result is in its lowest terms, divide by that whole number, and proceed as in *Case* 3. (Observe that *Case* 3 may be employed without this, if preferred.)

A number is divisible by

Two, when the last digit is divisible by *two*, or even; as in 66, 48, 132.

Three, when the sum of its digits is divisible by *three,* as 162, in which $1 + 6 + 2$ or 9 is divisible by 3.

Four, when the two last digits are divisible by *four,* as in 16864, in which 64 is divisible by 4.

Five, when the last digit is either 0 or 5, as in 180, 965. (To divide by 5, multiply by 2, and strike off the cipher.)

Six, when it is *even* and divisible by *three,* as 486.

Seven, according to no rule sufficiently simple to be useful.

Eight, when the three last digits are divisible by *eight,* as 2794216, in which 216 is divisible by 8.

Nine, when the sum of its digits is divisible by *nine,* as 729, in which $7 + 2 + 9$ or 18 is divisible by 9.

Ten, when the last digit is a cipher.

Eleven, when the two sets of sums made by taking alternate digits are either equal, or differ by a multiple * of 11, as 1034, in which $1 + 3$ is the same as $0 + 4$, 121 in which $1 + 1$ is the same as 2, 129382 in which $1 + 9 + 8$ or 18, differs from $2 + 3 + 2$ or 7, by 11.

Twelve, when it is divisible by *four* and *three.*

The preceding rules may be applied to the following fractions: find out which is employed in each.

$$\frac{5665}{5720}=\frac{1133}{1144}=\frac{103}{104} \text{ in the lowest terms.}$$

$$\frac{7944}{8916}=\frac{1986}{2229}=\frac{662}{743} \quad (\textit{Case } 3.)$$

$$\frac{8904}{4494}=\frac{1272}{642}=\frac{212}{107} \quad (\textit{Case } 3.)$$

Case 3. When there is no very evident divisor of the numerator and denominator, divide the greater by the less, the divisor by the remainder, the last-mentioned remainder by the new remainder, &c., &c., (as afterwards

* A multiple of *a* is any number which can be divided by *a* without remainder.

shown,) until there is no remainder, or until it is evident that two successive remainders have no common divisor. In the first case, the last divisor used will divide both terms of the given fraction, and will reduce it to its lowest terms; in the second case, the fraction is already in its lowest terms.

*** Observe that whatever divides two numbers divides their difference: therefore 102 and 107 can have no common divisor; if they had, it would be either 5 or would divide 5. This will often be useful.

Reduce $\frac{4466}{1856}$ to its lowest terms.

```
1856)4466(2
     3712
      754)1856(2
          1508
           348)754(2
               696
                58)348(6
                   348
                     0
```

therefore 58 is the greatest divisor which 4466 and 1856 have in common.

```
58)4466(77        58)1856(32
   406               174
    406               116
    406               116
      0                 0
```

Therefore $\frac{4466}{1856} = \frac{77}{32}$

Reduce $\frac{1847}{8209}$ to its lowest terms.

```
1847)8209(4
     7388
      821)1847(2
          1642
           205)821(4
               820
                 1)205(205
                   205
                     0
```

This tells us the greatest common divisor of 1847 and 8209 is 1, or that there is no divisor which will reduce the fraction to lower terms.

$$\frac{2433}{13787} = \frac{3}{17} \qquad \frac{156933}{19557} = \frac{329}{41}$$

$$\frac{314175}{100005} = \frac{355}{113} \qquad \frac{100110}{31866} = \frac{355}{113}$$

$$\frac{7992}{11544} = \frac{9}{13} \qquad \frac{54369}{73355} = \frac{63}{85}$$

instances of higher numbers,

$$\frac{7241379310344827586206896551}{9999999999999999999999999999} = \frac{63}{87}$$

$$\frac{42614574994432}{149720237927424} = \frac{16807}{59049}$$

The following is a table containing some *prime* numbers (or numbers which have no whole divisors greater than 1) by which examples may be formed.

23	367	857	1637	3299	8443	18583
29	397	883	1709	3389	8573	20611
83	433	947	1759	4591	8669	32801
149	509	953	1831	4673	9011	43717
179	541	967	1847	5189	9151	58573
181	619	971	1861	5407	9181	60013
191	647	977	2081	6329	9403	72053
257	709	983	2111	6449	9521	84229
271	761	991	2287	7237	9631	97073
311	809	997	2749	7321	9967	99991

Take any two of the preceding numbers, say 23 and 149; multiply both by any number, say 8, giving 184 and 1192, then

$$\frac{184}{1192} \text{ reduced to lowest terms, gives } \frac{23}{149}$$

Many thousands of examples may be thus formed.

II.—*To reduce Fractions to a common Denominator,—*

That is, to find fractions having the same denominator which shall be respectively equal to a set of fractions having different denominators.

Case 1. When the fractions have denominators, of which all the divisors can be easily seen. An example of this case will be better than any rule.

To reduce to a common denominator the following fractions,

$$\frac{1}{2}\quad\frac{2}{3}\quad\frac{1}{4}\quad\frac{3}{5}\quad\frac{5}{6}\quad\frac{1}{7}\quad\frac{1}{8}\quad\frac{2}{9}\quad\frac{3}{10}\quad\frac{1}{12}\quad\frac{13}{16}$$

write down all the denominators which are not evidently divisors of some of the rest.

$$7,\ 9,\ 10,\ 12,\ 16$$

Write these down in prime factors, that is, make them by multiplication.

$$7\qquad 3\times3\qquad 5\times2\qquad 2\times2\times3\qquad 2\times2\times2\times2$$

Take each prime number as often as it occurs in that one of the preceding which has it most often.

$$2\ 2\ 2\ 2\ 3\ 3\ 5\ 7$$

Multiply all these together, which gives

$$2\times2\times2\times2\times3\times3\times5\times7 = 5040;$$

Divide this by all the denominators in succession.

$$5040 \div 2 = 2520 \qquad 5040 \div 7 = 720$$
$$5040 \div 3 = 1680 \qquad 5040 \div 8 = 630$$
$$5040 \div 4 = 1260 \qquad 5040 \div 9 = 560$$
$$5040 \div 5 = 1080 \qquad 5040 \div 10 = 504$$
$$5040 \div 6 = 840 \qquad 5040 \div 12 = 420$$
$$5040 \div 16 = 315$$

These need not all be formed by actual division, for it is clear that to divide by 9, we may take the third part of 1680, in which 5040 has been already divided by 3.

Now look to the original fractions: multiply every numerator by the result of its denominator in the preceding list, and we shall thus have the numerators of the fractions required, while 5040 will be the common denominator, as follows:—

$$1 \times 2520 = 2520 \qquad \frac{1}{2} \text{ is } \frac{2520}{5040}$$

$$2 \times 1680 = 3360 \qquad \frac{2}{3} \text{ is } \frac{3360}{5040}$$

$$1 \times 1260 = 1260 \qquad \frac{1}{4} \text{ is } \frac{1260}{5040}$$

Similarly

$$\frac{3}{5} \text{ is } \frac{3240}{5040} \qquad \frac{2}{9} \text{ is } \frac{1120}{5040}$$

$$\frac{5}{6} \text{ is } \frac{4200}{5040} \qquad \frac{3}{10} \text{ is } \frac{1512}{5040}$$

$$\frac{1}{7} \text{ is } \frac{720}{5040} \qquad \frac{1}{12} \text{ is } \frac{420}{5040}$$

$$\frac{1}{8} \text{ is } \frac{630}{5040} \qquad \frac{13}{16} \text{ is } \frac{4095}{5040}$$

Fractions given.	The same reduced to a common denominator.
$\frac{3}{8}\quad\frac{5}{12}\quad\frac{7}{100}$	$\frac{225}{600}\quad\frac{250}{600}\quad\frac{42}{600}$
3* $\quad\frac{1}{12}\quad\frac{5}{16}\quad\frac{7}{30}$	$\frac{720}{240}\quad\frac{20}{240}\quad\frac{75}{240}\quad\frac{56}{240}$

* Consider this as $\frac{3}{1}$.

The most convenient common denominator is the *least* number which is divisible by all the denominators or their *least* common multiple; but any common multiple will answer. The least common multiple is found in the preceding process.

Case 2.—Where the least common multiple of all the denominators is evident. This, generally speaking, is when the denominators are very low numbers, and the least common multiple is found by multiplying the denominators together, rejecting any factor out of each, which is evidently contained in a preceding one. For instance, the least common multiple of 4 and 6 is not 4×6 but 4×3, because the factor 2, which is thrown out when 6 is made 3, is already in 4. The following are instances:—

Numbers given.	Least common multiple.
2, 3, 4, 6,	$2 \times 3 \times 2 \times 1 = 12$
4, 9, 10, 12, 18,	$4 \times 9 \times 5 \times 1 = 180$
2, 6, 10,	$2 \times 3 \times 5 = 30$
6, 8, 10,	$6 \times 4 \times 5 = 120$
3, 7, 9,	$3 \times 7 \times 3 = 63$
5, 8, 10, 15,	$5 \times 8 \times 3 = 120$
7, 9, 12, 15,	$7 \times 9 \times 4 \times 5 = 1260$
2, 4, 6, 8, 10,	$2 \times 2 \times 3 \times 2 \times 5 = 120$
18, 20, 24,	$18 \times 10 \times 2^* = 360$
16, 18, 22	$16 \times 9 \times 11 = 1584$

When the least common multiple has been found, perception derived from practice, rather than rules, must be the guide; if that fails, go back to *Case* 1.

Fractions given.	Reduced to a common denominator.
$\frac{1}{2}$ $\frac{2}{3}$ $\frac{1}{4}$ $\frac{5}{6}$	$\frac{6}{12}$ $\frac{8}{12}$ $\frac{3}{12}$ $\frac{10}{12}$
$\frac{3}{7}$ $\frac{3}{14}$ $\frac{1}{4}$	$\frac{12}{28}$ $\frac{6}{28}$ $\frac{7}{28}$
$\frac{9}{2}$ $\frac{5}{6}$ $\frac{1}{18}$	$\frac{81}{18}$ $\frac{15}{18}$ $\frac{1}{18}$
$\frac{7}{15}$ $\frac{3}{10}$ $\frac{5}{9}$	$\frac{42}{90}$ $\frac{27}{90}$ $\frac{50}{90}$
$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$ $\frac{1}{6}$	$\frac{30}{60}$ $\frac{20}{60}$ $\frac{15}{60}$ $\frac{12}{60}$ $\frac{10}{60}$

Case 3.—When there are only two fractions with complicated denominators, either multiply numerator and denominator of each by the denominator of the other; or, if considered worth while, find the greatest common measure of the two denominators, and their quotients when divided by it; multiply each numerator and denominator by the quotient of the other denominator.

To reduce $\frac{33}{82}$ and $\frac{11}{25}$ to a common denominator.

$$\frac{33}{82} = \frac{33 \times 25}{82 \times 25} = \frac{825}{2050}$$

$$\frac{11}{25} = \frac{11 \times 82}{25 \times 82} = \frac{902}{2050}$$

To reduce $\frac{81}{7700}$ and $\frac{37}{1540}$ to a common denominator. Here the greatest common measure of 1540 and 7700 is 1540; therefore the fractions are $\frac{81}{7700}$ and $\frac{185}{7700}$

Fractions given.		Reduced to a common denominator.	
$\frac{53}{181}$	$\frac{27}{936}$	$\frac{49608}{169416}$	$\frac{4887}{169416}$
$\frac{113}{355}$	$\frac{355}{113}$	$\frac{12769}{40115}$	$\frac{126025}{40115}$

* For 24 write 2, because 6 is already a factor of 18, and of the residuary factor 4, 2 is already in 20.

III.—*Estimation of the Value of Fractions.*

Rule 1. When fractions have a common denominator, the greater has the greater numerator.

$$\frac{16}{12} > \frac{9}{12} \qquad \frac{14}{11} > \frac{13}{11}$$

Rule 2. When fractions have a common numerator, the greater has the less denominator.

$$\frac{16}{7} > \frac{16}{8} \qquad \frac{21}{11} > \frac{21}{15}$$

Rule 3. If the numerators of two fractions be added for a numerator, and the denominators for a denominator, the resulting fraction lies between the two first.

$\frac{2}{5}$ lies between $\frac{1}{2}$ and $\frac{1}{3}$

$\frac{8}{17}$ lies between $\frac{3}{10}$ and $\frac{5}{7}$

$\frac{23}{60}$ lies between $\frac{20}{40}$ and $\frac{3}{20}$

Rule 4. By adding the same number to both numerator and denominator of a fraction, the fraction is brought nearer to 1; by subtracting the same number from the numerator and denominator, the fraction is removed farther from 1; that is, addition to both decreases fractions greater than 1, and increases fractions less than 1; subtraction from both increases fractions greater than 1, and decreases fractions less than 1.

$$\frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6},\ \&c.$$

is a continually increasing series.

$$\frac{3}{2}\ \frac{4}{3}\ \frac{5}{4}\ \frac{6}{5}\ \frac{7}{6},\ \&c.$$

is a continually decreasing series.

$\frac{11}{4}$ is less than $\frac{10}{3}$,

$\frac{28}{29}$ is greater than $\frac{27}{28}$.

IV.—*To add and subtract Fractions.*

Rule. Reduce the fractions to a common denominator; do with the numerators what is directed to be done with the fractions; let the result be the numerator; let the common denominator be the denominator. Reduce the result to its lowest terms, if thought worth while.

What is $\frac{1}{2}+\frac{1}{3}$? These are $\frac{3}{6}$ and $\frac{2}{6}$; hence $\frac{1}{2}+\frac{1}{3}$ has $3+2$ for numerator, and 6 for denominator, or is $\frac{5}{6}$. Similarly, $\frac{1}{2}-\frac{1}{3}$ is $\frac{3-2}{6}$, or $\frac{1}{6}$.

$$\frac{7}{3}+\frac{2}{11}=\frac{77}{33}+\frac{6}{33}=\frac{83}{33}$$

$$\frac{7}{3}-\frac{2}{11}=\frac{77}{33}-\frac{6}{33}=\frac{71}{33}$$

$$2-\frac{4}{5}=\frac{6}{5} \qquad \frac{12}{13}+\frac{2}{7}=\frac{110}{91}$$

$$\frac{7}{10}+\frac{8}{15}=\frac{37}{30} \qquad 1+\frac{1}{2}=\frac{3}{2}$$

$$1+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{15}{8}$$

$$\frac{1}{8}+\frac{1}{63}+\frac{1}{560}+\frac{1}{5040}=\frac{1}{7}$$

$$\frac{4}{17}+\frac{7}{50}+\frac{21}{850}-\frac{11}{90}=\frac{5}{18}$$

$$\frac{1}{3}+\frac{2}{5}+\frac{4}{21}=\frac{97}{105}$$

$$2-\frac{1}{7}+\frac{12}{13}=\frac{253}{91}$$

$$\frac{1}{3}+\frac{1}{5}+\frac{6}{7}-\frac{4}{15}=\frac{118}{105}$$

$$\frac{53}{89}+\frac{19}{347}=\frac{20082}{30883}$$

$$\frac{6}{197}+\frac{11}{12}=\frac{2239}{2364}$$

$$\frac{44}{3}-\frac{153}{427}=\frac{18329}{1281}$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{137}{60}$$

$$1-\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}=\frac{163}{60}$$

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}=\frac{23}{60}$$

$$\frac{1}{2}+\frac{4}{3}+\frac{3}{4}+\frac{6}{5}+\frac{5}{6}=\frac{277}{60}$$

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}=\frac{213}{60}$$

Such reductions as the following are particular cases which often occur:

$$2\frac{1}{3}=2+\frac{1}{3}=\frac{6}{3}+\frac{1}{3}=\frac{7}{3}.$$

Rule. Multiply the whole number by the denominator, and add it to the numerator; let the denominator remain.

$$3\frac{1}{4}=\frac{13}{4} \qquad 7\frac{1}{9}=\frac{64}{9}$$

$$4\frac{4}{5}=\frac{24}{5} \qquad 8\frac{5}{12}=\frac{101}{12}$$

$$16\frac{3}{100}=\frac{1603}{100} \qquad 2\frac{93}{100}=\frac{293}{100}$$

$$14\frac{11}{25}=\frac{361}{25} \qquad 12\frac{9}{17}=\frac{213}{17}$$

V.—*To multiply or divide Fractions by a Whole Number.*

Rule. Do as directed with the numerator, or the contrary with the denominator; that is, *to multiply*, multiply the numerator, or divide the denominator; *to divide*, divide the numerator, or multiply the denominator.

Multiply $\frac{6}{35}$ by 7,

either $\frac{6\times 7}{35}$ or $\frac{6}{35\div 7}$, that is,

either $\frac{42}{35}$ or $\frac{6}{5}$;

the latter is the more simple.

Divide $\frac{6}{35}$ by 3,

either $\frac{6\div 3}{35}$ or $\frac{6}{35\times 3}$, that is,

either $\frac{2}{35}$ or $\frac{6}{105}$;

the former is the more simple.

$$\frac{7}{15}\times 5=\frac{7}{3} \qquad \frac{7}{15}\div 5=\frac{7}{75}$$

$$\frac{8}{21}\times 10=\frac{80}{21} \qquad \frac{3}{19}\div 4=\frac{3}{76}.$$

$$\frac{144}{107}\times 2=\frac{288}{107} \qquad \frac{144}{107}\div 12=\frac{12}{107}.$$

When the multiplier is composed of factors, it may happen that some factors may be most conveniently used in one way, some in the other. The student must render himself very familiar with the following:

Multiplication of Numerator } is Multiplication,
Division of Denominator

Division of Numerator } is Division.
Multiplication of Denominator

Multiply $\frac{4}{77}$ by 14, or 7×2,

$$\frac{4\times 2}{77\div 7}=\frac{8}{11}.$$

Divide $\frac{60}{77}$ by 25 or 5×5,

$$\frac{60\div 5}{77\times 5}=\frac{12}{385}.$$

$$\frac{4}{21}\times 28=\frac{16}{3} \qquad \frac{4}{21}\div 28=\frac{1}{147}$$

$$\frac{12}{39}\times 26=\frac{24}{3} \qquad \frac{12}{39}\div 8=\frac{3}{78}$$

$$\frac{108}{35}\times 20=\frac{432}{7} \qquad \frac{108}{35}\div 24=\frac{9}{70}$$

$$\frac{55}{63}\times 45=\frac{275}{7} \qquad \frac{55}{63}\div 110=\frac{1}{126}$$

$$\frac{25}{43}\times 86=50 \qquad \frac{25}{43}\div 300=\frac{1}{516}.$$

VI.— *To multiply and divide Fractions by one another.*

Definition 1. The product of $\frac{2}{3}$ and $\frac{4}{5}$. This is the answer to such questions as the following: What is two-thirds of $\frac{4}{5}$? What is four-fifths of $\frac{2}{3}$? A. gave B. $\frac{2}{3}$ of his share, and B. gave C. $\frac{4}{5}$ of what he got. How much of A.'s share did C. get? If 1 gallon cost $\frac{2}{3}$ of a shilling, how much of a shilling does $\frac{4}{5}$ of a gallon cost? What is $\frac{4}{5}$ taken two-thirds of a time? What is twice the third part of $\frac{4}{5}$? &c.

Definition 2. The quotient of $\frac{2}{3}$ divided by $\frac{4}{5}$. This is the answer to such questions as the following: What number of times, or what parts of a time, does $\frac{2}{3}$ contain $\frac{4}{5}$? How must $\frac{4}{5}$ be treated, so as to give $\frac{2}{3}$; that is, into how many parts must $\frac{4}{5}$ be divided, and how many of these parts must be taken, so that $\frac{2}{3}$ may result? If 1 gallon cost $\frac{2}{3}$ of a shilling, how many gallons, or how much of a gallon, may be bought for $\frac{4}{5}$ of a shilling?

Rule. To *multiply*, multiply numerators by numerators, and denominators by denominators. To *divide*, divide numerator by numerator, and denominator by denominator; or invert the divisor, and multiply. Or to perform either operation, invert the multiplier or divisor, and proceed as in the other.

Multiply $\frac{2}{3}$ by $\frac{4}{5}$, divide the product by $\frac{5}{7}$, and multiply the result by $\frac{7}{11}$.

$$\frac{2}{3}\times\frac{4}{5}=\frac{2\times 4}{3\times 5}=\frac{8}{15},\quad \frac{8}{15}\div\frac{5}{7}=\frac{8}{15}\times\frac{7}{5}=\frac{56}{75},\quad \frac{56}{75}\times\frac{7}{11}=\frac{392}{825}.$$

$$\frac{16}{3}\times\frac{20}{7}=\frac{320}{21}\qquad \frac{4}{11}\div\frac{3}{5}=\frac{20}{33}.$$

$$\frac{20}{7}\times\frac{16}{3}=\frac{320}{21}\qquad \frac{3}{5}\div\frac{4}{11}=\frac{33}{20}.$$

Before the *multiplication* is made, strike out any factors which are common to a numerator and a denominator; before the *division* is made, strike out any factors which are common to both numerators, or to both denominators.

$$\frac{16}{21}\times\frac{35}{24}=\frac{8\times 2}{7\times 3}\times\frac{7\times 5}{8\times 3}=\frac{2}{3}\times\frac{5}{3}=\frac{10}{9}$$

$$\frac{22}{35}\div\frac{33}{25}=\frac{11\times 2}{7\times 5}\div\frac{11\times 3}{5\times 5}=\frac{2}{7}\div\frac{3}{5}=\frac{10}{21}$$

$$\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}=\frac{1}{120}\qquad \frac{1}{2}\div\frac{1}{3}\times\frac{1}{4}\div\frac{1}{5}=\frac{15}{8}$$

$$\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{1}{5}\qquad \frac{1}{2}\times\frac{1}{3}\times\frac{3}{4}\div\frac{2}{7}=\frac{7}{16}$$

$$\frac{21}{16}\times\frac{10}{7}=\frac{15}{8}\qquad \frac{7}{18}\times\frac{11}{10}\times\frac{27}{14}\times\frac{5}{2}=\frac{33}{16}$$

$$\frac{51}{96}\times\frac{51}{96}\times\frac{51}{96}=\frac{132651}{884736}\qquad \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}=\frac{64}{729}$$

$$\frac{113}{355} \times \frac{221}{118} = \frac{24973}{41890} \qquad \frac{412}{167} \times \frac{397}{777} = \frac{163564}{129759}$$

$$\frac{1}{169} \times \frac{228}{963} = \frac{228}{162747} \qquad \frac{93}{4600} \times \frac{125}{662} = \frac{11625}{30452}$$

$$\frac{60}{61} \times \frac{60}{61} \times \frac{60}{61} \times \frac{60}{61} \times \frac{60}{61} \times \frac{60}{61} = \frac{46656000000}{51520374361}$$

$$\frac{6}{7} \times \frac{1}{2} \times \frac{4}{3} \times \frac{2}{11} \times \frac{7}{9} \times \frac{18}{25} = \frac{16}{275} \qquad \frac{14}{13} \times \frac{13}{14} = 1$$

$$\frac{17}{2} \div \frac{6}{5} = 7\frac{1}{12} \qquad \frac{25}{18} \div \frac{25}{18} = 1$$

$$\frac{25}{18} \div \frac{18}{25} = \frac{625}{324} \qquad \frac{119}{27} \div \frac{338}{113} = \frac{13447}{9126}$$

$$\frac{3163}{468} \div \frac{799}{25} = \frac{79075}{373932} \qquad 3\frac{1}{2} \div \frac{1}{2} = 7$$

$$\left(\frac{7}{6} \times \frac{1}{2} \times \frac{3}{10}\right) \div \left(\frac{6}{11} \times \frac{7}{8} \times \frac{9}{2}\right) = \frac{11}{135}$$

$$\left(\frac{1}{7} \times \frac{4}{9} \times 3\right) \div \left(\frac{6}{11} \times \frac{8}{9} \times 4\right) = \frac{11}{112}$$

$$\left(\frac{1}{2} + \frac{1}{3}\right) \div \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{10}{7} \qquad 12 \div 8 = \frac{3}{2}$$

$$\left(2\frac{1}{2} + \frac{1}{6}\right) \div \left(3\frac{1}{2} - \frac{1}{8}\right) = \frac{64}{81} \qquad 6\frac{1}{8} \div 2\frac{1}{3} = 2\frac{5}{8}$$

$$17 \div \frac{1}{3} = 51 \qquad 1 \div \frac{1}{10} = 10$$

$$9 \div 1\frac{1}{2} = 6 \qquad 10\frac{1}{2} \div \frac{3}{4} = 14$$

$$\frac{4}{5} \div \frac{7}{5} = \frac{4}{7} \qquad \frac{100}{3} \div \frac{101}{3} = \frac{100}{101}$$

$$6 \times \frac{2}{3} \div \frac{3}{2} = \frac{8}{3} \qquad 10 \times \frac{7}{11} \div \frac{7}{11} = 10$$

$$9 = 1 + 2 \times \frac{4}{3} + 3 \times \frac{16}{9}$$

VII.—*Fractions having Fractions in the Numerator or Denominator, or both.*

Rule. To reduce such fractions to equivalent simple fractions, multiply the numerator and denominator by the least common multiple of the denominators of the fractions contained in them.

To reduce $\dfrac{3\frac{1}{2}}{4\frac{1}{3}}$ to a simple fraction.

$$\frac{3\frac{1}{2}}{4\frac{1}{3}} = \frac{3\frac{1}{2} \times 6}{4\frac{1}{3} \times 6} = \frac{18 + 3}{24 + 2} = \frac{21}{26}$$

To reduce $\dfrac{4\frac{1}{7} - 2\frac{1}{4}}{6\frac{1}{2} - 2\frac{1}{7}}$ to a simple fraction.

The least common multiple of 7, 4, 2, and 7, is 28:

$$\frac{4\frac{1}{7} - 2\frac{1}{4}}{6\frac{1}{2} - 2\frac{1}{7}} = \frac{4\frac{1}{7} \times 28 - 2\frac{1}{4} \times 28}{6\frac{1}{2} \times 28 - 2\frac{1}{7} \times 28} = \frac{116 - 63}{182 - 60} = \frac{53}{122}$$

That is, $4\frac{1}{7}$, when diminished by $2\frac{1}{4}$, is the same proportion of $6\frac{1}{2}$ diminished by $2\frac{1}{7}$ which 53 is of 122.

$$\frac{3\frac{1}{4}}{4\frac{1}{4}} = \frac{13}{17} \qquad \frac{2\frac{1}{9}}{3\frac{1}{5}} = \frac{95}{144} \qquad \frac{2\frac{1}{12}}{8\frac{1}{4}} = \frac{25}{99}$$

$$\frac{8\frac{2}{3}}{9\frac{3}{4}} = \frac{104}{117} \qquad \frac{1\frac{1}{2}}{1\frac{1}{8}} = \frac{4}{3} \qquad \frac{\frac{22}{3}}{1\frac{1}{4}} = \frac{88}{15}$$

$$\frac{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}}{1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}} = \frac{12 - 6 + 4 - 3}{12 + 6 - 4 - 3} = \frac{7}{11}$$

$$\frac{2\frac{1}{4} - 1\frac{1}{3}}{2\frac{1}{8} + 1\frac{1}{4}} = \frac{54 - 32}{51 + 30} = \frac{22}{81}$$

VIII.—*Miscellaneous Exercises in the preceding Rules.*

$$\frac{\frac{1}{4} - \frac{1}{9}}{\frac{1}{2} + \frac{1}{3}} \times \frac{3}{5} + \frac{\frac{2}{3} + \frac{3}{2}}{1 + \frac{4}{9}} \times \frac{1}{18} = \frac{11}{60}$$

$$\frac{7 - 3\frac{1}{4}}{7 + 3\frac{1}{4}} \times \frac{\frac{2}{3}}{\frac{4}{7}} - \frac{1}{2} \times \frac{1}{5}\,\frac{1 + \frac{1}{10}}{1000} = \frac{1749547}{4200000}$$

$$\frac{\frac{2}{19} + \frac{1}{3}}{3 - \frac{1}{3}}\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{5}{57}$$

$$\frac{18}{17} \times \left(1 - \frac{64}{81}\right) + \frac{.8}{11} \times \frac{1}{6} \times \left(\frac{1}{2} + \frac{5}{12}\right) = \frac{1}{3}$$

$$\frac{2}{7} \times \frac{1 - \frac{2}{7}}{2} + \frac{4}{5} \times \frac{1}{10} + \frac{3}{5}\left(\frac{1}{2} + \frac{11}{14}\right) + \frac{3}{70}\left(\frac{2}{7} + \frac{4}{5}\right) = 1$$

$$\frac{\frac{13}{21} \times \frac{1}{2} - \frac{11}{14} \times \frac{1}{3}}{\frac{16}{21} \times \frac{1}{2} - \frac{13}{14} \times \frac{1}{3}} = \frac{2}{3}$$

$$\frac{2\frac{1}{2} \times 2\frac{1}{2} \times 2\frac{1}{2} - 1}{2\frac{1}{2} \times 2\frac{1}{2} - 1} = 2\frac{1}{2} + \frac{2}{7}$$

$$\frac{6\frac{1}{4} \times 6\frac{1}{4} \times 6\frac{1}{4} - 8}{6\frac{1}{4} \times 6\frac{1}{4} - 4} = 6\frac{1}{4} + \frac{4}{8\frac{1}{4}}$$

$$\frac{3}{7} + \frac{16}{49} = \frac{4}{7} + \frac{9}{49} = 5\frac{2}{7} \times \frac{1}{8} + \frac{37}{392}$$

IX.—*Verification of Algebraical Processes.*

The following are some algebraical equations which are always true, whatever numbers or fractions may be placed instead of the letters; provided only, that wherever a subtraction occurs, such as $a - b$, a must be greater than b. The student must attempt to verify them; and the proof that he is correct consists in his finding the same number on each side of the equation. For instance, in the first example, let a stand for $\frac{1}{2}$, and b for $\frac{1}{3}$: then

$$\frac{b}{a+b} = \frac{\frac{1}{3}}{\frac{1}{2}+\frac{1}{3}} = \frac{2}{5}\ \frac{a-b}{b} = \frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}} = \frac{1}{2}$$

$$\frac{b}{a+b} + \frac{a-b}{b} = \frac{2}{5} + \frac{1}{2} = \left(\frac{9}{10}\right)$$

Again,

$$\frac{aa}{ab+bb} = \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}} = \frac{\frac{1}{4}}{\frac{1}{6}+\frac{1}{9}} = \left(\frac{9}{10}\right)$$

The following list may be considered long, but it must be remembered that every one is also an example in algebra, and that the young student cannot* be more usefully employed at this stage of his progress, whether the operation be considered with reference to arithmetic or algebra. The student should first try each expression with some whole numbers, before he proceeds to use fractions, in order to be certain that he understands the meaning of the terms.

$$\frac{b}{a+b} + \frac{a-b}{b} = \frac{aa}{ab+bb}$$

$$\frac{a-b}{a+b} + \frac{a+b}{a-b} = \frac{2aa+2ab}{aa-bb}$$

$$\frac{1}{1+a} - \frac{1}{1+2a} = \frac{a}{1+3a+2aa}$$

$$(a+b) \times (a-b) = aa - bb$$

$$(a+b) \times (a+b) = aa + 2ab + bb$$

$$(a-b) \times (a-b) = aa - 2ab + bb$$

$$\frac{1+x}{1+\frac{1}{x}} = x \qquad \frac{ax}{a+x} = a - \frac{aa}{a+x}$$

$$(x+a)\times(x+b) = xx+(a+b)\times x+ab$$

$$(x-a)\times(x-b) = xx-(a+b)\times x+ab$$

$$(ax - by)(ax - by) = (aa + bb)(xx + yy) - (ay + bx)(ay + bx)$$

$$\frac{a+b}{2} + \frac{a-b}{2} = a \qquad \frac{a+b}{2} - \frac{a-b}{2} = b$$

$$\frac{xx - 2x + 1}{xx - 1} = \frac{x-1}{x+1} \qquad \frac{xx - 3x + 2}{xx - 10x + 9} = \frac{x-2}{x-9}$$

$$\frac{xxx + 9xx + 26x + 24}{xxx + 6xx + 11x + 6} = \frac{x+4}{x+1}$$

$$\frac{(n+1)\ (n+2)}{2} = \frac{n(n+1)}{2} + n + 1$$

$$\frac{(n+1)(n+2)(n+3)}{6} = \frac{n(n+1)(n+2)}{6} + \frac{n(n+1)}{2} + n + 1$$

* The very little power which even advanced students generally possess, of turning their algebraical into arithmetical results, is one of the principal features of school instruction, as it exists at present, and is a very serious impediment to higher studies.

$$\frac{c}{a-b}=\frac{c}{a}+\frac{bc}{aa}+\frac{bbc}{aaa}+\frac{bbbc}{aaa\,(a-b)}$$

$$\frac{c}{a+b}=\frac{c}{a}-\frac{bc}{aa}+\frac{bbc}{aaa}-\frac{bbbc}{aaa\,(a+b)}$$

$$\frac{1+x}{1-x}-\frac{1-x}{1+x}-\frac{3x}{1-xx}=\frac{x}{1-xx}$$

$$a+\frac{1}{a}=\frac{(a+1)(a+1)}{a}-2 \qquad a-\frac{1}{a}=\frac{(a+1)(a-1)}{a}$$

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c)=$$
$$=2aabb+2aacc+2bbcc-aaaa-bbbb-cccc$$

X.—*Algebraical Theorems of Approximation, for Verification.*

If p be very nearly equal to 1, then the following theorems are *nearly* true :

$$pp=2p-1 \qquad 1\div p=2-p$$
$$ppp=3p-2 \qquad 1\div pp=3-2p$$
$$pppp=4p-3 \text{ \&c.} \qquad 1\div ppp=4-3p \text{ \&c.}$$

If x be very small, the following theorems are *nearly* true.

$$\frac{1}{1+x}=1-x \qquad \frac{1}{1-x}=1+x \qquad \frac{1+x}{1-x}=1+2x$$

$$\frac{1-3x}{1-2x}=1-x \qquad \frac{1+6x}{1-4x}=1+10x \qquad \frac{6-4x}{3+2x}=2-\frac{8}{3}x$$

If x be very great, the following theorems are *nearly* true :

$$\frac{x}{x+1}=1-\frac{1}{x} \qquad \frac{x-1}{x+1}=1-\frac{2}{x} \qquad \frac{1}{1+x}=\frac{1}{x}-\frac{1}{xx}$$

Section 2.—*Decimal Fractions.*

I.—*Exercises on the Meaning of the Decimal Notation.*

·1 is read *decimal, one.*
·123 is read *decimal, one, two, three.*
36·012 is read *thirty-six, decimal, nought, one, two.*

The student should now write the following and similar tables :—

·1 means $\frac{1}{10}$	·01 means $\frac{1}{100}$	·001 means $\frac{1}{1000}$
·2 . . . $\frac{2}{10}$	·02 . . . $\frac{2}{100}$	·002 . . . $\frac{2}{1000}$
·3 . . . $\frac{3}{10}$	·03 . . . $\frac{3}{100}$	·003 . . . $\frac{3}{1000}$
·4 . . . $\frac{4}{10}$	·04 . . . $\frac{4}{100}$	·004 . . . $\frac{4}{1000}$

&c., &c.

·123 is $\frac{1}{10}+\frac{2}{100}+\frac{3}{1000}$; which is $\frac{100}{1000}+\frac{20}{1000}+\frac{3}{1000}$; which is $\frac{123}{1000}$

·0104 is $\frac{1}{100}+\frac{4}{10000}$; which is $\frac{100}{10000}+\frac{4}{10000}$; which is $\frac{104}{10000}$

6·7 is $6+\frac{7}{10}$ which is $\frac{60}{10}+\frac{7}{10}$; which is $\frac{67}{10}$

$$35\cdot013 = 35 + \frac{1}{100} + \frac{3}{1000} = \frac{35103}{1000}$$

$$2\cdot008 = 2 + \frac{8}{1000} = \frac{2008}{1000}$$

$$\cdot0174 = \frac{1}{100} + \frac{7}{1000} + \frac{4}{10000} = \frac{174}{10000}$$

$$12\cdot11 = 12 + \frac{1}{10} + \frac{1}{100} = \frac{1211}{100}$$

$$12345 = 10000 + 2000 + 300 + 40 + 5$$

$$1234\cdot5 = 1000 + 200 + 30 + 4 + \frac{5}{10}$$

$$123\cdot45 = 100 + 20 + 3 + \frac{4}{10} + \frac{5}{100}$$

$$12\cdot345 = 10 + 2 + \frac{3}{10} + \frac{4}{100} + \frac{5}{1000}$$

$$1\cdot2345 = 1 + \frac{2}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000}$$

$$\begin{aligned}
12345 &= 1234\cdot5 \times 10 = 123\cdot45 \times 100 = 12\cdot345 \times 1000 \\
1234\cdot5 &= 123\cdot45 \times 10 = 12\cdot345 \times 100 = 1\cdot2345 \times 1000 \\
123\cdot45 &= 12\cdot345 \times 10 = 1\cdot2345 \times 100 = \cdot12345 \times 1000 \\
1\cdot2345 &= \cdot12345 \times 10 = \cdot012345 \times 100 = \cdot0012345 \times 1000 \\
\cdot12345 &= \cdot012345 \times 10 = \cdot0012345 \times 100 = \cdot00012345 \times 1000
\end{aligned}$$

$$\cdot12345 = \frac{1\cdot2345}{10} = \frac{12\cdot345}{100} = \frac{123\cdot45}{1000} = \frac{1234\cdot5}{10000}$$

$$1\cdot2345 = \frac{12\cdot345}{10} = \frac{123\cdot45}{100} = \frac{1234\cdot5}{1000} = \frac{12345}{10000}$$

$$12\cdot345 = \frac{123\cdot45}{10} = \frac{1234\cdot5}{100} = \frac{12345}{1000} = \frac{123450}{10000}$$

$$123\cdot45 = \frac{1234\cdot5}{10} = \frac{12345}{100} = \frac{123450}{1000} = \frac{1234500}{10000}$$

$$8\cdot2 = 8\cdot20 = 8\cdot200 = 8\cdot2000 = 8\cdot20000, \text{\&c.}$$

Instances like the preceding should be continued until the student is so familiar with the changes of the decimal point as instantly to point out the effect produced by it, without recurring to a rule.

II.—*To find a Decimal Fraction which shall be nearly equal to a given Common Fraction.*

Principle. No common fraction has a decimal fraction exactly equal to it, unless its denominator is divisible by nothing but 2 or 5, or is composed of the product of some numbers of *twos* and *fives*. But a decimal fraction can be found, which shall be as near to a given common fraction as we please, though not exactly equal to it.

Rule. Annex ciphers to the numerator, divide by the denominator, and neglect the remainder. Cut off as many places from the quotient as there were ciphers annexed to the numerator, for decimals. If one cipher was annexed, the decimal so obtained is within $\frac{1}{10}$ of the given fraction; if two ciphers, within $\frac{1}{100}$; if three ciphers, within $\frac{1}{1000}$; and so on.

In this and all other decimal operations, when directions are given to cut off a certain number of places, and there

are not places enough to be so cut off, affix ciphers to the beginning, in sufficient number to make up the deficiency. Thus, to cut off three decimal places from 25, write ·025; to cut off ten decimal places from 118, write ·0000000118.

Find a decimal fraction which shall be within $\frac{1}{10000}$ of $\frac{18}{23}$.

Annex *four* ciphers to 18, and divide by 23.

23)180000(7826
rem. 2.

Cut off *four* places from 7826, and the answer, ·7826, is within $\frac{1}{10000}$ of $\frac{18}{23}$.

Verification. $\frac{18}{23} - \frac{7826}{10000} = \frac{2}{230000}$, and $\frac{2}{230000}$ is $\frac{1}{115000}$, which is less than $\frac{1}{10000}$.

Find a fraction which shall be within $\frac{1}{1000000}$ of $\frac{1}{913}$.

913)1000000(1095
rem. 265.

Make *six* decimal places in 1095, which gives ·001095, the fraction required.

Definition. A decimal is said to be *true to the (first, second, third, &c.) place of figures* when any alteration in the (*first, second, third, &c.*) place of figures would remove it farther from the truth than it is as it stands. For instance:

$$\frac{61}{99} = 61616 \text{ very nearly.}$$

It is also very nearly ·6161, but not quite so near to this as to ·6162. The second is a little too great, the first a little too small; but the second is not so much in excess as the first is in defect.

Rule. To make a decimal true to the last figure, find one more figure than is wanted; if the last figure be 5, or upwards, increase the preceding by 1. Thus:

·18829976.

If we wish to retain one place only, write ·2
" " two places - - ·19
" " three - - - - ·188
" " four - - - - ·1883
" " five - - - - ·18830
" " six - - - - ·188300
" " seven - - - - ·1882998.

What is the nearest decimal fraction to $\frac{1}{309}$ true to *five* places of decimals.

Annex *six* ciphers to 1, and divide by 309.

309)1000000(3236

Answer: ·003236, which, made true to five places, is ·00324.

The following examples of decimal fractions are all true to the last place:

$\frac{7}{24} = \cdot 2917$	$\frac{370}{873} = \cdot 42383$
$\frac{11}{28} = \cdot 3929$	$\frac{61}{827} = \cdot 07376$
$\frac{1}{29} = \cdot 0345$	$\frac{447}{257} = 1\cdot 73930$
$\frac{1}{940} = \cdot 0011$	$\frac{1012}{582} = 1\cdot 73883$
$\frac{41}{741} = \cdot 0553$	$\frac{410}{555} = \cdot 73874$
$\frac{16}{289} = \cdot 0554$	$\frac{355}{113} = 3\cdot 14159$
* $\frac{33}{596} = \cdot 0554$	$\frac{1}{3090} = \cdot 00032$
$\frac{700}{793} = \cdot 8827$	$\frac{100}{633} = \cdot 15798$

* The student must not be surprised at this fraction having the same decimal (to four places) as the preceding. The two fractions do not differ by so much as *one ten-thousandth.*

Instances of the above process carried to a greater number of places:

$$\frac{653}{633} = 1\cdot 03159557661927330173775671406003159 55, \text{ \&c.}$$

$$\frac{43}{953} = \cdot 0451206715634837355718782791185729275970619$$

$$\frac{1}{561} = \cdot 00178253119429959001782531194295900, \text{ \&c.}$$

Instances of fractions which can be exactly expressed decimally; that is, in which the denominators are made by multiplying *twos* or *fives*, or both:

$$\frac{1}{2} = \cdot 5 \quad \frac{1}{5} = \cdot 2 \quad \frac{1}{4} = \cdot 25 \quad \frac{1}{25} = \cdot 04$$

$$\frac{1}{8} = \cdot 125 \quad \frac{1}{16} = \cdot 0625 \quad \frac{1}{32} = \cdot 03125$$

$$\frac{1}{64} = \cdot 015625 \quad \frac{1}{125} = \cdot 008 \quad \frac{1}{128} = \cdot 0078125$$

$$\frac{1}{512} = \cdot 001953125 \quad \frac{7}{512} = \cdot 013671875$$

$$\frac{63}{64} = \cdot 984375 \quad \frac{101}{8192} = \cdot 0123291015625.$$

III.—*Reduction of Decimal Fractions to a common Denominator.*

Rule. Annex ciphers to all which have a less number of places than are in that which has the greatest number of places, so that all shall have the same number of places. Thus ·1, ·12, ·123, reduced to a common denominator, are ·100, ·120, ·123.

Fractions given.			Reduced to a common Denominator.		
·06,	·031,	·0148	·0600,	·0310,	·0148
12·3,	2·4,	·197	12·300,	2·400,	·197.

IV.—*Addition and Subtraction of Decimal Fractions.*

Rule. Proceed in every respect as in whole numbers, but keep decimal points under one another, and place the decimal point of the result under the other points. (See page 1 for methods.)

```
  12              From 66·112
  12·1            Take  2·01783
   1·42                ────────
    ·0081              64·09417
 ───────
  25·5281
```

Add 12, 12·1, 1·42, and ·0081.

$$1 + \cdot 1 + \cdot 01 + \cdot 001 + \cdot 0001 = 1\cdot 1111$$
$$1 + \cdot 2 + \cdot 03 + \cdot 004 + \cdot 0005 = 1\cdot 2345$$
$$67 + 7\cdot 8 + \cdot 89 - 1\cdot 2168 = 74\cdot 4732$$
$$6\cdot 718909 - 2\cdot 1488 = 4\cdot 570109.$$

V.—*Multiplication of Decimals.*

Throw away the decimal points, and all preliminary ciphers; multiply the results together, and take as many decimal places in the result as there are in both multiplier and multiplicand.

Multiply together the following:

```
 1·2      6·3      2·99      ·001      6·0  Multiplicands.
 1·1      ·84      ·011      ·01       ·5  Multipliers.

 12       63       299        1        60
 11       84        11        1         5
 ───     ────     ────       ─        ───
 121      252     3289        1       300
         504
         ────
         5292

 1·21    5·292    ·03289    ·00001      3.  Answers.
```

8 × 8 = 64   8 × ·8 = 6·4   ·8 × ·8 = ·64   ·08 × ·8 = ·064
80 × ·8 = 64   ·008 × ·08 = ·00064   800 × ·0008 = ·64
15·94 × 254·0836 = 4050·092584
·004716 × ·22240656 = ·00104886933696
·923521 × ·28629151 = ·26439622160671
·155 × 24·025 = 3·723875   14·2 × ·142 = 2·0164.

VI.—*Division of Decimals.*

Rule. Case 1. When the divisor has no decimals, or is a whole number, proceed as in common division, and let the *first decimal place* of the quotient be that figure, in the making of which the *first decimal place* of the dividend is brought down; but if more than one decimal place of the dividend is used in making the first figure of the quotient, put the decimal point first, and then a cipher for every decimal place after the first which is used in making the first quotient-figure.

```
9)173·43        18)·0041(·0002        23)4·61(·2
  19·27              36                  4 6
                      5                    1
```

Case 2. When the divisor has decimal places, strike out the decimal point, and remove the point in the dividend as many places to the right as the number of places which have been thus destroyed in the divisor, previously annexing ciphers to the right of the dividend, if necessary.

In both cases, ciphers may be annexed at pleasure to the right of the dividend, and used in forming additional quotient-figures.

```
·09)1·68(            ·4)·0192(           ·11)3
  9)168·00...         4) ·192            11)300·00...
    16·22...            ·048                27·27...   Quotients.

      2·5)·1793                     ·0025)179·3
        25)1·793(7172                 25)1793000(71720
           1 75                          175
             43                           43
             25                           25
             180                          180
             175                          175
               50                           50
               50                           50
                0                            0
           ·07172                         71720   Quotient.
```

Case 3. If the dividend be a number followed by ciphers, as 86400, strike out the ciphers, proceed as before, and when the process is finished, remove the decimal point one place to the left for every cipher so struck out.

2500)1·793($\frac{1·793}{25} = ·07172$

25)1·793(·07172 $\frac{1·793}{2500} = ·0007172$

Dividend.	Divisor.	Altered Dividend.	Altered Divisor.	First Quotient-Figure.	Part of the Altered Dividend which gives it.	Column in which it must stand.
1·9628	64·19	196·28	6419	3	196·28	2nd Decimal Place.
·0019	·134	1·9	134	1	1·90	2nd Decimal Place.
674	·012	674000	12	5	67	Ten Thousands Column.
6·221	·9136	62210	9136	6	62210	Units Column.
·7021	123·65	70·21	12365	5	70·210	3rd Decimal Place.
1	·001	1000	1	1	1	Thousands Column.
118	190·5	1180	1905	6	1180·0	1st Decimal Place.
1	116·4	10	1164	8	10·000	3rd Decimal Place.

$$\frac{.6}{6} = \cdot1 \qquad \frac{6}{\cdot6} = 10 \qquad \frac{\cdot06}{60} = \cdot001 \qquad \frac{\cdot006}{\cdot6} = \cdot01$$

$$\frac{600}{\cdot6} = 1000 \qquad \frac{600}{\cdot06} = 10000 \qquad \frac{\cdot006}{600} = \cdot00001$$

$$\frac{8\cdot4}{12} = \cdot7 \qquad \frac{8\cdot4}{1\cdot2} = 7 \qquad \frac{8\cdot4}{\cdot12} = 70 \qquad \frac{8\cdot4}{\cdot012} = 700$$

$$\frac{\cdot84}{12} = \cdot07 \qquad \frac{\cdot084}{\cdot12} = \cdot7 \qquad \frac{\cdot084}{1\cdot2} = \cdot07 \qquad \frac{\cdot0084}{\cdot0012} = 7$$

$$\frac{1}{\cdot159} = 6\cdot289308 \qquad \frac{\cdot2}{23\cdot2} = \cdot00862069$$

$$\frac{8792}{937\cdot6567} = 9\cdot37657 \qquad \frac{6821691\cdot97627}{88\cdot03} = 77492\cdot809$$

$$\frac{37\cdot96416}{\cdot156} = 243\cdot36 \qquad \frac{\cdot00636056}{\cdot86} = \cdot007396$$

$$\frac{\cdot59}{79800} = \cdot000007393483 \qquad \frac{61000}{\cdot825} = 73939\cdot393939$$

$$\frac{\cdot59}{80000} = \cdot000007375 \qquad \frac{23}{\cdot000579} = 39723\cdot66148532$$

When the student has acquired sufficient knowledge of the meaning of decimals, and expertness in using them, he will need no other rule for all the cases than the following:—Put a semi-colon in the place where the decimal point ought to be, in order that the result should contain no higher or lower denomination than *units*, that is, should lie between 1 and 10; pass from the semi-colon to the decimal point as it stands, repeating '*tens, hundreds, thousands*, &c., as successive figures are passed over, *if to the left*, and TENTHS, HUNDREDTHS, THOUSANDTHS, &c., as successive figures are passed over, IF TO THE RIGHT. Let the first figure of the quotient have the denomination last named. We give underneath the place of the semi-colon, and the value of the first place of the quotient.

Divisions required.			Places of the semi-colon.			Value of first places of the quotient.		
$\frac{84}{\cdot19}$	$\frac{\cdot31}{22}$	$\frac{5630}{11\cdot9}$	$\frac{84}{\cdot19;}$	$\frac{\cdot31}{;22\cdot}$	$\frac{5630}{11\cdot90;}$	400	·01	400
$\frac{369\cdot7}{49872\cdot3}$	$\frac{216\cdot4}{193\cdot2}$		$\frac{369\cdot7}{49;872\cdot3}$	$\frac{216\cdot4}{193;2}$		·007	1	

VII.—*Contracted Multiplication of Decimals.*

Rule.—To multiply two decimals together, so as to retain only a certain number of places in the product, without the trouble of finding the rest—invert the order of the figures of the multiplier, and write them under those of the multiplicand in such a way that what was the units figure of the multiplier may come under the last place of decimals, which is to be retained. Multiply as usual, with this exception, that each figure of the multiplier begins with the figure of the multiplicand which comes immediately over it, the figure next to that being only used to carry from (as in the subsequent example). Put the several lines directly under one another, instead of removing each one place to the left.

⁂ As it is almost impossible to make this rule clear in words, we subjoin an example at length.

Ex. To multiply 147·3861 by ·6457, retaining only *three* places of decimals. The second factor, written so as to show a unit's place, is 0·6457, and in reversing, the 0 must fall under the *third* decimal place of the other factor, thus:—

1473861 75460	Multiplier reversed; units place 0 falling under third decimal 6 of the upper line.
88432	Multiplier 6; figure to begin with, 8, figure to carry from, 6. Six times 6 is 36, nearest ten, *four* tens, carry *four*. Six times 8 is 48, aod 4 is 52, put down 2 and carry 5. The rest as usual.
5895	Multiplier 4; figure to begin with, 3; figure to carry from 8. Four times 8 is 32; nearest ten, *three* tens, carry *three*. Four times 3 is 12 and 3 is 15, &c. The rest as usual.
737	Multiplier 5; figure to begin with, 7; figure to carry from, 3. Five times 3 is 15; nearest* ten, *two* tens, carry *two*. Five times 7 is 35 and 2 is 37, &c. The rest as usual.
103	Multiplier 7; figure to begin with, 4; figure to carry from, 7. Seven times 7 is 49, nearest ten, *five* tens, carry *five*. Seven times 4 is 28 and 5 is 33, &c. The rest as usual.
95·167	Add as usual, and mark off three places; (the number proposed) for decimals.

The full product of 147·3861 and ·6457 is 95·16720477, which in thousandths only is nearest to 95·167, our result.

The following multiplications have the proper arrangement and result given. *No* decimal places means that the whole number of the result is required, without fractions.

Multiplicatioo required.	No. of Decimals retained.	Arrangement of Multiplier and Multiplicaod.	Result.
36·3771 × 9·99339	three	36·3771 933 999	363·529
19·081137 × 523·36	two	19·081137 6 3325	9986·30
·0699268 × ·9975641	seven	·0699268 1 4657990	·0697565
13763819 × ·05877853	one	13763819·0 35877850 0	809017·0
753554·1 × 7·986355	none	753554·1 5536897	6018150
1·2799416 × ·6156615	seven	1·2799416 5 1665160	·7880108

Where the figures of the multiplier extend to the left of the multiplicand, continue as long as there is either multiplication or carriage. Thus in the first example, the first 9 of the arranged multiplier has no figure above it; but the carriage from the 3 ($9 \times 3 = 27$) is *three* tens, and three must be written under the right hand column of the preceding lines.

VIII.—*Contracted Division of Decimals.*

Rule.—Proceed as usual, until the number of quotient figures remaining to be found does not exceed the number of figures in the divisor. Then, instead of annexing a cipher, or bringing a figure down from the dividend, cut off the last figure of the divisor; that is, do not employ it except to carry from, as in the last rule. See how often this abridged divisor is contained in the remainder; multiply, carrying from the figure cut off; find a new remainder; cut off another figure from the divisor, and repeat the process until all the figures of the divisor are cut off. When the abridged divisor is not contained in the remainder, cut off a second figure from the divisor, put a cipher in the quotient, and proceed. We subjoin a detailed example.

To divide ·1299494 by ·9915206, as far as nine decimal places. The first quotient figure being a decimal, and there being seven places in the divisor, two quotient figures must be found by the usual method; after which, the process is explained.

* Io this case, 15 is equally oear to oae teo and *two* teus. It is usual, aod geoerally more correct, to take the higher of the two.

Divisor, afterwards abridged.		Dividend.	Quotient.	
9915206		) 12994940 (	·131060717	
		9915206		
		30797340		
		29745618		
991520	6	1051722	. .	Cut off the 6, reserving it to carry from; 991520 is contained in 1051722 once; once 6 is 6, nearest ten, one ten, carry one. The rest as usual.
		991521		
99152	06	60201	. .	Figure to carry from, 0; 99152 not contained in 60201, cut off another figure from divisor, and put 0 in quotient. Figure to carry from, 2; 9015 contained in 60201, six times. Six times 2 is 12; nearest ten, one ten, carry 1. Six times 5 is 30, and 1 is 31. The rest as usual.
9915	206			
		59491		
991	5206	710	. .	991 not contained in 710; cut off one more figure, and put 0 in quotient. Carrying figure 1; 99 contained in 710, seven times. Seven times 1 is 7, nearest ten, one ten, carry one. Seven times 9 is 63, and 1 is 64. The rest as usual.
99	15206			
		694		
9	915206	16	. .	Carrying figure 9. Divis. 9, contained once in 16. Once 9 is 9; nearest ten, one ten, carry one. Once 9 is 9, and 1 is 10.
		10		
	9915200	6	. .	No divisor, carrying figure 9. What number of times 9 will carry 6, or be most nearly 60? Seven times 9 is 63; put 7 in the quotient, and carry 6, which finishes the process.
		6		
		0		

Dividend.	Divisor.	No. of Decimals to be retained.	Quotient.
1	3·14159265	7	·3183099
1	2·7182818	7	·4342944
2992·9	51·77717	5	57·80347
171·8	414·487636	9	·414487636
·273	74·529	9	·003663004
·0008202	·67272804	8	·00121922

When the divisor itself contains more places than are required in the quotient, as many places may be cut from the right as will make the two the same; and the dividend may be cut down in the same way until no more places are left than will give one figure in the quotient, to the abridged divisor, remembering the rule for increasing the last figure. Thus ·1648267 ÷ ·7263, to two places only, may be found from ·16483 ÷ 73, by the rule exemplified above.

Both in multiplication and division, it is best to retain one more place than is absolutely required to be correct.

Section 3.—*Extraction of the Square Root. Examples of Surds and Irrational Quantities.*

I.—*Extraction of the Square Root.*

The rule for this will be better understood by a detailed example than by any verbal explanation. Though the quantities operated upon are decimal, it is to be understood that a whole number may be used in the same way. For 5, for instance, is 5·0000, &c.

The following contains the working of the rule at length for the extraction of the square root of 32·19, to four places of decimals. Annex so many ciphers that the decimal point shall be followed by twice as many places (eight) as there are to be decimals in the root (four). This gives 32·19000000. Point the unit's place, and every *other* place from it, to the right and left, which give $\dot{3}2\dot{1}9\dot{0}0\dot{0}0\dot{0}0$.

Divisors.	Given number pointed.	Root found, figure by figure, as below.
	3219900000	(5·6736
	25	First *period* 32; nearest square, 25; root 5. Put 5 in the root, and subtract 25 from 32.
106)	719	Remainder 7; bring down next period, 19. Double 5 (10), which place in divisor.
	636	Cut off one figure from 719,—71. This contains the divisor 10 seven times; *try* 7, as follows: annex it to divisor, 107; multiply by it, 107 × 7 is 749: this is greater than 719: 7 will not do. Try* 6. Then 106 × 6, is 636—less than 719. Put 6 in the root and in the divisor, and snhtract 636 from 719; remainder, 83.
1127)	8300 7889	Bring down next period, 00: add 6 last found to 106, giving new divisor, 112. Cut one figure from 8300—830. This contains 112 seven times. Try 7, and 1127 × 7 is 7889. Put 7 in the root and in the divisor, and subtract 7889 from 8300.
11343)	41100 34029	Remainder, 411. Bring down next period, 00; add 7 last fonnd to 1127, giving new divisor 1134. Cut one figure from 41100, —4110. This contains 1134 three times; trial * no louger necessary. Put 3 in the root, annex 3 to divisor, giving 11343. Subtract 11343 × 3, or 34029.
113466)	707100 680796	Remainder, 7071. Bring down last period, 00; add 3 last found to 11343, giving new divisor 11346. Cnt one figure from 707100;—70710. This contains 11346 six times. Put six in the root; annex 6 to divisor, giving 113466. Subtract 113466 × 6, or 680796.
	26304	Remainder, 26304; less than half of 113466, which shows that there is no occasion to change the last found 6 into 7, to have the nearest decimal of four places.

The required root is therefore 5·6736; by which we mean, that though 32·19 has no exact square root, yet 5·6736, multiplied by itself, will give a result *nearer to* 32·19 than any other nnmber with four decimal places. This we will try. Multiply the three successive fractions, 5·6735, 5·6736, 5·6737, each by itself, retaining five decimal places in the product.

5·67350	5·67360	5·67370
53765	63765	73765
2836750	2836800	2836850
340410	340416	340422
39715	39715	39716
1702	1702	1702
284	340	397
32·18861	32·18973	32·19087

Find the difference between each of these, and the quantity which we first set out with, and we have

·00139   ·00027   ·0008,

of which the second is the smallest.

When, after cutting off one figure from the altered remainder, the divisor is not contained in the result, bring down a second period, and place a cipher in the root and the divisor. The following is an instance in the extraction of the square root of 100406552374249. Where the calculator would simply annex a cipher or period to a line, we write the line again with the cipher, that the student may see the several steps.

* This trial will rarely be necessary after the second step. So that having cut one figure from the increased remainder, the nnmber of times which the divisor is therein contained may be written down on the right, and the whole divisor, *thus altered*, multiplied by its last figure.

```
         100406552374249(1
         1
      2) 00
     20) 0040            10      *
   2002) 004065          1002
           4004
   2004)     6152
   200403    615237       100203
             601209
   200406)   1402842
   20040607) 140284249    10020307
             140284249
                 0
```

Wherever a dotted line occurs, the augmented remainder, with the last figure cut off, is found not to contain the dividend, a new period is brought down below the line, a cipher is annexed to the divisor and to the root (also brought down), and the figure which, after this, answers the purpose, appears at the end of the divisor and of the root. There being no remainder at last, the exact square root required is 10020307.

The student should perform the preceding operation in this form:

```
           100406552374249(10020307
           1
   2002)   004065
             4004
   200403)   615237
             601209
   20040607) 140284249
             140284249
                 0
```

We must notice one more case in which a cipher may occur. We will first write the beginner's attempt, as it would be if he were not cautious. To extract the square root of 2034:

First Attempt.	Corrected Process.
203400, &c. (45·1	203400, &c. (45·09, &c.
16	16
85) 434	85) 434
425	425
901) 900	9009) 90000
901	81081
	9018) 892900
	&c.

In the first he has gone wrong, for though 900, stripped of its last figure, contains 90 once exactly, yet 901 (the new figure being annexed) is not contained in 900. He therefore puts a cipher in the divisor and the root, and brings down another period. The decimal point of the root always precedes that root figure in forming which the first decimal period was used, annexed

* The preliminary ciphers may be omitted; 20 is not contained in 004 or 4.

ciphers being always considered as decimals. If periods of ciphers be thrown away in the beginning of the operation, the root is all decimal, and has a cipher at the beginning for every period so thrown away; but this rule does not apply to the throwing away of a single cipher (not a whole period) at the beginning, or to a cipher in the unit's place.

In the following examples, the number whose root is to be extracted is in the first column; the pointing at full length in the second; the same with the decimal point and preliminary ciphers, if any, thrown away, in the third; and the answer in the last.

No. given.	Do., pointed.	Do., simplified.	Square Root, nearly.
·1	0·1000 &c.	1000 &c.	·31622776602
·85	0·8500 &c.	8500 &c.	·92195444573
·0683	0·068300 &c.	68300 &c.	·261342686907
·0068	0·006800 &c.	6800 &c.	·082462112512
9·79	9·7900 &c.	97900 &c.	3·12889756943
97·9	97·9000	979000 &c.	9·89444288

The preceding method may be shortened, as soon as half the decimal places required have been found, by substituting a contracted division.

The rule is, when *half* the number of (decimal and other) places have been obtained, instead of forming a new period, let the remainder stand, strike off a figure from the divisor, and proceed as in contracted division.

The following is the extraction of the square root of 12 to 12 decimal places by this method:

```
          12·00 . . . . . (3·464101615138
           9
     64)   3 00
           2 56
    686)    4400
            4116
     6924)  28400
             27696
    69281)   70400
             69281
  6928201)   11190000
              6928201
  692820|2)   4261799
              4156921
               104878
                69282
                35596
                34641
                  955
                  693
                  262
                  208
                   54
                   55  The nearest. The 8 not so much too great as 7
                                    would be too small.
```

The student may furnish himself with examples to any amount by the following principle: If A has the square-root B, *four* times A has the square-root *twice* B, *nine* times A has the square-root *three* times B, and so on. Let him then choose a number or fraction, and extract the square-root, say of four times that number, as well as of the number itself. His first result should be twice the second. The last figures only cannot be expected to agree.

The extraction of the cube-root is a long and useless process. When the student becomes acquainted with logarithms, he will always use them for the extraction of all roots, the square-root included.

II.—*Definition and Notation of Powers and Roots.*

Operation.	Denoted by	Commonly called
7×7	7^2	The square or second power of 7.
$7 \times 7 \times 7$	7^3	The cube, or third power of 7.
$7 \times 7 \times 7 \times 7$	7^4	The fourth power of 7.
$7 \times 7 \times 7 \times 7 \times 7$	7^5	The fifth power of 7.
&c.	&c.	&c.

By analogy, 7 is written 7^1 and called the first power of 7.

Condition fulfilled by p.	Manner of denoting p.	Name of p.
$pp = 7$	$\sqrt{7}$ or $7^{\frac{1}{2}}$	Square, or second root of 7, or 7 to the power of one-half.
$ppp = 7$	$\sqrt[3]{7}$ or $7^{\frac{1}{3}}$ . . .	Cube, or third root of 7, or 7 to the power of one-third.
$pppp = 7$	$\sqrt[4]{7}$ or $7^{\frac{1}{4}}$	Fourth root of 7, or 7 to the power of one-fourth.
$ppp = 7^2$	$\sqrt[5]{7^2}$ or $7^{\frac{2}{3}}$ . . .	Cube root of the square of 7, or 7 to the power of two-thirds.
$ppppp = 7^6$	$\sqrt[5]{7^6}$ or $7^{\frac{6}{5}}$	Fifth root of the sixth power of 7, or 7 to the power of six-fifths.
$pp = 7^{11}$	$\sqrt{7^{11}}$ or $7^{\frac{11}{2}}$. . .	Square root of the eleventh power of 7, or 7 to the power of eleven halves.

Verify the following equations by multiplication:

$$16 = 2^4 = 4^2 = 8^{\frac{4}{3}} = 16^1 = 32^{\frac{4}{5}} = 64^{\frac{2}{3}}$$

$$9 = 3^2 = 9^1 = 27^{\frac{2}{3}} = 81^{\frac{1}{2}} = 243^{\frac{2}{5}} = 729^{\frac{1}{3}}$$

$$32 = 8^{\frac{5}{3}} = 4^{\frac{5}{2}} \qquad 256 = 32^{\frac{8}{5}}$$

In such an equation as $32 = 4^{\frac{5}{2}}$, $\frac{5}{2}$ has two names; one referring it to the 4, the other to the 32.

$\frac{5}{2}$ is called the *exponent* of 4.

$\frac{5}{2}$ is called the *logarithm* of 32 *to the base* 4.

Verify the following assertions:

The number undermentioned	is the logarithm of the corresponding number undermentioned	to the base
2, 3, 4	100, 1000, 10000	10
3, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$	262144, 8, 4, 2	64
6, 5, 4	64, 32, 16	2
$\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{2}{5}$	1024, 32, 16, 256	1048576
$\frac{3}{4}$, $\frac{3}{5}$	32768, 4096	
$\frac{4}{5}$, $\frac{1}{10}$, $\frac{3}{10}$	65536, 4, 64	
$\frac{7}{10}$, $\frac{9}{10}$, $\frac{1}{20}$	16384, 262144, 2	

III.—*Particular cases of Propositions which are proved by Algebraic Reasoning.*

The student should go through the whole of these, adding others similar to them if necessary, until he performs all such operations by habit, without rules.

$$10^6 = 10 \times 10^5 = 10^2 \times 10^4 = 10^3 \times 10^3 = 1000000$$

$$2^8 = 2 \times 2^7 = 2^2 \times 2^6 = 2^3 \times 2^5 \text{ \&c.} = 256$$

$$2^8 \times 2^7 = 2^{15} \qquad 12^9 \times 12^{14} = 12^{23}$$

$$8^3 = \frac{8^4}{8} = \frac{8^5}{8^2} = \frac{8^6}{8^3} = \frac{8^7}{8^4} \text{ \&c.} = 512$$

$\frac{9^{16}}{9^5} = 9^{11}$ $\qquad \frac{163^{21}}{163^{15}} = 163^6$ $\qquad \frac{4^{90}}{4^{33}} = 4^{57}$

$$2^{12} = (2^2)^6 = (2^3)^4 = (2^4)^3 = (2^6)^2 = 4096$$

$(3^8)^4 = 3^{32}$ $\qquad (7^{11})^5 = 7^{55}$ $\qquad (100^9)^9 = 100^{81}$ &c.

$$2^3 = 2^{\frac{6}{2}}, \text{ or } \sqrt{2^6}, = 2^{\frac{9}{3}} \text{ or } \sqrt[3]{2^9}, = 2^{\frac{12}{4}} \text{ or } \sqrt[4]{2^{12}}$$

$$2^{\frac{1}{2}} = 2^{\frac{2}{4}} = 2^{\frac{3}{6}} = 2^{\frac{4}{8}} = 2^{\frac{5}{10}}$$

$$\text{or, } \sqrt{2} = \sqrt[4]{2^2} = \sqrt[6]{2^3} = \sqrt[8]{2^4} = \sqrt[10]{2^5} \text{ \&c.}$$

$\sqrt[4]{33^{18}} = \sqrt{33^9}$ $\qquad \sqrt[12]{36^{12}} = 36$ $\qquad \sqrt[3]{15^{15}} = 15^5$

$\sqrt{\sqrt{9}}$, or $(9^{\frac{1}{2}})^{\frac{1}{2}} = 9^{\frac{1}{4}}$ $\qquad \sqrt[2]{\sqrt[3]{9}}$ or $(9^{\frac{1}{3}})^{\frac{1}{2}} = 9^{\frac{1}{6}}$

$\sqrt[4]{\sqrt[5]{8}}$ or $(8^{\frac{1}{5}})^{\frac{1}{4}} = 8^{\frac{1}{20}}$ $\qquad \sqrt[5]{\sqrt[4]{8}}$ or $(8^{\frac{1}{4}})^{\frac{1}{5}} = 8^{\frac{1}{20}}$

$\sqrt[5]{32^2} = (\sqrt[5]{32})^2 = 4$ $\qquad \sqrt[5]{32^3} = (\sqrt[5]{32})^3 = 8$

$$\sqrt[5]{\left\{\sqrt[3]{10^2}\right\}^4} = (10^{\frac{2}{3}})^{\frac{4}{5}} = 10^{\frac{8}{15}} = \left(\left((10^2)^{\frac{1}{3}}\right)^4\right)^{\frac{1}{5}}$$

$$\sqrt[3]{\left\{\sqrt[5]{10^4}\right\}^2} = (10^{\frac{4}{5}})^{\frac{2}{3}} = 10^{\frac{8}{15}} = \left(\left((10^4)^{\frac{1}{5}}\right)^2\right)^{\frac{1}{3}}$$

$$6^{\frac{1}{2}} \times 6^{\frac{1}{3}} = 6^{\frac{1}{2}+\frac{1}{3}} = 6^{\frac{5}{6}}, \text{ or } \sqrt{6} \times \sqrt[3]{6} = \sqrt[6]{6^5}$$

$$7^{\frac{2}{3}} \times 7^{\frac{4}{5}} = 7^{\frac{2}{3}+\frac{4}{5}} = 7^{\frac{22}{15}}, \text{or } \sqrt[3]{7^2} \times \sqrt[5]{7^4} = \sqrt[15]{7^{22}}$$

$$8^{\frac{1}{2}} \div 8^{\frac{1}{3}} = 8^{\frac{1}{2}-\frac{1}{3}} = 8^{\frac{1}{6}}, \text{ or } \sqrt{8} \div \sqrt[3]{8} = \sqrt[6]{8}$$

$$9^{\frac{4}{5}} \div 9^{\frac{2}{3}} = 9^{\frac{4}{5}-\frac{2}{3}} = 9^{\frac{2}{15}}, \text{or } \sqrt[5]{9^4} \div \sqrt[3]{9^2} = \sqrt[15]{9^2}$$

$$10^4 \div 10^9 = \frac{10^4}{10^9} = \frac{10^4 \div 10^4}{10^9 \div 10^4} = \frac{1}{10^5} = \frac{1}{10^{9-4}}$$

$6^7 \div 6^{18} = \frac{1}{6^{18-7}} = \frac{1}{6^{11}}$ $\qquad 7^{20} \div 7^{33} = \frac{1}{7^{13}}$

$3^{\frac{1}{4}} \div 3^{\frac{1}{3}} = \frac{1}{3^{\frac{1}{3}-\frac{1}{4}}} = \frac{1}{3^{\frac{1}{12}}}$ $\qquad 5^{\frac{3}{7}} \div 5^{\frac{5}{7}} = \frac{1}{5^{\frac{2}{7}}}$

$$(10^{\frac{2}{3}})^{\frac{3}{4}} \times (10^{\frac{1}{2}})^4 \div (10^{\frac{1}{3}})^{\frac{1}{2}} = 10^{\frac{1}{2}} \times 10^2 \div 10^{\frac{1}{6}} = 10^{\frac{1}{2}+2-\frac{1}{6}} = 10^{\frac{7}{3}}$$

$$(5^{\frac{1}{2}})^7 \times (5^3)^{\frac{1}{2}} \div (5^4)^{\frac{1}{8}} = 5^{\frac{7}{2}} \times 5^{\frac{3}{2}} \div 5^{\frac{1}{2}} = 5^{\frac{9}{2}}$$

$6^{2\frac{1}{2}}$ means $6^{\frac{5}{2}}$ $\qquad 3^{4\frac{1}{5}}$ means $3^{\frac{21}{5}}$

$(2{\cdot}36)^{1{\cdot}01}$ means $(2{\cdot}36)^{\frac{101}{100}}$ $\qquad (6^{{\cdot}4})^{{\cdot}5}$ is $6^{{\cdot}2}$

$$7^{\cdot 1} \text{ means } 7^{\frac{1}{10}} \qquad 8^{\cdot 014} \text{ means } 8^{\frac{14}{1000}}$$

$$2^3 \times 4^3 = 8^3 \qquad 3^4 \times 10^4 = 30^4 \qquad 6^{\frac{1}{2}} \times 7^{\frac{1}{2}} = 42^{\frac{1}{2}}$$

$$2^{\frac{1}{2}} \times 3^{\frac{1}{2}} \times 4^{\frac{1}{2}} = \overline{2 \times 3 \times 4}|^{\frac{1}{2}} = 24^{\frac{1}{2}} \text{ or } \sqrt{2} \times \sqrt{3} \times \sqrt{4} = \sqrt{24}$$

$$7^{\frac{2}{3}} \times 8^{\frac{2}{3}} = \overline{7 \times 8}|^{\frac{2}{3}} \text{ or } \sqrt[3]{7^2} \times \sqrt[3]{8^2} = \sqrt[3]{56^2}$$

$$\left(\frac{1}{2}\sqrt{2}\right)^2 = \left(\frac{1}{2}\right)^2 \times 2 = \frac{1}{2} \qquad \left(\frac{2}{3}\sqrt{\frac{13}{14}}\right)^2 = \frac{4}{9} \times \frac{13}{14} = \frac{26}{63}$$

$$\left(4\sqrt{10}\right)^2 = 160 \qquad \left(\frac{1}{2}\sqrt{8}\right)^2 = 2 \qquad \left(\frac{3}{7}\sqrt{\frac{14}{3}}\right)^2 = \frac{6}{7}$$

$$\sqrt{81} = 9 \qquad \sqrt{81 \times 2} = \sqrt{81} \times \sqrt{2} = 9\sqrt{2} \quad ; \quad \sqrt{156} = 2\sqrt{39}$$

$$\sqrt{\frac{2}{3}} \times \sqrt{\frac{15}{8}} = \frac{\sqrt{5}}{2} \qquad \sqrt{18} \times \sqrt{20} = 2\sqrt{90}$$

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2} \quad ; \quad \sqrt{32} = 4\sqrt{2}$$

$$\sqrt{44} = 2\sqrt{11} \qquad \sqrt{160} = 4\sqrt{10} \qquad \sqrt{1836} = 6\sqrt{51}$$

$$10\sqrt{9} = \sqrt{900} \qquad 7\sqrt{3} = \sqrt{147} \qquad 12\sqrt{12} = \sqrt{1728}$$

$$\sqrt[3]{56} = \sqrt[3]{8 \times 7} = \sqrt[3]{8} \times \sqrt[3]{7} = 2\sqrt[3]{7} \qquad \sqrt{168} = 2\sqrt[3]{21}$$

$$\sqrt[4]{288} = \sqrt[4]{16 \times 18} = \sqrt[4]{16} \times \sqrt[4]{18} = 2\sqrt[4]{18} \quad , \quad \sqrt[4]{6144} = 4\sqrt[4]{24}$$

$$\sqrt[4]{6966} = 3\sqrt[4]{86} \quad , \quad 4\sqrt[4]{3} = \sqrt[4]{768} \qquad \sqrt[7]{256} = 2\sqrt[7]{2}$$

$$\sqrt{\frac{3}{7}} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{3} \times \sqrt{3}}{\sqrt{7} \times \sqrt{3}} = \frac{3}{\sqrt{21}} \qquad \sqrt{\frac{6}{11}} = \frac{6}{\sqrt{66}}$$

$$\sqrt{\frac{3}{7}} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{3} \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{\sqrt{21}}{7} \qquad \sqrt{\frac{6}{11}} = \frac{\sqrt{66}}{11}$$

$$\sqrt{\frac{5}{49}} = \frac{\sqrt{5}}{\sqrt{49}} = \frac{\sqrt{5}}{7} \qquad \sqrt{\frac{3}{64}} = \frac{\sqrt{3}}{8} = \sqrt{\frac{121}{7}} = \frac{11}{\sqrt{7}}$$

$$\sqrt[3]{\frac{3}{5}} = \frac{\sqrt[3]{3}}{\sqrt[3]{5}} = \frac{\sqrt[3]{3} \times \sqrt[3]{3} \times \sqrt[3]{3}}{\sqrt[3]{5} \times \sqrt[3]{3} \times \sqrt[3]{3}} = \frac{3}{\sqrt[3]{5 \times 3 \times 3}} = \frac{3}{\sqrt[3]{45}}$$

$$\sqrt[3]{\frac{3}{5}} = \frac{\sqrt[3]{3}}{\sqrt[3]{5}} = \frac{\sqrt[3]{3} \times \sqrt[3]{5} \times \sqrt[3]{5}}{\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}} = \frac{\sqrt[3]{75}}{5} = \frac{1}{5}\sqrt[3]{75}$$

$$\sqrt[3]{\frac{4}{7}} = \frac{\sqrt[3]{196}}{7} = \frac{4}{\sqrt[3]{112}} \quad , \quad \sqrt{\frac{9}{10}} = \frac{\sqrt[3]{900}}{10} = \frac{9}{\sqrt[3]{810}}$$

The preceding operations occur perpetually in the higher applications of arithmetic; the student should repeat them on low numbers, till he is perfectly familiar with all of them. The following are the rules under which they may all be reduced; but they should be dispensed with if possible, by mere habit of performing the operations.

Rule 1. All roots may be treated as powers, that is, fall under the same rules as powers, when the fraction which has the order of the root in its denominator is used as the exponent. Call them *fractional* powers, so that the word power shall mean both power and root.

$$\sqrt[n]{a} \text{ is } a^{\frac{1}{n}} \qquad \sqrt[n]{a^m} \text{ is } a^{\frac{m}{n}}$$

Rule 2. To raise a power to a power, multiply together the exponents for a new exponent.

$$\left(2^{\frac{2}{3}}\right)^{\frac{3}{4}} = 2^{\frac{2}{4}} \qquad \left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = a^{\frac{mp}{nq}}$$

Rule 3. To raise a product, or quotient, or the result of several multiplications and divisions, to any power, raise every multiplier and divisor to the same power.

$$\left(\frac{a\,b}{c\,d}\right)^n = \frac{a^n\,b^n}{c^n\,d^n}$$

Rule 4. When several powers are raised successively, it is indifferent in what order the operations are performed.

$$\left((a^p)^q\right)^r = \left((a^q)^r\right)^p = a^{qrp}$$

Rule 5. To multiply together two powers of the same quantity, add the exponents for a new exponent.

$$a^m \times a^n = a^{m+n}$$

Rule 6. To divide one power of a quantity by another power of the same, take the difference of the exponents for a new exponent, and place the result in the numerator or denominator, according as the dividend or divisor has the greater exponent.

$$m \text{ greater than } n \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$m \text{ less than } n \qquad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}$$

IV. *Various Combinations of the preceding Propositions applied to the Use of the Square-Root.*

Rule 1. To square the sum of two quantities, square each of them, and to the sum of the squares *add* twice the product of the quantities. To square the difference of two quantities, *subtract* twice the product, instead of *adding*.

$$(a+b)^2 = a^2 + b^2 + 2\,a\,b \qquad (a-b)^2 = a^2 + b^2 - 2\,a\,b$$

$$(6+4)^2 = 36 + 16 + 2 \times 24 \qquad (6-4)^2 = 36 + 16 - 2 \times 24$$

$$(6+\sqrt{3})^2 = 6 + 3 + 12\sqrt{3} = 9 + 12\sqrt{3}$$

$$(6-\sqrt{3})^2 = 6 + 3 - 12\sqrt{3} = 9 - 12\sqrt{3}$$

$$(\sqrt{7}+\sqrt{3})^2 = 10 + 2\sqrt{21} \qquad \left\{\begin{aligned}(\sqrt{14}-\sqrt{2})^2 &= 16 - 2\sqrt{28}\\ &= 16 - 4\sqrt{7}\end{aligned}\right.$$

$$(\sqrt{12}+\sqrt{10})^2 = 22 + 2\sqrt{120}$$

$$\left(\frac{3}{2}+\sqrt{2}\right)^2 = \frac{17}{4} + 3\sqrt{2} \qquad \left(2\tfrac{1}{2}-\sqrt{3}\right)^2 = \frac{37}{4} - 5\sqrt{3}$$

$$\left(\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{7}}\right)^2 = \frac{1}{2}+\frac{2}{7} - 2\sqrt{\frac{1}{2}\times\frac{2}{7}} = \frac{11}{14} - \frac{2}{7}\sqrt{7}$$

$$\left(\sqrt{\frac{3}{8}}-\sqrt{\frac{5}{12}}\right)^2 = \frac{3}{8}+\frac{5}{12} - 2\sqrt{\frac{3}{8}\times\frac{5}{12}} = \frac{19}{24} - \frac{1}{4}\sqrt{10}$$

$(\sqrt{2\tfrac{1}{2}}+\sqrt{3\tfrac{1}{2}})^2 = 6+\sqrt{35}$ $\quad (\sqrt{7\tfrac{1}{2}} - \sqrt{1\tfrac{1}{4}})^2 = 8\tfrac{3}{4} - \tfrac{1}{2}\sqrt{150}$

$(2\sqrt{2}+3\sqrt{7})^2 = 71+12\sqrt{14}$ $\quad (3\sqrt{2}-2\sqrt{3})^2 = 30-12\sqrt{6}$

$\left(\frac{1}{2}\sqrt{3}-1\right)^2 = \frac{7}{4}-\sqrt{3}$ $\quad \left(\frac{1}{3}\sqrt{2}+\frac{1}{2}\sqrt{3}\right)^2 = \frac{35}{36}+\frac{1}{3}\sqrt{6}$

$\left(\frac{3}{7}\sqrt{5}-\frac{1}{7}\sqrt{10}\right)^2 = \frac{55}{49}-\frac{6}{49}\sqrt{50}$ $\quad \left(\frac{2}{3}\sqrt{\frac{1}{2}}-\frac{1}{4}\sqrt{\frac{1}{3}}\right)^2 = \frac{35}{144}-\frac{1}{18}\sqrt{6}$

$(1{\cdot}1-\sqrt{{\cdot}1})^2 = 1{\cdot}31 - 2{\cdot}2\sqrt{{\cdot}1}$ $\quad (\sqrt{{\cdot}6}-\sqrt{{\cdot}7})^2 = 1{\cdot}3 - 2\sqrt{{\cdot}42}$

Rule 2. The product of the sum, and difference of two quantities, is the difference of their squares:

$$(a+b)\times(a-b) = a^2 - b^2$$
$$(6+4)\times(6-4) = 36 - 16$$

Factors given.		Product.
$\sqrt{5}+\sqrt{3}$	$\sqrt{5}-\sqrt{3}$	2
$\frac{1}{2}\sqrt{\frac{1}{7}}+\frac{1}{10}$	$\frac{1}{2}\sqrt{\frac{1}{7}}-\frac{1}{10}$	$\frac{9}{350}$
$6+\sqrt{\frac{1}{2}}$	$6-\sqrt{\frac{1}{2}}$	$35\frac{1}{2}$
$\sqrt{\frac{8}{3}}+\sqrt{\frac{3}{8}}$	$\sqrt{\frac{8}{3}}-\sqrt{\frac{3}{8}}$	$\frac{55}{24}$
$\sqrt{\frac{1}{2}}+\frac{1}{2}$	$\sqrt{\frac{1}{2}}-\frac{1}{2}$	$\frac{1}{4}$
$\sqrt{10}+3$	$\sqrt{10}-3$	1

SECTION 4. *Miscellaneous Questions involving the Use of Fractions.*

1. If $\frac{2}{3}$ of a shilling buy $\frac{1}{4}$ of a gallon, how much will $\frac{3}{5}$ of a shilling buy?

If $\frac{2}{3}s.$ buy $\frac{1}{4}$ gall.

Then $2s.$ buy $\frac{3}{4}$ gall.

$1s.$ buys $\frac{3}{8}$ gall.

$3s.$ buy $\frac{9}{8}$ gall.

$\frac{3}{5}s.$ buys $\frac{9}{40}$ gall.

2. If $\frac{3}{17}$ of a pound sterling is paid for $\frac{2}{15}$ of a yard, how much must be paid for $3\frac{1}{4}$?

If $\frac{2}{15}$ yd. cost £$\frac{3}{17}$

2 yds. „ £$\frac{45}{17}$

1 yd. costs £$\frac{45}{34}$

13 yds. cost £$\frac{585}{34}$

$\frac{13}{4}$ yd. „ £$\frac{585}{136}$

3. If £$2\frac{1}{2}$ buy $3\frac{1}{3}$ gallons, how much will £$4\frac{1}{7}$ buy? *Answer*, $5\frac{11}{21}$.

4. If $3\frac{1}{5}$ acres let for £$10\frac{1}{4}$, how much will $11\frac{1}{3}$ acres let for? *Answer*, £$36\frac{29}{96}$.

5. If £$\frac{2}{3}$ be worth $\frac{1}{5}$ of a sheep, and $\frac{3}{7}$ of a sheep be worth $\frac{1}{14}$ of an ox, how much must be given for 100 oxen? *Answer*, £2000.

6. If 12 oxen be worth 29 sheep, 15 sheep worth 25 hogs, 17 hogs worth 3 loads of wheat, and 8 loads of wheat worth 13 loads of barley; how many loads of barley must be given for 20 oxen? *Answer*, 23 $\frac{41}{408}$ loads.

7. If 12 of A count for 13 of B, 6 of B for 18 of C, and 13 of C for 2 of D; how many of A count for 100 of D? *Answer*, 200.

8. A. is indebted $\frac{1}{14}$ of his whole property, and loses $\frac{7}{8}$ of it. He recovers as much as amounts to adding $\frac{1}{5}$ to what he then has, and afterwards loses $\frac{1}{2}$ of what he has got. Can he then pay his debts? *Answer*, Yes; after which $\frac{1}{280}$ of his original property will remain to him.

9. A. gains 3 per cent. (3 parts out of a hundred) on what he already has, and B. 7 per cent. But A. gains £100 less than B., and they started with the same sums. What were those sums? *Answer*, £2500 each.

10. There is a number to which 3 is added, and $\frac{1}{10}$ of the result taken. To this 5 is added, and $\frac{1}{15}$ of the result taken. The produce is then $1\frac{1}{3}$. What was the number? *Answer*, 172.

11. A woman bought 150 apples at three a-penny, and 100 at two a-penny, and found she neither lost nor gained by selling the whole lot at five for twopence. But on doing the same with a couple of other lots of 150 apples each, she found she was a loser. What was the reason of this?

12. How much per cent. is £62 of £75: that is, how many times does $\frac{62}{75}$ contain $\frac{1}{100}$? Answer, $82\frac{2}{3}$ per cent.

13. What decimal fraction of a pound is one farthing? Ans., ·001041666....

14. How many pounds are there in a hundred million of farthings? (See last question.)

15. What fraction is one pound avoirdupoise of a hundred weight, one day of a year, and one second of a day? Answer, ·00892857, ·002739726, ·000011574, nearly.

16. A cistern $\frac{2}{3}$ full has two cocks, which alone would empty *the whole cistern* in 7 and 5 minutes. How soon will they empty it together? (Show that this amounts to asking how often $\frac{1}{7}+\frac{1}{5}$ is contained in $\frac{2}{3}$.) Answer, in $1\frac{17}{18}$ minutes.

17. The tenth part of a number is increased by 1, the tenth part of the result by 1, and so on in succession five times, the result of which is 6·79829. What was the number? Answer, 568719.

18. Show that if any number be treated in the preceding manner a sufficient number of times, the result may be brought as near to $\frac{10}{9}$ as we please.

19. What is the reason of the following series of equations, the law of which will be immediately perceived:

$$1 + \frac{1}{2} + \frac{1}{4} = \frac{4 - 1}{4} \qquad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{8 - 1}{8}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{16 - 1}{16} \text{ \&c.}$$

20. From $\sqrt{3} = 1{\cdot}7320508$ deduce $\frac{1}{\sqrt{3}} = {\cdot}5773503$ in the most simple manner; and also

$$\frac{1}{2 + \sqrt{3}} = {\cdot}2679492,$$

without dividing by any decimal fraction.

SECTION 5.—*Useful approximative Rules applicable to cases which frequently occur.*

1. To find how much a certain sum per day amounts to in a year, and the converse.

Rule 1. To the number of pence per day add its half; call the result pounds, and this is the amount in 360 days; add a shilling for every penny in the half-day's allowance just found, and this is the result for a year and a day, or for leap-year. For a common year, diminish the above by one day's allowance.

How much does 3*s*. 4*d*. a-day amount to in a year?

40 pence
20 pence, its half
Sum 60 pence

£60 0 0
1 20 shillings
£61 0 0 in 366 days.

The correct amount is 3*s*. 4*d*. less than this.

How much does $7\frac{1}{2}$*d*. a-day amount to in a year?
$3\frac{3}{4}$ its half
Sum $11\frac{1}{4}$ pence

£11	5	0	$11\frac{1}{4}$ pounds
	3	9	$3\frac{3}{4}$ shillings
£11	8	9	for 366 days
		$7\frac{1}{2}$	
£11	8	$1\frac{1}{2}$	for 365 days.

The undermentioned Sum, per Day,	Is the undermentioned, per Year (365 Days).
6*s*. 4*d*.	£115 11 8
2*s*. $3\frac{1}{2}$*d*.	£ 41 16 $5\frac{1}{2}$
$9\frac{3}{4}$*d*.	£ 14 16 $6\frac{3}{4}$

Rule 2. Take the nearest pound to the year's allowance, subtract one-third of itself from it, and let the result be pence. If more exactness be required, subtract a penny for every six shillings. The result is within a penny of the sum per day.

How much per day is £100 a-year?

100
$33\frac{2}{3}$ $\frac{1}{3}$ of 100
$66\frac{1}{3}$ pence is
5*s*. 6*d*. nearly;

therefore 5*s*. 5*d*. is nearly the answer.

Per Year.	Per Day, about
£357	19*s*. 7*d*.
£ 27	1*s*. 6*d*.
£493	£1. 7*s*. 1*d*.

Rule 3. A number of shillings per *week* taken twice, and a half, and a tenth, is the number of pounds per year.

Thus, 16 shillings a week is $2 \times 16 + \frac{1}{2}$ of $16 + \frac{1}{10}$ of 16, or $32 + 8 + 1{\cdot}6$, or $41{\cdot}6$ pounds per year, or £$41\frac{6}{10}$, or £41. 12*s*.

Rule 4. A number of pounds per year is very nearly* one-third and one-twentieth in shillings per week.

Thus, £60 a-year is $\frac{1}{3}$ of 60*s*. + $\frac{1}{20}$ of 60*s*., or 23*s*. a-week (exactly 23*s*. 0*d*. $\frac{12}{13}$).

Similarly, £37 a-year is $\frac{1}{3}$ of 37*s*. + $\frac{1}{20}$ 37*s*., or 12*s*. 4*d*. + 1*s*. 10*d*., or 14*s*. 2*d*. per week.

II.—*To reduce Shillings, &c., to the Decimal of £1, and the converse.*

Rule 1. Annex two ciphers to the shillings, and halve the result. Turn the pence and farthings into farthings, adding one if that gives 24 or upwards. Add and make three decimal places.

What decimals of £1 are 15*s*. 9¾*d*., 6¾*d*., and 1*s*. 2¼*d*.?

2)1500		0*s*.	6¾	1*s*. 2¼*d*.	
750		27	6¾ × 4 + 1	5)100	
40	9¾ × 4 + 1	·027	Answer	50	
790				9	2¼ × 4
·790	Answer			59	
				·059	Answer.

4*s*. 11½*d*.	is	£ ·247
5*s*. 11½*d*.	is	£ ·297
1*s*. 0*d*.	is	£ ·050
2*s*. 0*d*.	is	£ ·100
16*s*. 7¼*d*.	is	£ ·830
17*s*. 4½*d*.	is	£ ·868
£15 7 6¼	is	£15·376

Rule 2. Given a decimal of a pound: take the three first places, double the first figure, and add one, if the second be 5 or upwards, for the shillings; take the second and third places, throwing out 5 from the second, if that be 5 or upwards, and 1 from the third, if the result of the last give 25 or upwards. This is the number of farthings, which must be turned into pence and farthings.

What shillings, pence, and farthings are there in £·177?

First figure × 2	= 2
Add 1 for 5 in second figure	= 1
	3 shillings
Second and third figures, with 5 struck out from the second	27
Take away 1, this being upwards of 24	1
	26 farthings
	= 6½*d*.
	3*s*. 6½*d*. Answer.

£ ·019	is	4½*d*.
£ ·076	is	1*s*. 6¼*d*.
£ ·342	is	6*s*. 10¼*d*.
£ ·969	is	19*s*. 4¾*d*.
£1·118	is	£1. 2*s*. 4½*d*.

* Within less than a penny in a pound.

The results of the last two rules are approximations which are sufficient for common purposes. The student should repeat them until he can solve both cases mentally. They give immediately the price of 10 things within three-pence, of 100 within about two shillings, and of 1000 within a pound, when the price of one is known. For example, if one thing cost £2. 14s. 4½d., or £2·718, ten cost £27·18, or £27·180, or £27. 3s. 7d. (within a penny or two), one hundred cost 271·8., or £271l. 16s. (within a few shillings), and 1000 cost £2718. (within a pound). The correct answers to the preceding cases are 27l. 3s. 9d., 271l. 17s. 6d. and 2718l. 15s. *Observe, that in the case of shillings and sixpences, without odd pence and farthings, these rules are exact.*

What is the interest on 157l. 17s. 6d. for one year, at 5 per cent.?

$$\begin{array}{r} 157\cdot875 \\ 5 \\ \hline 100)789\cdot375 \\ \hline 7\cdot89375 \end{array} \quad £7.\ 17s.\ 11d.$$

III.—*To reduce Miles per Hour to Feet per Second, and the converse.*

Rule 1. Half as much again as the number of miles per hour is, with sufficient exactness for common purposes, the number of feet per second. To be perfectly exact use the following:

$$\text{Number of miles} + \tfrac{1}{2}\text{ the number} - \tfrac{1}{30}\text{ the number.}$$

Thus 6 miles an hour is $6 + \frac{1}{2}$ of $6 - \frac{1}{30}$ of 6, or $8\frac{4}{5}$ feet per second.

Rule 2. Add half of the feet per second to its fifth (and if perfect accuracy be necessary, subtract one eleventh of the last); the result is the number of miles per hour. Thus 22 feet per second gives $\frac{1}{2}$ of $22 + \frac{1}{5}$ of 22, or $11 + 4\cdot4$, or $15\cdot4$ miles per hour nearly; $15\cdot4 - \frac{1}{11}$ of $4\cdot4$, or 15 miles exactly.

The preceding rules have been given because they frequently apply in practice. In no other case is it worth while to learn a special rule. But in every sort of occupation which has any reference to arithmetic, the necessity for multiplying or dividing by some particular decimal fraction will frequently occur. A calculator who does not meet with any one particular fraction oftener than another, will not need to take any other than common rules, since the trouble of learning and recollecting a particular rule will more than counterbalance its convenience, in the few instances in which he will have need to apply it; but where one particular fraction occurs frequently, the following hints may be useful.

1. The labour of calculation will be saved, and the chance of error almost destroyed, by a table, which may be more or less extensive according to circumstances. For example, a reader of French works of geography, travels, architecture, &c., will continually be obliged to convert metres into feet, and the converse: he should, therefore, make on a card such a table* as the following:—

Metres.	Feet.	Metres.	Feet.	Metres.	Feet.
1	3·2809	10	32·809	100	328·09
2	6·5618	20	65·618	200	656·18
3	9·8427	30	98·427	300	984·27
4	13·1236	40	131·236	400	1312·36
5	16·4045	50	164·045	500	1640·45
6	19·6854	60	196·854	600	1968·54
7	22·9663	70	229·663	700	2296·63
8	26·2472	80	262·472	800	2624·72
9	29·5281	90	295·281	900	2952·81

* In forming such tables, avoid, as much as possible, the necessity of altering what is taken from the table. An expert calculator needs only the first column; but of these there are not many.

This table is calculated from the following:

1 metre is 3·2809 feet,

and its use is as follows:—For example, what is 867·41 metres?

800 metres are	2624·72	feet.
60 - - -	196·85	„
7 - - -	22·97	„
·4 - - -	1·31	„
·01 - - -	·03	„
867·41 - -	2845·88	

2. For a less exact method, to be used when tables are not at hand, or when a great degree of correctness is not required, lay down the number of decimal places which are to be retained, and endeavour to separate these places into simpler fractions, somewhat in the manner followed in the rule of *Practice* in commercial arithmetic. For instance, in the preceding case, suppose that the metre is 3·281 feet, the error of which is less than one ten-thousandth part of a foot, that is, giving in the multiplication an error of less than one foot in ten thousand. The preceding is $3·25 + ·03 + ·001$, which gives the following rule:—To turn A metres into feet, take three times A, the hundredth part of this, and the quarter and thousandth of A, and add the results together. For instance, what number of feet are in 867·41 metres?

$$
\begin{aligned}
A &= 867·41 \\
3A &= 2602·23 \\
\tfrac{3}{100}A &= 26·02 \text{ nearly enough} \\
\tfrac{1}{4}A &= 216·85 \quad - \quad - \\
\tfrac{1}{1000}A &= ·87 \quad - \quad - \\
& 2845·97 \text{ nearly as before.}
\end{aligned}
$$

The student may employ himself in endeavouring to simplify other cases. All must depend on his expertness in separating the fractions.

3. Look for such simplifications as may be made by making the multiplier the sum or difference of two numbers or fractions. Thus a degree is $69\frac{1}{2}$ statute miles, or thereabouts. To turn degrees into miles, multiply the degrees by 70 and subtract one-half their number, instead of multiplying by 69 and adding one-half.

4. Multiplication by a number which often comes into use may be more safely done by division. Take the preceding instance of multiplication by 3·2809. Now,

$$3·2809 = \frac{1}{·3048} = \frac{10000}{3048}$$

very nearly. Hence, any one who has often occasion to turn metres into feet, should keep by him the following table of multiples of 3048.

1	3048	4	12192	7	21336
2	6096	5	15240	8	24384
3	9144	6	18288	9	27432

Hence the rule is, multiply by 10,000 and divide by 3048; which latter part, with the assistance of the table, is nothing but inspection and subtraction, as follows:—What is 867·41 metres in feet?

```
867·41 × 10000 = 8674100
  3048 )8674100(2845·83
        6096
        25781
        24384
         13970
         12192
          17780
          15240
           25400
           24384
            10160
```

The advantage of this method is, that with the table it is less liable to error than multiplication, and the figures of the result which are most wanted are first found.

We shall proceed in the next treatise to the use of Logarithms.

SECTION 6. *Meaning of Logarithms. Rules. Arrangement of Tables in common use. Method of taking out Logarithms, and Numbers to Logarithms.*

In the preceding treatise (page 23) we have said that if $a^b = c$, then b, which is the *exponent* of a, is called the *logarithm* of c or of a^b, to the base a. Thus $10^3 = 1000$, whence 3, the exponent of 10, is called the logarithm of 1000 to the base 10. Hence it follows that 3 may be the logarithm of all sorts of numbers, according to the base chosen. Thus:

$2^3 = 8$	$3 =$	log. 8	(base 2.)
$3^3 = 27$	$3 =$	log. 27	(base 3.)
$4^3 = 64$	$3 =$	log. 64	(base 4.)
&c.	&c.	&c.	

But as, in the practice of logarithms, no other base is used, except only 10, we shall, in this treatise, suppose no other base; and logarithms to this base are called *common* logarithms, *tabular* logarithms, or *Brigg's* logarithms. And because we have nothing to do with the method of constructing logarithms, but only with the use to be made of them when they have been found, we shall refer to works on algebra for the former part of the subject, and proceed to the latter, after we have stated in what the difficulty of finding them consists.

According to the common language of algebra, if we raise the mth power of 10, and extract the nth root of the result, we have what is called the $\frac{m}{n}$th power of 10, or

$$\sqrt[n]{10^m} = 10^{\frac{m}{n}}$$

We shall now simply write down some results, not expecting the student to verify them; because, though that might possibly be done by ordinary arithmetic, yet the process would be of very great length and trouble:

$$\sqrt[10]{10^3} \quad \text{or } 10^{\frac{3}{10}} \quad \text{or } 10^{\cdot 3} \quad = 1\cdot 9952623150 \text{ nearly.}$$

$$\sqrt[1000]{10^{301}} \quad \text{or } 10^{\frac{301}{1000}} \quad \text{or } 10^{\cdot 301} \quad = 1\cdot 9998618696 \text{ nearly.}$$

$$\sqrt[100000]{10^{30103}} \quad \text{or } 10^{\frac{30103}{100000}} \quad \text{or } 10^{\cdot 30103} \quad = 2\cdot 0000000200 \text{ nearly.}$$

So that we may get a result as near to 2 as we please; that is, we may find a decimal fraction x, which shall, as nearly as we please, satisfy the equation $10^x = 2$. The answer is $x = \cdot 30103$ nearly. And in the same way we may find an *approximate* logarithm for any other number or fraction. These approximate logarithms are arranged in tables, with certain modifications derived from the following fundamental rules, which are proved* in works on the subject.

1. *The logarithm of a product must be the sum of the logarithms of the factors.* Thus, 6, 8, and 10, *multiplied* together, give 480; the logarithms of 6, 8, and 10, *added* together, give the logarithm of 480. The following instances may be immediately verified from any tables:

$2 \times 5 = 10$

Log. 2 + Log. 5 = Log. 10

Log. 2 =	·3010300
Log. 5 =	·6989700
Log. 10 =	1·0000000

$4 \times 7 = 28$

Log. 4 + Log. 7 = Log. 28

Log. 4 =	·6020600
Log. 7 =	.8450980
Log. 28 =	1·4471580

2. *To find the logarithm of a quotient, subtract the logarithm of the divisor from the logarithm of the dividend.* Thus, 20 divided by 5 gives 4; the logarithm of 20, diminished by the logarithm of 5, is the logarithm of 4:

$100 \div 40 = 2\cdot 5$

Log. 100 =	2·0000000
Log. 40 =	1·6020600
Log. 2·5 =	0·3979400

$64 \div 16 = 4$

Log. 64 =	1·8061800
Log. 16 =	1·2041200
Log. 4 =	0·6020600

3. *The logarithm of a power, root, or combination of power and root, which is*

* The reader must recollect throughout, that we here lay down rules only, not demonstrations.

denoted in algebra by a fractional exponent, is found by multiplying the logarithm of the number given by the exponent in question. The following equations will set this in a clearer light:

$$\text{Log. } aa \text{ or Log. } a^2 = 2 \text{ Log. } a.$$
$$\text{Log. } aaa \text{ or Log. } a^3 = 3 \text{ Log. } a.$$
$$\text{Log. } \sqrt{a} \text{ or Log. } a^{\frac{1}{2}} = \tfrac{1}{2} \text{ Log. } a.$$
$$\text{Log. } \sqrt[3]{a} \text{ or Log. } a^{\frac{1}{3}} = \tfrac{1}{3} \text{ Log. } a.$$
$$\text{Log. } \sqrt[4]{a^3} \text{ or Log. } a^{\frac{3}{4}} = \tfrac{3}{4} \text{ Log. } a.$$
$$\text{Log. } \sqrt[n]{a^m} \text{ or Log. } a^{\frac{m}{n}} = \tfrac{m}{n} \text{ Log. } a.$$

What is the logarithm of the square root of 156?

Log. 156 = 2·1931246

½ Log. 156 = 1·0965623 *Ans.*

What is the logarithm of the fifth root of the fourth power of 2097?

$$\begin{array}{rl} \text{Log. } 2097 = & 3{\cdot}3215984 \\ & \underline{\qquad\qquad 4} \\ 5)\ & \underline{13{\cdot}2863936} \\ & 2{\cdot}6572787 \quad \textit{Ans.} \end{array}$$

4. *The logarithm of* 1 *is* 0; *that of the base* (which is here 10) *is* 1; *that of the square of the base* (here 100) *is* 2; *that of the cube of the base* (here 1000) *is* 3; and so on: or

Log. 1 = 0	Log. 1000 = 3
Log. 10 = 1	Log. 10000 = 4
Log. 100 = 2	Log. 100000 = 5
	&c.

5. *As the number increases the logarithm increases, and the greater the number the greater the logarithm: but the rate at which the logarithm increases is perpetually diminishing as the number increases.* Thus we see that, as the number passes from 10,000 to 100,000 (through *ninety thousand* units) the logarithm passes from 4 to 5, receiving no greater increase than takes place while the number passes from 1 to 10 (through *nine* units only).

6. *In any logarithm* (4·6183 for instance) *the whole number* (4) *is called the* characteristic, *and the remainder* (·6183) *the* decimal part *of the logarithm.*

7. *In any number* (368·414 for instance) *the figures which precede the decimal point* (the 3, the 6, and the 8,) *are called* integers, *and those which follow the point are called* decimals. *And figures, when opposed to ciphers, are called* significant. Thus, in 864000, 4 is the last significant figure; in ·000193, 1 is the first significant figure.

8. A fraction less than unity (·5 for instance) has none but a *negative* logarithm: but that students may use logarithms who have not studied algebra, we affix a meaning to the term negative, for this subject only. The term multiplication is extended in arithmetic to whole numbers and fractions, so that multiplication, in its extended meaning, includes the first meaning of division: thus, to *multiply* by $\frac{1}{10}$ is to *divide* by 10. But from the connection which exists between *multiplication* of numbers and *addition* of logarithms, and also between *division* of numbers and *subtraction* of logarithms, we cannot use the word *multiplication* in an extended sense, which includes *division, and keep rules* (1) *and* (2) *at the same time,** unless we also use the word *addition* in an extended sense, which includes *subtraction.* And this is done as follows: by $\bar{1}$ we mean a unit, with a warning, that in all operations performed upon this $\bar{1}$, we are to subtract where we should have added if the bar had been absent, and to add where we should have subtracted. And with this we say, that 1 being the logarithm of 10, $\bar{1}$ is the logarithm of $\frac{1}{10}$; 2 being the logarithm of 100, $\bar{2}$ is the logarithm of $\frac{1}{100}$: the following are instances of the use of this sign, with the corresponding real operations:—

Multiply	*Divide*
1000 by $\frac{1}{10}$	1000 by 10
Log. 1000 = 3	Log. 1000 = 3
Log. $\frac{1}{10}$ = $\bar{1}$	Log. 10 = 1
Add $\bar{2}$	*Subtract* $\bar{2}$

And 2 is log. 100

$1000 \times \frac{1}{10} = 100, \quad 1000 \div 10 = 100$

* The choice is, between making two rules, and using the words of one rule in a sense which will make that one include both. The latter is the more difficult at first, but the more convenient in the end.

Divide	*Multiply*
1000 by $\frac{1}{100}$	1000 by 100
Log. 1000 = 3	Log. 1000 = 3
Log. $\frac{1}{100}$ = $\bar{2}$	Log. 100 = 2
Subtract $\bar{5}$	*Add* $\bar{5}$

And 5 is log. 100,000.

$1000 \div \frac{1}{100} = 100{,}000 \quad 1000 \times 100 = 100{,}000$

When a subtraction appears which is impossible, invert the subtraction, and place the bar over the result. The following are instances, with the corresponding operations, the first line of each set containing logarithms, and the second the numbers and operations corresponding:—

$$\begin{cases} 3 - 5 = \bar{2} \\ 1000 \div 100{,}000 = \frac{1}{100} \end{cases}$$

$$\begin{cases} 2 - 3 = \bar{1} \\ 100 \div 1000 = \frac{1}{10} \end{cases}$$

$$\begin{cases} 0 - 3 = \bar{3} \\ 1 \div 1000 = \frac{1}{1000} \end{cases}$$

In all other cases, combinations of the preceding rules may be used: and it must be considered that $\bar{1}$ and $\bar{1}$ added make $\bar{2}$, and so on: the following instances will contain all the cases:—

$$\begin{cases} \bar{1} + \bar{1} = \bar{2} \\ \frac{1}{10} \times \frac{1}{10} = \frac{1}{100} \end{cases}$$

$$\begin{cases} \bar{2} + \bar{3} = \bar{5} \\ \frac{1}{100} \times \frac{1}{1000} = \frac{1}{100{,}000} \end{cases}$$

$$\begin{cases} \bar{2} - \bar{3} \text{ or } \bar{2} + 3 \\ \text{or } 3 + \bar{2} \text{ or } 3 - 2 = 1 \\ \frac{1}{100} \div \frac{1}{1000} = 10 \end{cases}$$

$$\begin{cases} \bar{2} - 2 \text{ or } \bar{2} + \bar{2} = \bar{4} \\ \frac{1}{100} \div 100 = \frac{1}{10{,}000} \end{cases}$$

$$\begin{cases} 1 + \bar{2} + \bar{3} = 1 - 2 - 3 = \bar{4} \\ 10 \times \frac{1}{100} \times \frac{1}{1000} = \frac{1}{10{,}000} \end{cases}$$

$$\begin{cases} \bar{4} - \bar{4} = 0 \\ \frac{1}{10{,}000} \div \frac{1}{10{,}000} = 1 \end{cases}$$

$$\begin{cases} 0 - \bar{3} = 3 \\ 1 \div \frac{1}{1000} = 1000 \end{cases}$$

What are the results of the following, and what are the corresponding operations in the numbers to which the terms are logarithms?

$$4 + \bar{3} - \bar{2} - \bar{1}$$
$$\bar{2} - \bar{3} + 5 - 1$$

What is the logarithm of ·5?

Log. 5 = ·69897
Log. 10 = 1·00000

Subtract. Impossible, therefore invert the subtraction, and place a bar over the whole; as follows,

$\overline{·30103}$

What is 20 × ·5?

Log. 20 = 1·30103
Log. ·5 = $\overline{·30103}$
Add 1·00000 *Ans.* 10.

What is 100 ÷ ·5?

Log. 100 = 2·00000
Log. ·5 = $\overline{·30103}$
Subtract 2·30103 which is log. 200 } *Ans.* 200.

But the necessity of using decimal places with a negative sign, can always be avoided, and the characteristic only made negative, as follows: for

log ·5 or log. $\frac{5}{10}$ or log. 5 − log. 10 or

·69897 − 1 write ·69897 + $\bar{1}$

or $\bar{1}$·69897;

in which the first figure only is to be used as a negative quantity. We repeat the preceding instances.

What is 20 × ·5?

Log. 20 = 1·30103
Log. ·5 = $\bar{1}$·69897
Add 1·*0000 *Ans.* 10.

Here the 1, which is carried after adding 1, 6, and 3, (where we have placed an asterisk instead of a cipher to mark the place) instead of increasing the $\bar{1}$, destroys it.

What is 100 ÷ ·5?

Log. 100 = 2·00000
Log. ·5 = $\bar{1}$·69897
Subtract 2·30103 as before.

To make any logarithm which is entirely negative, negative in the characteristic only, make that characteristic greater by 1, and subtract the decimal part from 1.

What is $\overline{40\cdot41372}$?

$$1 - \cdot41372 = \cdot58628$$

Answer $\overline{41}\cdot58628$

$\overline{0\cdot3} = \bar{1}\cdot7 \quad \overline{1\cdot21} = \bar{2}\cdot79 \quad \overline{0\cdot1} = \bar{1}\cdot9$

$\overline{1\cdot6141982} = \bar{2}\cdot3858018$

In the practice of logarithms, it will be necessary to appear to subtract the greater from the less, which is done by subtracting in the usual way till we come to the last place, inverting the subtraction which there occurs, and placing the negative sign over the result.

From	$1\cdot6936$	$20\cdot414$
Take	$3\cdot0177$	$29\cdot666$
Ans.	$\bar{2}\cdot6759$	$\overline{10}\cdot748$

$6\cdot4$	$12\cdot8$	$6\cdot6$	$0\cdot00$
$7\cdot9$	$14\cdot4$	$6\cdot7$	$4\cdot28$
$\bar{2}\cdot5$	$\bar{2}\cdot4$	$\bar{1}\cdot9$	$\bar{5}\cdot72$

$0\cdot0000$	$0\cdot00000$
$2\cdot1896$	$0\cdot12345$
$\bar{3}\cdot8104$	$\bar{1}\cdot87655$

In the following examples the negative characteristic is treated in the manner already described, namely, as to be subtracted in addition, and added in subtraction. The figure *carried* is always to be added, and, therefore, makes a negative characteristic less: thus 2 carried to $\bar{5}$ makes it $\bar{3}$.

Add.	Add.	Add.
$1\cdot48$	$9\cdot83$	$\bar{2}\cdot18$
$\bar{2}\cdot56$	$\bar{1}\cdot47$	$\bar{6}\cdot00$
$\bar{3}\cdot41$	$\bar{4}\cdot66$	$9\cdot14$
$\bar{3}\cdot45$	$5\cdot96$	$1\cdot32$

From	$\bar{1}\cdot616$	$8\cdot413$	$\bar{4}\cdot17$
Take	$2\cdot929$	$\bar{1}\cdot097$	$\bar{5}\cdot28$
	$\bar{4}\cdot687$	$9\cdot316$	$0\cdot89$

From	$0\cdot000$	$0\cdot00$	$\bar{2}\cdot66$
Take	$\bar{2}\cdot147$	$\bar{1}\cdot42$	$\bar{3}\cdot44$
	$1\cdot853$	$0\cdot58$	$1\cdot22$

As the difficulty lies entirely in the line of characteristics, we give some examples of that line only, the figure carried from the preceding line being written in Roman figures at the top.

	I	I	II	IV	III
Add	1	$\bar{2}$	$\bar{3}$	2	$\bar{1}$
	$\bar{3}$	10	$\bar{8}$	6	$\bar{2}$
	$\bar{7}$	$\bar{5}$	7	$\bar{9}$	$\bar{3}$
	4	$\bar{8}$	1	3	4
	$\bar{4}$	$\bar{4}$	$\bar{1}$	6	1

	O	I	I	I
From	2	0	$\bar{3}$	$\bar{1}$
Take	$\bar{3}$	$\bar{2}$	$\bar{4}$	0
	5	1	0	$\bar{2}$

To multiply a logarithm with a negative characteristic by a whole number, proceed in all respects as in common multiplication, except only in *subtracting*, instead of *adding* the figures which are to be carried, so soon as the characteristic comes to be multiplied.

$\bar{1}\cdot61$	$\bar{2}\cdot55$	$\bar{4}\cdot1$	$\bar{4}\cdot6$
4	3	8	2
$\bar{2}\cdot44$	$\bar{5}\cdot65$	$\overline{32}\cdot8$	$\bar{7}\cdot2$

When the multiplier exceeds 12, and the process is not performed in one line, the better way is to omit the characteristic altogether, at first, and subtract the product arising from it afterwards, as in the following multiplication of $\bar{2}\cdot136$ by 15.

$$\begin{array}{r} \cdot136 \\ 15 \\ \hline 680 \\ 136 \\ \hline 2\cdot040 \\ 4 \times 15 = 60 \\ \hline \text{Subtract } \overline{58}\cdot040 \end{array}$$

To divide a logarithm with a negative characteristic by a whole number, begin by *increasing* the characteristic until it is divisible by the whole number, make the quotient a negative characteristic for the result, and use the augment which was found necessary, as if it had been a remainder. Thus, to divide $\bar{1}\cdot4$ by 2, increase the first 1, and make it 2 (necessary augment, 1) and 2 being contained in 2 once, $\bar{1}$ is the characteristic of the quotient. Then, taking the augment 1, prefix it to the 4, giving 14, which contains 2 seven times. Therefore $\bar{1}\cdot7$ is the quotient.

$2)\bar{3}\cdot010$	$4)\bar{1}\cdot11$	$10)\bar{6}\cdot52$
$\bar{2}\cdot505$	$\bar{1}\cdot78$	$\bar{1}\cdot45$
$7)\bar{8}\cdot10$	$3)\bar{9}\cdot12$	$4)\bar{4}\cdot13$
$\bar{2}\cdot87$	$\bar{3}\cdot04$	$\bar{1}\cdot03$

As divisions by higher numbers rarely occur, we shall only give one instance, that the student may exercise himself in reconciling the process as it here appears, with the rule given. The asterisks mark where the process differs from common division.

$$13)\overline{165}\cdot61(\overline{13}\cdot35\ldots$$
13
35
* 39
* 46
39
71

We can now give a logarithm, by help of the tables, to any number or fraction, and can, by the above conventions, make the rules marked (1) (2) and (3) include all cases of logarithmic operations, by help of the following rules.

(*a*) An alteration in the position of the decimal point, alters only the characteristic, and not the decimal part of the logarithm, if the significant figures remain the same: thus all the following numbers and fractions have the same decimal part in their logarithms, with different characteristics.

·000256	2·56	25600
·00256	25·6	256000
·0256	256	2560000
·256	2560	25600000

(*b*) In every whole number, let a decimal point be understood after the unit's place. Thus 58 is 58·, or 58·0, or 58·00, &c.

(*c*) When there are figures before the decimal point, let the characteristic be *one* less than the number of places of those figures. Thus the logarithm of 26861·5 has the characteristic 4; so also has that of 26861 (or 26861·).

The decimal places of the logarithm of 21925 are ·3409396; hence

Log. 21925000	=	7·3409396
Log. 2192500	=	6·3409396
Log. 219250	=	5·3409396
Log. 21925	=	4·3409396
Log. 2192·5	=	3·3409396
Log. 219·25	=	2·3409396
Log. 21·925	=	1·3409396
Log. 2·1925	=	0·3409396

(*d*) When there are no figures (or only ciphers) before the decimal point, let the characteristic be negative, and let it tell in what place following the decimal point, the first significant figure is found. Thus, in ·0000136, the first significant figure being in the fifth place following the decimal point, the characteristic of the logarithm is $\bar{5}$. The decimal places in the logarithm of 324 being ·510545, we have

Log. ·324	=	$\bar{1}\cdot510545$
Log. ·0324	=	$\bar{2}\cdot510545$
Log. ·00324	=	$\bar{3}\cdot510545$
Log. ·000324	=	$\bar{4}\cdot510545$
Log. ·0000324	=	$\bar{5}\cdot510545$

The decimal places in the logarithm of 1 being 000, &c., we have the following logarithms, which consist entirely of characteristics:

Log. 1000	=	3·0000...
Log. 100	=	2·0000..
Log. 10	=	1·0000
Log. 1	=	0·0000
Log. ·1	=	$\bar{1}\cdot0000$
Log. .01	=	$\bar{2}\cdot0000$
Log. ·001	=	$\bar{3}\cdot0000$ &c.

and these are the only numbers to which logarithms can be exactly found; the decimal places of all others being approximations only.

Tables of logarithms (generally) contain the decimal part of the logarithm, which is evidently all that is necessary, as the characteristic can be found by the preceding rule. Being approximations, they are more or less correct according to the greater or smaller number of places which they give. Modern tables never have fewer than *four*, or more than *seven* decimal places. The following is the rule by which the power of a table of logarithms is to be judged.

The number of places of figures which may be obtained in a result derived from any table of logarithms, is the same as the number of decimals to which the logarithms are carried. But towards the end of the table, the last place thus obtained cannot always be depended upon within a unit.

We shall proceed to the description of the arrangements of several tables, such as are most likely to fall in the reader's way.

I. The tables which run to seven places of decimals are all arranged in one form, of which the following is a specimen.

No.	0	1	2	3	4	5	6	7	8	9
4550	6580114	0209	0305	0400	0496	0591	0687	0782	0877	0973
1	1068	1164	1259	1355	1450	1545	1641	1736	1832	1927
2	2023	2118	2213	2309	2404	2500	2595	2690	2786	2881
--	---	--	--	--	--	--	--	--	--	--
9	8696	8791	8886	8982	9077	9172	9267	9363	9458	9553
4560	9648	9744	9839	9934	0029	0125	0220	0315	0410	0506
1	6590601	0696	0791	0886	0982	1077	1172	1267	1362	1458

Diff.	
95	
1	10
2	19
3	29
4	38
5	48
6	57
7	67
8	76
9	86

The first column contains the first four places of the number, and over the head of the page is the fifth place of the number. The first three places of the logarithm (which throughout the specimen are either 658 or 659,) are not repeated with every logarithm, but only inserted at (or as near as may be to) the place where a change of the third figure takes place. But the best way to explain this table will be to destroy arrangement and abbreviation, and begin to write it down at full length. The student must account for every figure of the following out of the specimen. The characteristic need not be inserted, as what we here take out is merely the decimal part of the logarithm.

Log. 45500	·6580114	
Log. 45501	·6580209	
Log. 45502	·6580305	
Log. 45503	·6580400	
- - - -	- - - -	
- - - -	- - - -	
Log. 45509	·6580973	
Log. 45510	·6581068	
Log. 45511	·6581164	
- - - -	- - - -	
- - - -	- - - -	
Log. 45602	·6589839	
Log. 45603	·6589934	
Log. 45604	·6590029	{ change of 3rd fig.
Log. 45605	·6590125	

It would break the page to show that 658 becomes 659 in the middle of it; and various methods are used to remind the computer that the change has taken place. In different works, the line 4560, *after the change*, is varied thus:—

	$_{o}$029	$_{o}$125	&c.
or	$\bar{0}$029	$\bar{0}$125	&c.
or	0029	0125	&c.

all serving to remind that the first three places must be looked for *immediately below*, instead of more or less above, the line of the last four.

The column marked Diff. (for *difference*) shows how to find the logarithm of a number of six or seven places of figures. For instance, what is the logarithm of 4551132? Take out the decimal part of log. 45511; to this *add* what comes opposite to the *sixth* place in the column Diff.; (the sixth place is 3, and 29 is opposite to 3 in column Diff.); *add* the nearest number of tens in the number opposite to the seventh place (the seventh place is 2; opposite to 2 in col. Diff. is 19, nearest number of tens, 2 tens) and the result is the decimal part of the logarithm required: thus—

Log. 45511..	·6581164
3.	29
2	2
Log. 4551132	·6581195

Log. 455·2008 = 2·6582031

Log. 45603·97 = 4·6590027*

Log. ·4560444 = $\bar{1}$·6590071

In the earlier part of the tables, where columns of differences occur more thickly, several for the same line of logarithms, it is almost immaterial which is used; but for safety, take that column of differences which is headed by the difference between the logarithm taken out and the next following it.

To find the number corresponding to a given logarithm, look in the table for the decimal places, which are nearest below those of the given logarithm; take out this logarithm, and the *five* places of the number, subtract the logarithm taken out from the given logarithm. Look in the second column of the

* The change in the third figure takes place in the process.

differences for the number next below the result of subtraction just found, opposite to it will be found the *sixth* place of the number. Subtract the number used in the second column of the differences from the result of subtraction above-mentioned, annex a cipher, and repeat the process with the column of differences, taking the nearest this time, whether above or below; the result is the *seventh* place of the number. For instance, what is the number to the logarithm 1·6582554?

	·6582554
Nearest log. below No. 45525	6582500
	54
Opposite to 5 is	48
Subt. and annex 0	60

Opp. to 6 is 57, the nearest.

The first five places are 45525, the sixth is 5, and the seventh is 6, so that 4552556 is the number required; and because 1 is the characteristic, there must be two places before the decimal point; that is, 45·52556 is the answer.

The following useful numbers are mostly taken from the list at the end of Mr. Babbage's logarithms. They will serve as exercises, either in taking the logarithm to a number, or the converse.

	No.	Log.
Circumference of circle (diam. being 1)	3·141593	·4971499
Area of circle (do. do.)	·7853982	$\bar{1}$·8950899
Content of sphere (do. do.)	·5235988	$\bar{1}$·7189986
No. of seconds in 360°	1296000	6·1126050
No. of arcs of 1″ in the radius	206264·8	5·3144251
No. of arcs of 1′ in the radius	3437·747	3·5362739
No. of arcs of 1° in the radius	57·29578	1·7581226
Base of Naperian logarithms	2·718282	·4342945
Modulus of common logarithms	·4342945	$\bar{1}$·6377843
Metres in a toise	1·94904	·2898200
Yards in a toise	2·131531	·3286916
Feet in a toise	6·394593	·8058129
Yards in a metre	1·093633	·0388716
Feet in a metre	3·280899	·5159929
Inches in a metre	39·37079	1·5951741
Feet in a French foot	1·065765	·0276616
Acres in an are (French)	·02471143	$\bar{2}$·3928979
Lbs. troy in a gramme	·00268098	$\bar{3}$·4282928
Lbs. avoir. in a gramme	·00220606	$\bar{3}$·3436173
Cwts. in a kilogramme	·0196969	$\bar{2}$·2943993
Gallons in a litre	·2200969	$\bar{1}$·3426139
Seconds in 24 hours	86400	4·9365137
Diurnal acceleration of stars in mean solar seconds	235·9093	2·3727451
Common tropical year in mean solar days	365·2422	2·5625810
Grains in a cubic inch of water (barom. 30 inch, therm. 62, Fahr.)	252·458	2·4021891
Inches in the pendulum, which vibrates seconds in a vacuum in the latitude of London	39·1393	1·5926130

II. The second set of tables, which it will be worth while to describe, has five places of figures in the logarithms, and four places in the number, with a difference to find a fifth. We have not described logarithms of six places, partly because they are arranged much in the manner of those which have

seven places, and partly because tables of six places are of comparatively little use. For most practical purposes out of astronomy, and for very many of the details of calculation connected with the latter science, five places are amply sufficient: and where five are not sufficient, seven are much more frequently wanted than six; besides which, the arrangement of most tables of six places which we have seen is so defective, that those of seven are, in our opinion, more easily used.

The best tables of five places (though with a very singular and awkward defect, presently to be noticed), are those of Lalande*. The following is a specimen:—

Nomb.	0. 21' 30" Logarit.	D.
1290	3·11059	34
1291	3·11093	33
1292	3·11126	34
1293	3·11160	33
1294	3·11193	34
1295	3·11227	34
1296	3·11261	34
1297	3·11294	33
1298	3·11327	

The defect alluded to is the characteristic, which is inserted as if the logarithms of whole numbers of four places were always required, to the exclusion of all others. Thus, though the characteristic above given is correct for the logarithm of 1292, it is not so for those of 129·2, 12·92, &c. The best way for the student who uses this work, is never to think of the characteristic as anything but an addition to the boundary line; that is, to look upon the numbers as separated from the decimal part of their logarithms by a fanciful boundary, like

|3
|3 instead of simply |
|3

To find the logarithm of any four places, simply look in the table and choose the right characteristic. Thus:

Log. ·1295 = $\bar{1}$·11227

To find the logarithm of ·12956, take the difference which comes next under 11227, namely 34; multiply it by the new figure 6, but instead of writing down the first place, carry the ***nearest number of tens*** to the next place. Say, 6 times 4 is 24, carry 2; 6 times 3 is 18 and 2 is 20. So that

Log. ·1295·	=	$\bar{1}$·11227
6		20
Log. ·12956	=	$\bar{1}$·11247
Log. ·1295·	=	$\bar{1}$·11227
7		24
Log. ·12957	=	$\bar{1}$·11251

To find the number to a given logarithm, take out of the table the decimal part next below the given decimal part, and the four places opposite to it. Annex a cipher to the difference, and divide by the number in the column of differences, taking the nearest quotient of one figure. That one figure is the fifth figure of the number. For instance, what is the number to the logarithm $\bar{2}$·11178?

	$\bar{2}$·11178
1293	·11160
	33) 180 (5
Ans.	·012935

Given Log.	8·11153
1292	11126
	34) 270 (8
Ans.	129280000 nearly.

We cannot fill up the remaining places out of this table, and must place ciphers instead. The real number to the log. 8·11153 is

129279600·09 *very* nearly.

The numbers given in page 39 may be made exercises; but the nearest five significant figures of the number must be taken, and the nearest five decimal figures of the logarithm will be found.

Example. What is Log. 3·1416

Log. 3·141·	0·49707
6	8
Log. 3·1416	0·49715 *Ans.*

* The title-page of the best edition is as follows:—" *Tables des Logarithmes pour les nombres et pour les sinus. Avec les explications, &c. &c. &c. Edition Stéréotype gravée fondue et imprimée, par* Firmin Didot. A Paris, &c. 1805. (*tirage de* 1831)." The last four words should be particularly looked at.

III. There are logarithms of four places on the table given in the Treatise on Arithmetic and Algebra, which are sufficient for many purposes. These tables are arranged somewhat after the manner of those of seven places, with the exception of the column of differences being placed horizontally, with a common heading. A few examples of the method of taking out logarithms will suffice:

$$\begin{array}{lll} \text{Log.} & 16\cdot8 \,\cdot\cdot & = 1\cdot2253 \\ & \quad\;\; 4 & \quad\;\; \underline{\;\;11} \\ \text{Log.} & 16\cdot84 & \quad 1\cdot2264 \end{array}$$

$$\begin{array}{lll} \text{Log.} & 1\cdot69 \,\cdot\cdot & = 0\cdot2279 \\ & \quad\;\; 1 & \quad\;\; 3 \\ \hline \text{Log.} & 1\cdot691 & = 0\cdot2282 \end{array}$$

The number to a logarithm might be found by the reverse process. Thus:

$$\begin{array}{ll} \text{Given Log.} & \bar{1}\cdot2687 \\ \quad 185 \,\cdot\cdot & \quad \underline{2672} \\ \qquad 6 \text{ or } 7 & \quad\;\; 15 \\ \textit{Ans.} \;\; \cdot1856 \text{ or } \cdot1857 & \end{array}$$

But these tables are accompanied by an anti-logarithmic table, in which the numbers and logarithms change places; so that a number is found from its logarithm by the same process as that which finds the logarithm from the number. For instance, in the preceding example,

$$\begin{array}{ll} 268 \,\cdot\cdot & \quad 1854 \\ \underline{\quad 7} & \quad \underline{\quad 3} \\ \bar{1}\cdot2687 \text{ is Log. of} & \cdot1857 \end{array}$$

The table of anti-logarithms is more trustworthy than the inverse process with the table of logarithms.

IV. We subjoin a table of logarithms and anti-logarithms to three places only; partly because there is considerable power in such a table, and partly because it will be a guide to the beginner in consulting larger tables, as he may thus (while new to the subject) find out to what part of the larger table to turn, in either of the operations of taking out logarithms or numbers.

In the following table the differences between successive logarithms are placed in smaller figures in the intermediate space. As these tables are only an abbreviation of those already described, the first gives the log. of two places to three places; the second gives the proportional parts of the differences; the third gives the number of a log. of two places to three places. When the difference is 12 or under, the proportional part can be found at once by the process described in speaking of tables of five places:

$$\begin{array}{lll} & \text{Given No. } 16\cdot3 & \\ \text{Log. } 16 \,\cdot\cdot & = \;\; 1\cdot204 & \text{D } 26 \\ \qquad 3 & \qquad \underline{\;\;8} & \\ \text{Log. } 16\cdot3 & = \;\; 1\cdot212 & \end{array}$$

$$\begin{array}{lll} & \text{Given No. } 109 & \\ \text{Log. } 10 \,\cdot\cdot & = 1\cdot000 & \text{D } 41 \\ \qquad 9 & \qquad \underline{37} & \\ & \quad 1\cdot037 & \end{array}$$

$$\begin{array}{lll} & \text{Given Log. } \bar{2}\cdot308 & \\ \;\; 30 \,\cdot\cdot & \quad 200 & \text{D } 4 \\ \underline{\qquad 8} & \quad \underline{\quad 3} & \\ \bar{2}\cdot308 & \quad \cdot0203 & \end{array}$$

$$\begin{array}{lll} & \text{Given Log. } 1\cdot496 & \\ \;\; 49 \,\cdot\cdot & \quad 309 & \text{D } 7 \\ \underline{\qquad 6} & \quad \underline{\quad 4} & \\ 1\cdot496 & \quad 31\cdot3 & \end{array}$$

V. Finally, we recommend the student to commit to memory the following table of logarithms to two places:

No.	Log.	No.	Log.	No.	Log.
1	00	4	60	7	85
2	30	5	70	8	90
3	48	6	78	9	95

LOGARITHMS.

1	2	3	4	5	6	7	8	9
0 000	0 301	0 477	0 602	0 699	0 778	0 845	0 903	0 954
41	21	14	11	9	7	6	5	5
1 041	1 322	1 491	1 613	1 708	1 785	1 851	1 908	1 959
38	20	14	10	8	7	6	6	5
2 079	2 342	2 505	2 623	2 716	2 792	2 857	2 914	2 964
35	20	14	10	8	7	6	5	4
3 114	3 362	3 519	3 633	3 724	3 799	3 863	3 919	3 968
32	18	12	10	8	7	6	5	5
4 146	4 380	4 531	4 643	4 732	4 806	4 869	4 924	4 973
30	18	13	10	8	7	6	5	5
5 176	5 398	5 544	5 653	5 740	5 813	5 875	5 929	5 978
28	17	12	10	8	7	6	5	4
6 204	6 415	6 556	6 663	6 748	6 820	6 881	6 934	6 982
26	16	12	9	8	6	5	6	5
7 230	7 431	7 568	7 672	7 756	7 826	7 886	7 940	7 987
25	16	12	9	7	7	6	4	4
8 255	8 447	8 580	8 681	8 763	8 833	8 892	8 944	8 991
24	15	11	9	8	6	6	5	5
9 279	9 462	9 591	9 690	9 771	9 839	9 898	9 949	9 996
22	15	11	9	7	6	5	5	4

PROPORTIONAL PARTS.

	41	38	35	32	30	28	26	25	24	22	21	20	19	18	17	16	15	14	13	
1	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1
2	8	8	7	6	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3	2
3	12	11	11	10	9	8	8	8	7	7	6	6	6	5	5	5	5	4	4	3
4	16	15	14	13	12	11	10	10	10	9	8	8	8	7	7	6	6	6	5	4
5	21	19	18	16	15	14	13	13	12	11	11	10	10	9	9	8	8	7	7	5
6	25	23	21	19	18	17	16	15	14	13	13	12	11	11	10	10	9	8	8	6
7	29	27	25	22	21	20	18	18	17	15	15	14	13	13	12	11	11	10	9	7
8	33	30	28	26	24	22	21	20	19	18	17	16	15	14	14	13	12	11	10	8
9	37	34	32	29	27	25	23	23	22	20	19	18	17	16	15	14	14	13	12	9
	41	38	35	32	30	28	26	25	24	22	21	20	19	18	17	16	15	14	13	

ANTI-LOGARITHMS.

·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
0 100	0 126	0 158	0 200	0 251	0 316	0 398	0 501	0 631	0 794
2	3	4	4	6	8	9	12	15	19
1 102	1 129	1 162	1 204	1 257	1 324	1 407	1 513	1 646	1 813
3	3	4	5	6	7	10	12	15	19
2 105	2 132	2 166	2 209	2 263	2 331	2 417	2 525	2 661	2 832
2	3	4	5	6	8	10	12	15	19
3 107	3 135	3 170	3 214	3 269	3 339	3 427	3 537	3 676	3 851
3	3	4	5	6	8	10	13	16	20
4 110	4 138	4 174	4 219	4 275	4 347	4 437	4 550	4 692	4 871
2	3	4	5	7	8	10	12	16	20
5 112	5 141	5 178	5 224	5 282	5 355	5 447	5 562	5 708	5 891
3	4	4	5	6	8	10	13	16	21
6 115	6 145	6 182	6 229	6 288	6 363	6 457	6 575	6 724	6 912
2	3	4	5	7	9	11	14	17	21
7 117	7 148	7 186	7 234	7 295	7 372	7 468	7 589	7 741	7 933
3	3	5	6	7	8	11	14	18	22
8 120	8 151	8 191	8 240	8 302	8 380	8 479	8 603	8 759	8 955
3	4	4	5	7	9	11	14	17	22
9 123	9 155	9 195	9 245	9 309	9 389	9 490	9 617	9 776	9 977
3	3	5	6	7	9	11	14	18	23

Section 7.—*Application of Logarithms worked at Length.*

In most of the following examples, we shall use the tables of seven places; those who employ smaller tables can produce the same result, as far as their tables go. The following is an instance of the way in which the same question must be treated, according to different tables:

What is ·1234567 × 26813·92?

1. *With tables of seven places:*

Log. ·12345..	=	$\bar{1}$·0914911
6		212
7		25
Log. ·1234567	=	$\bar{1}$·0915148
Log. 26813..	=	4·4283454
9		146
2		3
Log. 26813·92	=	4·4283603
		$\bar{1}$·0915148
	Add	3·5198751
33103..		5198674
		77
5		66
8		110

Ans. 3310·358.

2. *With five places:* or what is ·12346 × 26814?

Log. ·1234.	=	$\bar{1}$·09132
6		21
Log. ·12346	=	$\bar{1}$·09153
Log. 2681.	=	4·42830
4		6
Log. 26814	=	4·42836
		$\bar{1}$·09153
		3·51989
3310·	·	51983
		13) 60 (5

Ans. 3310·5.

3. *With four places:* ·1235 × 26810:

Log. ·123	$\bar{1}$·0899
5	17
Log. ·1235	$\bar{1}$·0916

Log. 26800	4·4281
1	2
Log. 26810	4·4283
	$\bar{1}$·0916
	3·5199

·519·	3304	
9	7	
·5199	3311	*Ans.*

4. *With three places:* ·123 × 26800:

Log. ·12·	$\bar{1}$·079 D 35
3	11
Log. ·123	1·090

Log. 26000	4·415
8	13
Log. 26800	4·428
	$\bar{1}$·090
	3.518

51··	324	
8	6	
3·518	3300	*Ans.*

In future we shall give the logarithms to seven places, but without going through the detail of using the table of differences to find the sixth and seventh places, either of a number to a logarithm or of a logarithm to a number, except in a few particular cases.

Question 1. Find $\frac{1}{1084\cdot9}$

Log.	1	=	0·0000000
Log.	1084·9	=	3·0353897
·000921744			$\bar{4}$·9646103

Question 2. Find $\sqrt{\cdot1}$ and $\sqrt[5]{97.65625}$.

Log. ·1	2) $\bar{1}$·0000000
	$\bar{1}$·5000000
31622	4999893
	107
7	96
8	110

or $\sqrt{\cdot1}$ = ·3162278

Log. 97·65625	5) 1·9897000
2·5 *Ans.*	·3979400

In the last result, the exact coincidence (to seven places) of the answer with 2·5 may induce a supposition that 97·65625 is the exact fifth power of 2·5, which is really the case; but nothing can be inferred from the tables, except that the fifth root of 97·65625 lies between 2·4999995 and 2·5000005. It might be

2·499999576........
or 2·500000214......

and the answer of the tables would still be 2·5.

Question 3. $\dfrac{\sqrt[5]{(32\cdot92416 \times 10\cdot27251)^6}}{1084\cdot9}$

Log. 32·92416	1·5175147
Log. 10·27251	1·0116766
	2·5291913
	6
	5) 15·1751478
	3·0350296
Log. 1084·9	3·0353897
Ans. ·9991712	$\bar{1}$·9996399

Question 4. Find a fourth proportional to 1234, 2345, and 3456; or find 2345 × 3456 ÷ 1234:

Log. 2345	3·3701428
Log. 3456	3·5385737
	6·9087165
Log. 1234	3·0913152
Ans. 6567·518	3·8174013

We shall hereafter give a more expeditious way of solving this question.

Question 5. What is the thousandth power of 2?

Log. 2 = ·3010300
1000
Multiply 301·0300000

Now, this is the logarithm, as nearly as our tables will tell, of

107151900000.....

the number of ciphers being 294; that is, apparently, the thousandth power of 2 is a number of 302 places of figures, the first seven of which are 1071519. But it must be recollected that a thousand times ·3010300 is 301·0300, and that we only annex four more ciphers because we do not know with what figures to fill up the vacant places. We cannot, therefore, depend upon more than four places of the result, and should say that 2^{1000} is a number of 302 figures, of which the first four are 1071. If we would have the first seven places correct, we must go to a table of ten places at least. This gives

Log. 2 =	·3010,299957
Log. 2^{1000} =	301·0299957
1071508	0299922
	35

so that the first seven figures are 1071508.

Let us here observe, that by mere inspection of a logarithm, we answer questions which would take years of calculation. For instance, from the above logarithm of 2, we see that the tenth power of 2 has 4 figures (3+1); the hundredth power has 31 figures (30+1); the millionth power has 301,030 figures, and so on. Hence the simplest method *in theory*, of calculating a logarithm to seven places, is by the following formula:—

$$\text{Log. } x = \frac{\left\{\begin{matrix}\text{No. of fig. in ten-}\\ \text{millionth power of } x\end{matrix}\right\} - 1}{\text{ten million}}$$

but this, of course, would be practically impossible to use.

Question 6. What whole number is that which has 256 places of figures in its 70th power. The logarithm of that 70th power must be between

255·000.... and 255·999....

that is, the logarithm of the number itself must lie between

$$\frac{255\cdot000....}{70} \text{ and } \frac{255\cdot999....}{70}$$

or 3·6428571 and 3·6571428

Answer: All whole numbers between 4394 and 4540, both inclusive.

Question 7. What is the value of

$$\sqrt{\frac{5\sqrt[3]{138}}{\sqrt[5]{\cdot01}}}$$

Log.	138	3)2·1398791
		·7132930
Log.	5	·6989700
Log. $5\sqrt[3]{138}$		1·4122630
Log.	·01	5)$\bar{2}$·0000000
		$\bar{1}$·6000000
		1·4122630
		2)1·8122630
Ans. 8·056224		·9061315

Operations with logarithms may be divided into—1. Those in which a number need never be found to a logarithm

until the end of the process; 2. Those in which numbers must be found to logarithms as a subordinate part of the process. All the instances hitherto given, and all which involve only multiplication, division, raising of powers, and extraction of roots, fall under the first case; while all which contain addition or subtraction fall under the second. For instance, to find

$$\sqrt{\sqrt[3]{4}+\sqrt[3]{5}}$$

we must first find $\sqrt[3]{5}$, then $\sqrt[3]{4}$, then make the addition indicated, and find the square root of the sum.

Log. 4.	Log. 5.
3)·6020600	3)·6989700
·2006867	·2329900
1·587401	1·709976
1·709976	
3·297377...	2)·5151686
Ans. 1·809607	·2575843

Question 8.

$$\sqrt[10]{\left(\cdot01\sqrt[10]{(\cdot01\sqrt[10]{\cdot01})}\right)}\,?$$

Log. ·01	10)$\bar{2}$·0000000
	$\bar{1}$·8000000
	$\bar{2}$·0000000
	10)$\bar{3}$·8000000
	$\bar{1}$·7800000
	$\bar{2}$·0000000
	10)$\bar{3}$·7800000
Ans. ·5997911	$\bar{1}$·7780000

Repeat the process until the tenth root has been extracted seven times, and show that the result will then be very nearly equal to the ninth root of ·01.

Question 9. Supposing the earth to be 7916, and the moon 2160 miles in diameter, how many times does the bulk of the former contain the latter? [Spheres are to one another as the cubes of their diameters; that is, if one diameter contain another x times, the sphere on the first contains that on the second $x\,x\,x$ times.] The question is, what is $(7916 \div 2160)^3$?

Log. 7916	3·8985058
Log. 2160	3·3344538
	0·5640520
	3
Ans. 49·22163	$\bar{1}$·6921560

Answer—About 49¼ times.

Question 10. What is the number of cubic miles in the earth and moon, the diameters being as in the last question? [To find the cubic miles in a sphere, multiply the cube of the diameter by the cubical content of a sphere of one mile in diameter, page 39.]

Log. 7916		Log. 2160
3·8985058		3·3344538
3		3
11·6955174		10·0033614
$\bar{1}$·7189986		$\bar{1}$·7189986
11·4145160		9·7223600
259726400000	{*Answers nearly*}	5276671000

Question 11. To how much will 15*l.* 7*s.* 3½*d.* amount in fifty years, at 3 per cent. compound interest; or what is

$$\pounds 15 \quad 7 \quad 3\tfrac{1}{2} \times (1\cdot03)^{50}$$

The sum mentioned is £15·364.

Log.	1·03	·0128372
		50
		·0418600
Log.	15·364	1·1865043
	67·354	1·8283643

Answer—£67 7 1—very nearly.

Question 12. How many feet are there in 867·41 metres [page 39, log. No. of feet in metre = ·5159929.]

Log.	867·41	2·9382244
Log.	(feet in metre)	·5159929
		3·4542173

Answer—2845·885

Question 13. Taking it for granted, as is proved in a higher branch of mathematics, that when x is a large number, the product

$$1 \times 2 \times 3 \times 4 \ldots\ldots \times (x-1) \times x$$

is very nearly equal to

$$\sqrt{6\cdot2831854 \times x} \times \left\{\frac{x}{2\cdot7182818}\right\}^{x}$$

what is (nearly) the product of the first thousand numbers? is it greater or less than would be obtained by substituting the average for every one of the numbers, and how many times does the greater contain the less? Also how many figures are in each product?

[The average of 1, 2, 3......1000 is 500·5, and the products to be compared are therefore

$1 \times 2 \times 3 \times \ldots\ldots \times 1000$ & $(500\cdot5)^{1000}$]

Log. 6·283185	·7981799
Log. 1000	3·0000000
	2)3·7981799
	1·8990899*

Log. 1000	3·0000000
Log. 2·718282	·4342945
	2·5657055
	1000
	2565·7055
	1·8991*
	2567·6046

Hence the product of the first thousand numbers contains 2568 figures, of which the first four are 4023; and the best approximation we can make is—

4023000....(2564 ciphers).

Log. 500·5	=	2·6994041
		1000
		2699·4041
		2567·6046
		131·7995

It appears that the second product has 2700 figures, the first four of which are 2535: it is incomparably the greater of the two, and contains the first a number of times, having 132 figures, the first four of which are 6302. As some further examples of the preceding formula, let $[x]$ signify the product of all the numbers up to x inclusive; then—

Log. [1010]	=	2597·6284
Log. [1020]	=	2627·6952
Log. [1030]	=	2657·8046
Log. [1040]	=	2687·9561
Log. [1050]	=	2718·1493
Log. [1100]	=	2869·7278
Log. [1150]	=	3022·2933
Log. [1200]	=	3175·8028

Question 14. What whole power of 2 is nearer than any other to 100,000,000? That is, how many times does the logarithm of 100,000,000 contain the logarithm of 2?

Log. 2 Log. 100,000,000
·30103) 8·00000...... (26·57
Ans.—The 27th power.

To get examples by which the student may ascertain whether he has acquired the highest degree of accuracy in taking out logarithms, &c., the verification of cases such as those in page 27 (rule 2) will be useful. For instance:—

Question 15. Verify to seven places of figures, (if the logarithms to seven places will serve) the equation

$$\sqrt{18} + \sqrt{11} = \frac{7}{\sqrt{18} - \sqrt{11}}$$

Log. 18.		Log. 11.
2)1·2552725		2)1·0413927
·6276363		·5206964
4·242641		3·316625
3·316625		
7·559266 Sum	its log.	·8784795
0·926016 Diff.	its log.	$\bar{1}$·9666185
		0·8450980

which is correctly the logarithm of 7.

At the beginning of the tables (1000...) an alteration of a unit in the seventh figure of the number makes an alteration of 4 units in the seventh figure of the logarithm; so that two logarithms, which differ only in the seventh decimal, by less than 4, are for every practical purpose the same. But in the last half of the tables, a unit of difference in the seventh figure of the number causes less than a unit of difference in the seventh place of the logarithm, which renders the tables not so safe in the latter part as in the former. To illustrate this, we form the following table from the extreme end of the table to seven places, repeating only the figures which change.

No.	Dec. Part of Log.
9999900	·9999957
1	57
2	58
3	58
4	59
5	59
6	60
7	60
8	60
9	61

From this it appears that ·9999960 may belong to 999990, followed either by 6, 7, or 8, so that the number cannot be found within two units in the seventh place. But this is the extreme point; and, generally speaking, the results may be depended upon within one unit in the seventh place, which is always more than sufficient for practical purposes.

Question 16. What is the value of x in the equation?

$$(20)^x = 100$$

* It is useless to retain more than four places of this.

This is the same as asking, what is the logaritbm of 100 to the base 20? Taking the logarithms of both sides, we have

$$\text{Log. } 20^{x} \text{ or } x \times \text{Log. } 20 = \text{Log. } 100$$

$$x = \frac{\text{Log. } 100}{\text{Log. } 20} = \frac{2}{1{\cdot}30103}$$

$$= 1{\cdot}537244$$

Question 17. At what rate of compound interest will money double itself in ten years? That what is the solution of

$$(1 + x)^{10} = 2$$

$$x = \sqrt[10]{2} - 1 = {\cdot}071773$$

or 7·177 per cent.; that is £7 3 6½ per cent.

In working questions of compound interest for long periods of time*, it is sometimes necessary to have certain logarithms to more than seven places. The following will be sufficient.

No.	Dec. part of Log.	Rate per cent. in which this Log. is used.
10025	00108 43813	¼
10050	00216 60618	½
10075	00324 50548	¾
10100	00432 13738	1
10125	00539 50319	1¼
10150	00646 60422	1½
10175	00753 44179	1¾
10200	00860 01718	2
10225	00966 33167	2¼
10250	01072 38654	2½
10275	01178 18305	2¾
10300	01283 72247	3
10325	01389 00603	3¼
10350	01494 03498	3½
10375	01598 81054	3¾
10400	01703 33393	4
10425	01807 60636	4¼
10450	01911 62904	4½
10475	02015 40316	4¾
10500	02118 92991	5
10525	02222 21045	5¼
10550	02325 24596	5½
10575	02428 03760	5¾
10600	02530 58653	6

Question 18. What is the amount of one farthing, for 500 years, at 3 per cent. compound interest?

One farthing is £.·001041667, and the quantity to be found is

$$£.\ (1{\cdot}03)^{500} \times {\cdot}001041667$$

$$\begin{array}{lr} \text{Log. } 1{\cdot}03 & {\cdot}0128372247 \\ & 500 \\ \hline & 6{\cdot}41861235 \\ \text{Log. } {\cdot}001041667 & \bar{3}{\cdot}0177286 \\ \hline 2731{\cdot}121 & 3{\cdot}4363410 \end{array}$$

Ans. £2731 2*s.* 5¼*d.*

Question 19. Given the common logarithm to find the hyperbolic or Naperian logarithm.

This may be done by the following table:

1	023 025 851
2	046 051 702
3	069 077 553
4	092 103 404
5	115 129 255
6	138 155 106
7	161 180 957
8	184 206 807
9	207 232 658

This table is intended to abbreviate the operation of multiplying by 2·3025851, and its use will be evident from the following examples. What is, first, the Naperian logarithm of 56?

* Such, for example, as Dr. Price's celebrated problem about a farthing put out to compound interest at the beginning of the world.

Common Log. 56.

In table, we find opposite to	1·	7	4	8	1	8	8	0	
1	0	2	3	0	2	5	8	5	1
7		1	6	1	1	8	0	9	6
4			0	9	2	1	0	3	4
8				1	8	4	2	0	7
1					0	2	3	0	3
8						1	8	4	2
8							1	8	4
		4	0	2	5	3	5	1	7

Make seven decimal places, and the answer is 4·0253517, the Naperian logarithm of 56.

What is Nap. Log. 9828?

Common Log. 9828.

3· 9 9 2 4 6 5 1
0 6 9 0 7 7 5 5 3
2 0 7 2 3 2 6 6
2 0 7 2 3 2 7
0 4 6 0 5 2
0 9 2 1 0
1 3 8 2
1 1 5
0 2
9 1 9 2 9 9 0 7

Ans. 9·1929907.

Examples for practice:

Number.	Nap. Log.
3·141593	1·1447299
2349	7·7617450
156·3	5·0517778

When there is no characteristic, use one place less, and make seven places. When there is a negative characteristic, neglect it, and proceed as in last sentence; but subtract at the end the number opposite to the characteristic in the table with all its places, attending to page 35.

What is Nap. Log. ·008?

Common Log. ·008.

$\bar{3}$·9030900
20723266
069078
2072
20794416
069077553
$\bar{5}$1716863

Ans. $\bar{5}$·1716863

[N. B. As a check upon this rule, remember that the Naperian logarithm must be something more than twice the common logarithm.]

Question 20. To reduce the Naperian logarithm to the common logarithm, use the following table in the same manner:

1	0434 2945
2	0868 5890
3	1302 8834
4	1737 1779
5	2171 4724
6	2605 7669
7	3040 0614
8	3474 3559
9	3908 6503

The Naperian logarithm being 9·1929907, what is the common logarithm?

9·1929907
39086503
0434295
390865
08686
3909
391
3
39924652

Ans. 3·9924652

Section 8.—*Examples of the Application of Logarithms for Practice.*

Before proceeding to give any examples, we shall explain why we have deviated from the usual practice, and in a manner which some of our readers will consider rather singular. In working rules by examples, which are presumed to be quite correctly answered in the book, the student is apt to work by the answer—that is, to look at the answer from time to time, and judge, or at least guess, whether he is proceeding correctly. Very few have the resolution to shut the book, and not look at the answer until they have produced their own. The consequence is, that no confidence is gained, and the student has to learn how to be independent after he has left his elementary treatise, and has to solve questions which occur in practice. To give no answers at all, would be depriving him of an assistance which, to a certain extent, is useful, and even necessary. We have therefore made some figure or figures, or positions of the decimal point (perhaps many or all, but the student must find this out), intentionally incorrect; so that while there will be enough to

assist the student who is disposed to learn how to shift for himself, there will be enough to perplex the one who has no assurance of being correct, except what he derives from the printed answer. When a figure or figures of the answer are found to differ from those here given, let the process be thoroughly re-examined, until the student is satisfied that he has obtained the correct answer.*

$$\frac{441\cdot5059}{89\cdot72584} = 4\cdot921610 \qquad \frac{1}{6\cdot729} = \cdot1486105$$

$$\frac{13052\cdot62}{\cdot9914449} = 13169\cdot25 \qquad \frac{1}{51\cdot88} = \cdot01927535$$

$$\frac{6115\cdot27}{79122\cdot35} = \cdot0772889 \qquad \frac{1}{291\cdot2} = \cdot003444066$$

$$\frac{7\cdot466382}{66\cdot52304} = \cdot1122380 \qquad \frac{1}{\cdot1239} = 8\cdot071025$$

$$\frac{\cdot9326154}{1-(\cdot4663077)^2} = 1\cdot191654$$

$$\sqrt{100 + (8\cdot09784)^2} = 12\cdot86759$$

$$\sqrt{(629\cdot3203)^2 + (777\cdot144)^2} = 999\cdot9998$$

$$\sqrt{\cdot00001215} = \cdot003485685$$

$$\sqrt{.027} = \cdot1653168 \qquad \sqrt{\cdot27} = \cdot5196162$$

$$\sqrt[3]{4\cdot355} = 1\cdot633137 \qquad \sqrt[3]{436} = 7\cdot582787$$

$$(13\cdot22869)^{\frac{3}{2}} = 48\cdot11445 \qquad \sqrt[100]{6} = 1\cdot0181$$

$$\frac{\cdot018594 \times 763^{\frac{13}{15}}}{7654\cdot3 \times 794} = \cdot000002337$$

$$\sqrt[7]{8} = 1\cdot3859 \qquad \sqrt[4]{3\cdot5246} = 1\cdot371179$$

$$\sqrt[8]{172\tfrac{5}{6}} = 1\cdot904169 \qquad \sqrt[13]{\frac{3348}{569}} = 1\cdot146156$$

$$\left(\frac{9}{8}\right)^{21} = 11\cdot87322 \qquad \left(\frac{643}{637}\right)^{183} = 31\cdot69104$$

$$\left(\frac{167}{53}\right)^{\cdot32} = 1\cdot44378 \qquad \left(\frac{5}{7}\right)^{\cdot0537} = \cdot982693$$

$$\frac{(52072)^{13} \times \sqrt{(\cdot000734)^9}}{(255608)^8} = 9930\cdot834$$

$$\left(\frac{42666}{1147}\right)^{12} \times \left(\frac{765}{19432}\right)^{10} = 627568\cdot8$$

$$\sqrt[5]{\left(\frac{7}{3}\sqrt[4]{6}\right)} = 1\cdot215695$$

$$\sqrt[3]{\left(\cdot26\sqrt{\tfrac{2}{3}}\right)} = \cdot596544$$

$$\sqrt[6]{\left(\frac{3425\sqrt[7]{136}}{\cdot00034}\right)} = 28\cdot94619$$

$$253\sqrt[3]{\frac{716\cdot5}{\sqrt{2}}} = 2016\cdot014$$

$$\sqrt{\frac{132\,(7\cdot356)^2}{\sqrt{(3\cdot25)^5}}} = 144\cdot5972$$

* Many of the examples are taken, *mutatis mutandis*, from Meier Heirsch's *Sammlung von Beispielen, Formeln*, &c., Berlin, 1816; others from trigonometrical tables, &c.

$$\frac{(466871)^{\frac{6}{7}} \times (3576)^{\frac{16}{9}}}{(996003) \times (\cdot 0071)^{\frac{1}{5}}} = 1780845$$

$$\sqrt[3]{(21 + \sqrt[6]{19})} = 1\cdot 470075$$

$$\sqrt[6]{5\cdot 03 + \sqrt[3]{\cdot 2})} = 1\cdot 792929$$

$$\sqrt[5]{(9\cdot 921 - 3\sqrt{5\cdot 02})} = 1\cdot 261866$$

$$\frac{1 + \sqrt{3}}{2\sqrt{2}} = \cdot 9659258 \qquad \frac{\sqrt{(5 + \sqrt{5})}}{2\sqrt{2}} = \cdot 9510565$$

$$\frac{1 + \sqrt{3}}{8\sqrt{2}}(\sqrt{5} + 1) - \frac{\sqrt{3} - 1}{8}\sqrt{(5 - \sqrt{5})} = \cdot 6293204$$

$$\frac{\sqrt{3} - 1}{8\sqrt{2}}(\sqrt{5} - 1) + \frac{\sqrt{3} + 1}{8}\sqrt{(5 + \sqrt{5})} = \cdot 9986295$$

$$\sqrt[16]{\left(\frac{43 + 5\sqrt[3]{278}}{\sqrt[6]{17}}\right)} = 1\cdot 264848$$

$$\sqrt[7]{\cdot 01 \sqrt[6]{(\cdot 02 \sqrt[5]{\cdot 03})}} = \cdot 4640688$$

What is the diameter of the sphere which shall have the same content as a cube of 21·16 yards in length? *Ans.* 26·25 yards.

Find the number of cubic *feet* in a cube of 15 *inches* long. *Ans.* 1·953.

What is the diameter of a circle whose circumference is 25000 miles? *Ans.* 7958 miles nearly.

What is the circumference of a circle whose diameter is 7958 miles? *Ans.* 25000$\frac{3}{4}$ miles.

What is the area of a circle whose circumference is 22 feet? *Ans.* 38·517 square feet.

What is the area of a circle whose diameter is 15·25 inches? *Ans.* 20·2949 square inches.

The surface of a sphere being four times the area of its largest circle, what is the surface of a sphere of 4·5 feet in diameter? *Ans.* 63·6174 square feet.

A sphere of six feet in diameter is painted at the rate of a halfpenny per square inch, what is the cost? *Ans.* £33 18*s.* 7*d.*

The diameter of a sphere being 7 feet, what is the side of the cube of equal solidity? *Ans.* 5·64228.

If a sphere be 3 feet in diameter, how long is the side of a square of the same surface? *Ans.* 5·31736 feet.

The squares of the times of revolution of different planets being as the cubes of their mean distances from the sun, and the mean distances of Saturn and Jupiter being in the proportion of 9538786 to 5202776, and the time of revolution of Jupiter 4332·585 days, what is that of Saturn? *Ans.* 10759·22.

The diameter of a sovereign being ·87 of an inch, how many miles would 600,000,000 sovereigns extend, if placed side by side? *Ans.* 8238·64.

SECTION 9.—*Arithmetical Complement. Trigonometrical Tables: their Use in common Calculations.*

The arithmetical complement of a number is the number by which it falls short of the unit of the next higher denomination. It is abbreviated into *Ar. co.* Thus:

Ar. co.	6	= 10	−	6	=	4
Ar. co.	893	= 1000	−	893	=	107
Ar. co.	·669	= 1	−	·669	=	·331

The lowest denomination considered is the unit. Thus:

Ar. co. ·0094 = 1 − ·0094
not ·01 − ·0094

The most expeditious way of finding the arithmetical complement is as follows:—Begin from the left, subtract every figure from 9, *up to the lowest significant figure, which subtract from 10.* Repeat the ciphers at the end, if any.

No.	156·142	·0013754
Ar. co.	843·858	·9986246
No.	1708000	4009000
Ar. co.	8292000	5991000

When there is a negative characteristic, *add* it to 9, instead of subtracting it from 9.

No.	$\bar{1}\cdot 439$	$\bar{2}\cdot 33$	$\bar{3}\cdot 108$
Ar. co.	10·561	11·67	12·892

The student should now practise taking out from the tables, not the logarithms there written, but their arithmetical complements, without first taking out the logarithms themselves. The operation above described can be correctly performed in the head, with a little practice. For instance, looking in the table, and seeing ·6123180, he should say—6 and 3 make 9, put down 3; 1 and 8 make 9, put down 8, &c., up to 8 and 2 make *ten*, which will give ·3876820.

To subtract a number, add its arithmetical complement; the result will be too great by a unit of the kind which was used in making the arithmetical complement. Thus, 9 − 4 may be thus found:

$$9 + \text{Ar. co. } 4 - 10$$

and subtractions may be reduced to the subtractions of single units from the results of an addition. In the following examples, the first is the common method, the second the one just described.

From	9·66813	9·66813
Take	3·44210	6·55790
	6.22603	16·22603 − 10
From	2·30746	2·30746
Take	$\bar{3}$·42815	12·57185
	4·87931	14·87931 − 10
From	$\bar{1}$·21769	$\bar{1}$·21769
Take	$\bar{2}$·30999	11·69001
	0·90770	10·90770 − 10

The better way will be always to write, after an arithmetical complement, the unit which must be subtracted, after addition has been substituted for subtraction by means of that complement.

What is 1835 + 968 − 1036, and 21648 − 9763 − 144?

1835	21648
968	237 − 10,000
8964 − 10,000	856 − 1000
11767	22741
10000	11000
1767	11741

Required a fourth proportional to 117·1097, 17·36482, 9510·565:

Log. 9510·565	3·9782063
Log. 17·36482	1·2396702
Ar. co. Log. 117·1097	7·9314071 − 10
	13·1492836 − 10

Ans. 1410·209.

The student may now try any of the preceding examples, with addition of arithmetical complements instead of subtraction. But we recommend him rather to avoid this method, which is very subject to error, except in the hands of a practised computer.

The trigonometrical tables have already been described (for trigonometrical purposes) in the treatise on that science (page 51). The logarithms there given are generally made too great by ten; that is, instead of the subtractive characteristic $\bar{1}$, we have the characteristic 9, &c.; or, instead of subtracting 1, we add 9, which makes the result too great by 10. In trigonometrical operations this is convenient; but principally because the extraction of roots very seldom occurs. If we had, for example, to extract the square root of the sine of 46°, which we find in the tables to be ·7193398, and the tabular logarithm of which is 9·8569341 (but, in reality, $\bar{1}$ ·8569341), the following process will be wrong in the characteristic:

$$2)\ \underline{9\cdot8569341}$$
$$4\cdot9284671$$

for the dividend being 10 too much, the quotient will be 5 too much; or, rather, the addition of the dividend being intended to be followed by a subtraction of 10, the addition of the quotient must be followed by a subtraction of 5. In extracting the cube root, the following process gives characteristic and decimal part both wrong:

$$3)\ \underline{9\cdot8569341}$$
$$3\cdot2836447$$

for, the dividend being too great by 10, the quotient is too great by $3\frac{1}{3}$, or 3·3333333, and must be set right by subtracting this. But, to reduce the result to the tabular logarithm, the logarithm should be made 20, 30, 40, &c., too great before dividing by 2, 3, 4, &c., as in those cases the results will severally be 10 too great. But, perhaps, the better way is to restore the proper negative characteristic, and proceed in the way already described (page 37).

What we have here to do with the trigonometrical tables is to observe that they may be considered as registers of the value of certain expressions, which, being already calculated, may be referred to, and thus the trouble of fresh calculation saved. We shall proceed to explain the following table:

Sine.	Cosine.	Tangent.	Cotangent.	Secant.	Cosecant.
a	$\sqrt{1-a^2}$	$\frac{a}{\sqrt{1-a^2}}$	$\frac{\sqrt{1-a^2}}{a}$	$\frac{1}{\sqrt{1-a^2}}$	$\frac{1}{a}$
$\sqrt{1-a^2}$	a	$\frac{\sqrt{1-a^2}}{a}$	$\frac{a}{\sqrt{1-a^2}}$	$\frac{1}{a}$	$\frac{1}{\sqrt{1-a^2}}$
$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{\sqrt{1+a^2}}$	a	$\frac{1}{a}$	$\sqrt{1+a^2}$	$\frac{\sqrt{1+a^2}}{a}$
$\frac{1}{\sqrt{1+a^2}}$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{a}$	a	$\frac{\sqrt{1+a^2}}{a}$	$\sqrt{1+a^2}$
$\frac{\sqrt{a^2-1}}{a}$	$\frac{1}{a}$	$\sqrt{a^2-1}$	$\frac{1}{\sqrt{a^2-1}}$	a	$\frac{a}{\sqrt{1-a^2}}$
$\frac{1}{a}$	$\frac{\sqrt{a^2-1}}{a}$	$\frac{1}{\sqrt{a^2-1}}$	$\sqrt{a^2-1}$	$\frac{a}{\sqrt{1-a^2}}$	a

By this we mean, that if a be the sine of an angle, or if we can find a in the table of sines, we find $\sqrt{1-a^2}$ in the corresponding line of the table of cosines, $\frac{a}{\sqrt{1-a^2}}$ in that of the table of tangents: if a be found in the table of tangents, we have opposite to it $\frac{a}{\sqrt{1+a^2}}$ in that of sines, and $\frac{1}{\sqrt{1+a^2}}$ in that of cosines. If the tables only give *logarithms** of sines, &c., we must look for the logarithm of a, and we find the logarithms of the above quantities.

To use this method with great accuracy would give very nearly if not quite as much trouble as the common logarithmic process, but by mere inspection a few places of any result may be obtained; so that when very considerable accuracy is not required, calculation is altogether saved. For instance—

$$a \text{ being } \cdot 6346 \text{, what is } \frac{a}{\sqrt{1+a^2}}$$

if the first be a tangent, the second is the corresponding sine: we look in the table of tangents (Hutton's), and find as follows, under 32° 24′, that the tangent being ·6346193, the sine is ·5358268, so that ·5358 will be very near the answer. To get a nearer answer, we must use the method which we proceed to describe.

Definition. When a number is to be found in a table opposite to another number, the second, by means of which we know where to go in the table, is called the *argument*. Thus, in finding the logarithm of 56, we *enter* the column of numbers with the argument 56. In the last example, we *enter* the table of tangents with the *argument* ·6346. The result obtained we shall call the *resultant*†.

Let a be the given argument not exactly to be found in the tables. Let M and m (M being the greater) be the arguments in the tables, between which a is found to lie. Let R and r be the corresponding resultants to M and m. Thus,

Column of arguments.	Column of resultants.
M	R
a	ρ
m	r

or the following—

m	r
a	ρ
M	R

according as the arguments decrease or increase the resultant of the argument a is

$$r + \frac{(a-m)(\mathrm{R}-r)}{\mathrm{M}-m}$$

when arguments and resultants both increase or both decrease together; and

$$r - \frac{(a-m)(r-\mathrm{R})}{\mathrm{M}-m}$$

* Hutton's tables, which give *sines*, &c., as well as their logarithms, are decidedly the best of which we know for the engineer or mechanic. There are more useful tables for the astronomer, to which we need not here allude. For logarithms of numbers only, we believe those of Mr. Babbage to be best for general use, of all those which contain seven places.

† There is no correlative to *argument* in common use. The reader may take his choice of *resultant*, *inference*, *extract*, *function*, &c.; either of which would be better than no word at all.

when one of the two increases and the other decreases. This process is called *interpolation.*

In the above example, we are to find the sine to the tangent ·6346000 (the argument). Looking in the tables, we find as follows—

Arguments. Tangents.	Resultants. Sines.
$m = 6342113$	$5355812 = r$
$a = 6346000$	?
$M = 6346193$	$5358268 = R$

Arguments and resultants increase together.

$M - m = 4080 \quad a - m = 3887 \quad R - r = 2456$

$\frac{3887 \times 2456}{4080} = 2340$ (nearest whole No.)

$$\begin{array}{r} 5355812 \\ 2340 \\ \hline ·5358152 \end{array}$$ *Answer.*

This is within a unit, in the last place, of the truth.

An unlimited number of examples of the preceding process may be obtained by taking an argument and resultant from the tables themselves, and finding the latter by means of those which come before and after. Suppose, for instance, that the cotangent of 19° 31′ had been erased or blotted so as not to be visible, and that it were required to fill it up by using the table of tangents, the process would be as follows:—

Arguments. Tangents.	Resultants. Cotangents.
$m = 3541186$	$28239129 = r$
$a = 3544460$	?
$M = 3547734$	$28187003 = R$

Arguments increase and resultants decrease.

$M - m = 6548 \quad a - m = 3274 \quad r - R = 52126$

$\frac{3274 \times 52126}{6548} = 26063$

$$\begin{array}{r} 2·8239129 \\ 26063 \\ \hline 2·8213066 \end{array}$$ *Ans.*

which is incorrect by a unit in the sixth decimal place.

This process may be applied to tables of logarithms; or, in fact, to any tables. It is substantially what is done in finding the logarithm of a number intermediate to two numbers in the tables, or *vice versa*, and also in finding the logarithmic sine, &c. of an angle intermediate to two angles in the table, as the following examples will prove, if the student compare them with the usual process.

Required log. 6416958. We shall only consider the decimal part.

Arguments. Numbers.	Resultants. Logarithms.
$m = 6416900$	$8073253 = r$
$a = 6416958$	?
$M = 6417000$	$8073320 = R$

Arguments and resultants increase together.

$M - m = 100 \quad a - m = 58 \quad R - r = 67$

$\frac{58 \times 67}{100} = 39$ (nearest whole No.)

$8073253 + 39 = 8073292$

Answer ·8073292

Required the logarithm of the sine of 36° 18′ 47″·6.

Arguments. Angles.	Resultants. Log. sines.
$m = 36° 18' 0''$	$9·7723314 = r$
$a = 36° 18' 47''·6$	?
$M = 36° 19' 0''$	$9·7725033 = R$

Arguments and resultants increase together.

$M - m = 60'' \quad a - m = 47''·6 \quad R - r = 1719$

$\frac{47·6 \times 1719}{60} = 1364$ (nearest wh. No.)

$$\begin{array}{r} 9·7723314 \\ 1364 \\ \hline 9·7724678 \end{array}$$ *Answer.*

What is the angle whose logarithmic sine is 9·9475008?

Arguments. Log. sines.	Resultants. Angles.
$m = 9·9474674$	$62° 23' 0'' = r$
$a = 9·9475008$	?
$M = 9·9475335$	$62° 24' 0'' = R$

Arguments and resultants increase together.

$M - m = 661 \quad a - m = 334 \quad R - r = 60$

$\frac{334 \times 60}{661} = 30''·3$ (nearest tenth)

Answer 62° 23′ 30″ ·3

What is the logarithm of the cosine of 57° 5′ 9″ ·8?

Arguments. Angles.	Resultants. Log cosines.
$m = 57° 5' 0''$	$9·7351345 = r$
$a = 57° 5' 9''·8$	?
$M = 57° 6' 0''$	$9·7349393 = R$

Arguments increase and resultants decrease.

$M - m = 60'' \quad a - m = 9''·8 \quad r - R = 1952$

$\frac{9·8 \times 1952}{60} = 319$ (nearest wh. No.)

$$\begin{array}{r} 9·7351345 \\ 319 \\ \hline 9·7351026 \end{array}$$ *Answer.*

What is the angle whose logarithmic cosine is 8·8852331?

Arguments. Log. cosines.	Resultants. Angles.
m = 8·8849031	85° 36′ 0″ = r
a = 8·8852331	?
M = 8·8865418	85° 35′ 0″ = R

Arguments increase and resultants decrease.

M $-m$=16387 $a-m$ 3300 $r-$R=60″

$$\frac{3300 \times 60}{16387} = 12''\cdot 1 \text{ (nearest tenth)}$$

85° 36′ − 12″·1 = 85° 35′ 47″·9

The accuracy of the preceding method depends upon the tabular differences continuing the same, or nearly the same, for the resultants of several successive arguments. This is the case for the most part throughout the tables; but where it is not so, a small error is committed. For instance, in the last example, if we look at the table, we find—

Log. cos. 85° 34′ − log. cos. 85° 35′ = ·0016325
Log. cos. 85° 35′ − log. cos. 85° 36′ = ·0016387
Log. cos. 85° 36′ − log. cos. 85° 37′ = ·0016450

An error of some tenths of a second in the answer is the consequence. In the tables of sines and tangents of angles under 2°, or of cosines of angles above 88°, the disparity of the differences is so great, that it is useless to apply the above process; and it is usual, therefore, to give a separate table of sines and tangents for the first two degrees, in which the angles increase by seconds.

In all other applications of the trigonometrical tables to calculation of common algebraical formulæ, no very great advantage is gained, where much accuracy is required, by any tables which give the logarithms to minutes only. The reason is the length of the processes of interpolation. Where extreme accuracy is not required, the common tables, which go to minutes, are often advantageous, as in the following instance:—

To find $\sqrt{a^2 + b^2}$, let tan $\theta = \frac{b}{a}$ then

$$\sqrt{a^2 + b^2} = \frac{a}{\cos\theta}.$$

EXAMPLE. What is $\sqrt{(92\cdot 736)^2 + (64\cdot 018)^2}$?

Log. 64·018 =	1·8062343
Log. 92·736 =	1·9672484
Log. tan. 34° 37′	9·8389859 *
	1·9672484
Log. cos. 34° 37′	$\bar{1}$·9153846 †
Ans. 112·68	2·0518638

The more exact process, beginning from the third line of the above, is the following:—

Log. tan. 34° 36′ 51″·8	9·8389859
	1·9672484
Log. cos. 34° 36′ 51″·8	$\bar{1}$·9153965
Ans. 112·6813	2·0518519

The following method of verifying any term in a table may be useful when an error is suspected, and no other table is at hand for comparison.

Let z be the suspected term, and let A, B, C, &c., be those which precede, and a, b, c, &c. those which follow; so that the table proceeds thus:

$$\cdots D, C, B, A, z, a, b, c, d\cdots$$

When the tabular differences appear uniform in the neighbourhood of the term z, use the first of the formulæ at the head of next page; but where this is not the case, count the number of places of the differences which alter at each step, and use the formula opposite to that number in the next page.

For instance, suppose we wish to verify the logarithm of the sine of 1° 12′, we have—

A = 8·3149536	B = 8·3087941
a = 8·3270163	b = 8·3329243
16·6419699	16·6417184
15	6
249·6295485	99·8503104
16·6412989	
266·2708474	C = 8·3025460
99·8503104	c = 8·3387529
20) 166·4205370	16·6412989
8·32102685	
8·3210269 in the tables.	

* 9 is written for $\bar{1}$, that this result may be too great by 10, as are all the logarithms in the table of tangents.

† It is more convenient here to restore the real characteristic.

0.	$z = \frac{1}{2}\ \ (A + a)$
1, 2	$z = \frac{1}{6}\{4\ (A + a) - (B + b)\}$
3, 4	$z = \frac{1}{20}\{15\ (A + a) - 6\ (B + b) + C + c\}$
5, 6	$z = \frac{1}{70}\{56\ (A + a) - 28\ (B + b) + 8\ (C + c) - (D + d)\}$

The trigonometrical tables may be made to furnish a solution of equations of the second degree*, in cases where the coefficients are too complicated to admit of easy multiplication and division. The rules are as follows :—

$$\left.\begin{array}{l} 1\ ax^2 + bx + c = 0 \\ 2\ ax^2 - bx + c = 0 \end{array}\right\}\text{Let } \sin\theta = \frac{2\sqrt{ac}}{b}$$

$$\left.\begin{array}{l} 3\ ax^2 + bx - c = 0 \\ 4\ ax^2 - bx - c = 0 \end{array}\right\}\text{Let } \tan\theta = \frac{2\sqrt{ac}}{b}$$

Then the two *numerical* values of x will be

$$\frac{\sqrt{ac}}{a}\tan\frac{\theta}{2} \quad \text{and} \quad \frac{\sqrt{ac}}{a}\cot\frac{\theta}{2}$$

and the signs will be as follows :—

1. Both negative.
2. Both positive.
3. Greater numerical root negative.
4. Lesser numerical root negative.

The instance on the opposite side will admit of easy verification in the common way.

The process which we have put in brackets is one of those little artifices of calculation, which occur in hundreds, but which it is impossible to reduce to rule, and which nothing but practice can teach. We have got the first root by means of the

$$3x^2 - 8x + 4 = 0$$

Log. a	0·4771213
Log. c	0·6020600
Log. ac	2)1·0791813
Log. $\sqrt{ac}$	·5395907
Log. 2	·3010300
Ar. Co. log. b	9·0969100 − 10
Log. sin 60° 0′ 0″·0	9·9375307 − 10

$$\theta = 60^\circ\ 0'\ 0''\cdot 0$$
$$\tfrac{1}{2}\theta = 30^\circ\ 0'\ 0''\cdot 0$$

Log. tan $\frac{1}{2}\theta$ =	9·7614394 − 10
Log. $\sqrt{ac}$ =	·5395907
Ar. co. log. a =	9·5228787 − 10
·6666667	19·8239088 − 20†
[2 Log. tan $\frac{1}{2}\theta$	19·5228788 − 20
2	0·3010300]

formula $\frac{\sqrt{ac}}{a}\tan\frac{1}{2}\theta$; and which is ·6666667, as far as seven places of decimals. [In fact, the real root is $\frac{2}{3}$.] It remains to find the second root from the formula $\frac{\sqrt{ac}}{a}\cot\frac{1}{2}\theta$, so that, in like manner as we found the first root from the calculation of

$$\text{Log. tan.}\tfrac{1}{2}\theta + \log.\sqrt{ac} + \text{Ar. co. log. } a - 10$$

we should proceed to find the second from

$$\text{Log. cot.}\tfrac{1}{2}\theta + \log.\sqrt{ac} + \text{Ar. co. log. } a - 10$$

But since the tangent and cotangent of the same angle are reciprocals, or

$$\cot.\tfrac{1}{2}\theta = \frac{1}{\tan.\frac{1}{2}\theta}$$

we have

$$\log.\cot.\tfrac{1}{2}\theta = -\log.\tan.\tfrac{1}{2}\theta$$

consequently the second formula is the same as

$$\log.\sqrt{ac} + \text{Ar. co. log. } a - 10 - \log.\tan\tfrac{1}{2}\theta$$

which, calling P the result of the first, is

$$P - 2\log.\tan.\tfrac{1}{2}\theta$$

and it is thus that we have proceeded.

The preceding example, given merely for the sake of illustrating the rule, is one which might have been more easily done by the usual method. But the following is a case in which much trouble will be saved by the trigonometrical method.

$$\begin{array}{ccc} a & b & c \\ 1\cdot0082\,x^2 & +\ 6\cdot4347\,x & -\ 2\cdot4566 = 0 \end{array}$$

* In this process we suppose the student to understand algebra, as far as the solution of equations of the second degree.

† 19—20 gives the characteristic $\bar{1}$.

Log. a	$0 \cdot 0035467$
Log. c	$0 \cdot 3903344$
Log. ac	2) $0 \cdot 3938811$
Log. $\sqrt{ac}$	$0 \cdot 1969406$
Log. 2	$0 \cdot 3010300$
Ar. co. log. b	$9 \cdot 1914717 - 10$
Log. tan. $26^\circ\ 3'\ 56'' 1 \cdot$	$9 \cdot 6894423 - 10$
$\theta =$	$26^\circ\ 3'\ 56'' \cdot 1$
$\frac{1}{2}\theta =$	$13^\circ\ 1'\ 58'' \cdot 1$
Log. tan. $\frac{1}{2}\theta$	$9 \cdot 3644973 - 10$
Log. $\sqrt{ac}$	$0 \cdot 1969406$
Ar. co. log. a	$9 \cdot 9964533 - 10$
$\cdot 3613193$	$19 \cdot 5578912 - 20$
	$18 \cdot 7289946 - 20$
$6 \cdot 743675$	$0 \cdot 8288966$

The numerical roots being therefore $\cdot 3613193$ and $6 \cdot 743675$, we find, by the rule of signs before given, that the real roots are

$$\cdot 3613193 \text{ and } -6 \cdot 743675.$$

To form examples of this method, let the student proceed as follows: choose any two numbers for roots; for instance, $1 \cdot 308$ and $- \cdot 486$, and any value of a, for instance, $2 \cdot 709$. Take the algebraical sum and product of the roots; that is

$$1 \cdot 308 - \quad \cdot 406 = \quad \cdot 902$$
$$1 \cdot 308 \times - \cdot 486 = - \cdot 635688$$

Multiply these by $2 \cdot 709$, the chosen value of a, giving $2 \cdot 443518$ and $-1 \cdot 722079$. Change the sign of the first; then the roots of

$$2 \cdot 709\, x^2 - 2 \cdot 443518\, x - 1 \cdot 722079$$

should be $1 \cdot 308$ and $- \cdot 486$.

The only way to obtain any security in the use of logarithms is to work many examples, beginning with simple cases. Repeated failures will take place at first; but the student will finally acquire that sort of habit which suggests the right method independently of rules.

We shall now proceed to examples of algebraical operations.

SECTION 10. *Division of Algebraical Operations. Algebraical Reduction, Addition, and Subtraction of Integral Quantities.*

The operations of algebra may be divided into two classes:—1. Those which present no forms distinct from the results of ordinary arithmetic. 2. Those which require the use of the negative or other symbol in the manner peculiar to algebra.

The operations of algebra may be subdivided again as follows:—1. Those which are usually left to the student in the higher parts of the subject. 2. Those which are usually inserted at length. Among the infinite number of classes of examples which might be chosen, we shall confine ourselves to those which it is most necessary the student should be able to perform, in order to read any work on the higher parts of algebra, or on the differential calculus. We suppose the student to gain the demonstrations of the rules here exemplified from some other source.

I. *Questions illustrative of the meaning of the fundamental symbols + and −, to be answered without rules.*

1. By how much does $a + b$ exceed a? By how much does $a + b$ exceed $a - 1$? By how much does $a + b$ exceed $a - m$, $a - b$, and $a - 2b$?

2. On what does it depend whether $a + b$ or $a + c$ is the greater? When $a + b$ is greater than $a + c$, by how much does the first exceed the second? By how much does $a + 2b$ exceed $a + b$?

3. On what does it depend whether $a + b - c$ is greater than, equal to, or less than a? By how much does $a + b - c$ fall short of $a + b + 1$, and of $a + b + c$?

4. On what does it depend whether $a - b$ is greater than, equal to, or less than $a - c$?

5. If a lie between b and c, c being the greater, what must a be in order that $c - a$ and $b + a$ may be equal?

6. If a be greater than b, and x less than y, which of the following must be true, which must be false, and which may be either true or false?

$a + x$	is greater than	$b + y$
$a + y$	. . . .	$b + x$
$a - x$	. . . .	$b - y$
$a + x$	is less than	$b + y$
$a + y$	. . .	$b + x$

II.—*On the Use of Brackets.* Distinguish between the meanings of—

$a + (b + c)$ and $a + b + c$
$a - (b + c)$ and $a - b + c$
$a - (b - c)$ and $a - b - c$
$(a - b) - c$ and $a - (b - c)$.
$2(a + b)$ and $2a + b$.

$$\left\{\begin{array}{l} a - \{b - (c - d)\} \\ a - b - c - d. \\ a - (b - c) - d \text{ and} \\ a - (b - c - d). \end{array}\right.$$

III.—*Simple Reductions.* How is an expression altered if $8a$, and afterwards $5a$ be added to it? *Answer,* $13a$ is added to it, which is thus expressed—

$$P + 8a + 5a = P + 13a$$

Establish the following, and express the question asked, in words, as above.

$P + a + a = P + 2a$
$P + a - a = P$
$P - a + a = P$
$P + 2a - a = P + a$
$P - 3a + 4a = P + a$
$P - 16a - 3a = P - 19a$
$P + 16a - 20a = P - 4a$
$P - 21a + a = P - 20a$

Give all the ways of expressing $P + 3a$ derived from the preceding.

$P + a + a + a$ $\quad$ $P - a + 4a$
$P + 2a + a$ $\quad$ $P - 2a + 5a$
$P + a + 2a$ $\quad$ $P - 3a + 6a$ &c.

$P + 4a - a$ $\quad$ $4a + (P - a)$
$P + 5a - 2a$ $\quad$ $5a + (P - 2a)$
$P + 6a - 3a$ $\quad$ $6a + (P - 3a)$
&c. $\quad$ &c.

$4a - (a - P)$ $\quad$ $5a - (2a - P)$ &c.

In the last two sets of the preceding, point out the cases in which they are possible, or impossible, and why. Repeat the whole process with $P - a$, $P + 6a$, &c.

IV.—*General expressions derived from the preceding.*

$$P + ma + na = P + (m + n)a$$
$$P + ma - na = P + (m - n)a$$
$$\text{or} = P - (n - m)a$$
$$P - ma + na = P - (m - n)a$$
$$\text{or} = P + (n - m)a$$

V.—*Simple Reductions including Fractions.*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

$$\frac{1}{2}a + \frac{1}{3}a + \frac{1}{4}a = \frac{13}{12}a \text{ or}$$

$$\frac{a}{2} + \frac{a}{3} + \frac{a}{4} = \frac{13a}{12}$$

$$x + 6x - \frac{1}{2}x + 2x = 8\tfrac{1}{2}x = \frac{17x}{2}$$

$$3ab - ab + \frac{6}{7}ab = \frac{20ab}{7}$$

$$x^2 + 2x^2 + 3x^2 - \tfrac{1}{2}x^2 = \frac{11}{2}x^2$$

$$xyz - \frac{1}{2}xyz + \frac{1}{3}xyz = \frac{5}{6}xyz$$

VI.—*Reductions in cases where the arrangement of the terms makes the operations appear impossible.*

$a - 2a$ is impossible, but $a - 2a + 3a$ is to be considered as a misplacement of $a + 3a - 2a$, which is $2a$.

$6a - 10a + 4a = 0$
$2a - 9a - 3a + 16a = 6a$

$xy - 4xy + 10xy = 7xy$
$\tfrac{1}{2}p^3 - p^3 + \tfrac{3}{4}p^3 = \tfrac{1}{4}p^3$
$a^2c - 3a^2c + 100a^2c = 98a^2c$

$$m + 4m - 11\tfrac{1}{2}m + 12m = \frac{11}{2}m$$

VII.—*Generalization of the two preceding Articles.*

$$m\,a + n\,a = (m + n)a \qquad m\,x + n\,x - p\,x = (m + n - p)\,x$$
$$m\,a - n\,a = (m - n)\,a \qquad \frac{m\,x}{n} + \frac{p\,x}{q} - \frac{r\,x}{s} = \left(\frac{m}{n} + \frac{p}{q} - \frac{r}{s}\right)x$$
$$m\,a + n\,a + p\,a = (m + n + p)\,a$$

VIII.—*Additions in which subsequent Reduction is impossible.*

Add together $a + b$, $c - d$, $e - f$, and $a\,b - c\,f$. *Answer*, $a + b + c - d + e - f + a\,b - c\,f$.

Add together $a - b + c$, $a\,c - 1 + f$, $c\,f + 6\,d$, and $p + q^2$. *Answer*, $a - b + c + a\,c - 1 + f + c\,f + 6\,d + p + q^2$.

IX.—*Additions in which subsequent Reduction is possible.*

Add $a - b$, $a - c$, $c - 7$, and $c + 6$. *Answer*, $a - b + a - c + c - 7 + c + 6$, or $2\,a - b + c - 1$.

Add $a + b$ and $a - b$. *Answer*, $a + b + a - b$, or $2\,a$.

Add

$$\begin{array}{l} a\,b\,c - a + x\,y + 12 - p^2 - q^2 \\ 2\,a\,b\,c - 4\,p^2 - x\,y - q^2 + 20 + a \\ 10\,p^2 - 100 + 40\,a - 14\,x\,y \\ 4\,q^2 - 4\,p^2 + a\,b\,c \\ \hline 4\,a\,b\,c + 40\,a - 14\,x\,y - 68 + p^2 + 2\,q^2 \end{array}$$

Add

$$\begin{array}{l} 13\,x^3 + 20\,x^2 - 45\,x + \frac{1}{2} \\ \frac{1}{2}\,x^3 - x^2 + 50\,x + 3\frac{1}{4} \\ 30\,x^2 - 5\,x^3 + 15\,x - 8 \\ 2\,x - x^2 + x^3 - 10 \\ \hline 9\frac{1}{2}\,x^3 + 48\,x^2 + 22\,x - 14\frac{1}{4} \end{array}$$

Further examples are deferred for the present.

X.—*Rule of Addition.*

Let the first term of each expression be considered as having the sign +

Annex all the expressions with their signs.

Make such reductions as are practicable.

Rule of Subtration.

Let the first terms of both expressions have the sign +

Change the signs in the expression which is to be subtracted.

Annex the expressions, and make reductions.

XI.—*Subtractions in which Reduction is impracticable.*

From a take $b - c$. *Ans.* $a - b + c$.

From $2\,a - c$ take $4\,b - d$. *Ans.* $2\,a + d - c - 4\,b$.

From $p - q + r - s$ take $z - p\,q$. *Ans.* $p - q + r - s - z + p\,q$.

XII. *Subtractions in which Reduction is practicable.*

From a take $a - c$. *Ans.* $a - a + c$, or c.

From $a + b$ take $a - b$. *Ans.* $a + b - a + b$, or $2\,b$.

From $p - a$ take $q - a$. *Ans.* $p - a - q + a$, or $p - q$.

$$\begin{array}{ll} \text{From} & a\,b + 3\,a - 4\,b + 16 - z \\ \text{Take} & 6\,a\,b - 12\,a - 5\,b + 12\,z - 8 \\ \hline \textit{Ans.} & 15\,a - 5\,a\,b + b + 24 - 13\,z \end{array}$$

$$\begin{array}{ll} \text{From} & 6\,a^2 b - 12\,a + \frac{1}{2} - 3\,z - 22\,a\,b \\ \text{Take} & a^2 b + 100 - 40\,a\,b + a - 3\,z \\ \hline \textit{Ans.} & 5\,a^2 b - 13\,a - 99\frac{1}{2} + 18\,a\,b \end{array}$$

$$\begin{array}{ll} \text{From} & 3\,x^2 - 18\,x^3 + 6\,x - 150 \\ \text{Take} & 12\,x^2 - 40\,x^3 + 5\,x + 40 \\ \hline \textit{Ans.} & 22\,x^3 - 9\,x^2 + x - 190 \end{array}$$

$$\begin{array}{ll} \text{From} & a - b + c - d + e - f + g - h \\ \text{Take} & a - 2\,b + 3\,c + 4\,d - 5\,e + f + g - 8\,h \\ \hline \textit{Ans.} & b - 2\,c - 5\,d + 6\,e - 2\,f + 7\,h \end{array}$$

$$(\tfrac{1}{2}\,a + b) - (a + \tfrac{1}{2}\,b) = \tfrac{1}{2}\,b - \tfrac{1}{2}\,a$$
$$(\tfrac{1}{2}\,a - b) - (a - b) + c = c - \tfrac{1}{2}\,a$$
$$(2\,a + b) - (a - 3\,b) = a + 4\,b$$

Let there be any series of quantities, a, b, c, d, e, f, &c., and let another series be obtained by taking the first from the second, the second from the third, the third from the fourth, and so on. Let this process be repeated with the second series, giving a third, with the third series, giving a fourth, and so on. What will be the several series?

1st Series.	2nd Series.	3rd Series.
a	$b-a$	$c-2b+a$
b	$c-b$	$d-2c+b$
c	$d-c$	$e-2d+c$
d	$e-d$	$f-2e+d$
e	$f-e$	$g-2f+e$
f	$g-f$	$h-2g+f$
&c.	&c.	&c.

4th Series.	5th Series.
$d-3c+3b-a$	$e-4d+6c-4b+a$
$e-3d+3c-b$	$f-4e+6d-4c+b$
$f-3e+3d-c$	$g-4f+6e-4d+c$
$g-3f+3e-d$	$h-4g+6f-4e+d$
&c.	&c.

1st term of the Series.	
6	$f-6e+10d-10c+5b-a$
7	$g-6f+15e-20d+15c-6b+a$
&c.	&c.

One way to form results which shall give examples of addition and subtraction combined, is as follows:—Take any number of quantities at pleasure, subtract each one from that which follows, add all the differences *and the first of the quantities* together; the result should be the *last of the quantities.*

For instance, let the quantities be $a-b$, $2a+b-c$, $3a+2b-4c$, and $7a+6b$. From the second take the first, which gives $a+2b-c$; from the third take the second which gives $a+b-3c$; from the fourth take the third, which gives $4a+4b+4c$. Add together the following—

$$\begin{array}{l} a-b \\ a+2b-c \\ a+b-3c \\ 4a+4b+4c \\ \hline \end{array}$$

which gives $7a+6b$, the last of the quantities first mentioned.

As questions of mere addition and subtraction are of little use by themselves, we shall now proceed to another point.

Section 11. *Exercises in the use of the Algebraical symbols + and − as distinguished from Arithmetical* symbols.*

In whatever way the symbols $+a$ and $-a$ may be explained, the method of using them is as follows:—

$+(+a)$ is $+a$ $\quad +(-a)$ is $-a$
$-(-a)$ is $+a$ $\quad -(+a)$ is $-a$
or, *like signs produce +, unlike signs −.*

$$a+(-b)=a-b=-b+(+a)$$
$$a-(-b)=a+b=\quad b-(-a)$$
$$+\{+\{+a\}\}=+a \qquad -\{-\{-a\}\}=-a$$
$$a-b=-(b-a)\quad =-b-(-a)$$
$$1-2=-1 \qquad 2-3=-1 \qquad a-2a=-a$$
$$6-15=-9 \qquad 4-7-8=-11$$
$$x-(x+y)=x-x-y=-y$$
$$(2a+3b)-(4a+7b)=-(2a+4b)$$
$$a-b+3c-d=-(b+d-a-3c)$$
$$-a-4a-7a-12a=-24a$$
$$3a-7b-4a+5b=-a-2b$$
$$+a\times+b=+ab \qquad +a\times-b=-ab$$
$$-a\times-b=+ab \qquad -a\times+b=-ab$$
$$+a\times+b\times-c\times+d=-abcd$$
$$-a\times-b\times-c\times+d=-abcd$$

* The student may omit this section until he has read an explanation of the negative sign.

$$-3 \times -4 = +12 \qquad -3 \times +4 = -12$$
$$a\,b \times -c = a\,c \times -b = b\,c \times -a$$
$$\frac{a}{b} = \frac{a \times -1}{b \times -1} = \frac{-a}{-b}, \quad \text{If } a = -b,\ b = -a$$
$$\frac{p-q}{x-y} = \frac{-(p-q)}{-(x-y)} = \frac{q-p}{y-x}$$
$$\frac{+a}{+b} = +\frac{a}{b} \qquad \frac{-a}{+b} = -\frac{a}{b}$$
$$\frac{-a}{-b} = +\frac{a}{b} \qquad \frac{+b}{-b} = -\frac{a}{b}$$
$$\frac{p-q}{x-y} = \frac{-(q-p)}{x-y} = -\frac{q-p}{x-y}$$
$$\frac{p-q}{x-y} = \frac{p-q}{-(y-x)} = -\frac{p-q}{y-x}$$
$$a - b + c = a - (b - c) = a + (c - b)$$
$$= -(b - a - c) = -(b - a) + c = -(b - c) + a$$
$$= -(-a + b - c) = a + (-b) + c$$

What suppositions will make the following expressions identically the same?

$$a\,x^2 + b\,x\,y + c\,y^2 + d\,y + e$$
$$3\,x^2 - 4\,x\,y - 2\,y^2 + 7\,y - 6$$
Ans. $a = 3 \quad b = -4 \quad c = -2$
$d = 7 \quad e = -6;$

$$a\,x - b\,y + c\,z$$
$$3\,x + y + z$$
$a = 3 \quad b = -1 \quad c = 1$

and the following—

$$p\,x^2 + q\,x + r$$
$$\tfrac{1}{2}\,x^2 - x + 1$$
Ans. $p = \frac{1}{2} \quad q = -1 \quad r = 1$

If $x = -y$ $\qquad x^4 = y^4 \qquad x^7 = -x^7$
$x^2 = y^2 \qquad x^5 = -y^5 \qquad x^8 = y^8$
$x^3 = -y^3 \qquad x^6 = y^6 \qquad x^9 = -y^9$

Which of the following pairs are the same, and which differ in sign?

$-x^8$ and $(-x)^8$ $\qquad -x^{13}$ and $(-x)^{13}$
x^6 and $(-x)^6$ $\qquad x^7$ and $(-x)^7$
$x^3 \times (-y)^4 \times (-z)^5$ and $(-x)^3 \times (-y)^4 \times z^5$

Change the sign of x in the following expression, that is, write $-x$ instead of $+x$, and $+x$ instead of $-x$.

$$a + b\,x + c\,x^2 + d\,x^3 - \frac{1}{x}$$

it becomes $a + b\,(-x) + c\,(-x)^2 + d\,(-x)^3 - \dfrac{1}{-x}$

or, $a - b\,x + c\,x^2 - d\,x^3 + \dfrac{1}{x}$

Of the following expressions, the second in each set arising from changing the sign only of some letter in the first, let the student find out which letter it is, and account for every change.

1. $a + b\,x + \dfrac{1-x}{1+x} - x^2\,(1 + x^3)$

   $a - b\,x + \dfrac{1+x}{1-x} - x^2\,(1 - x^3)$

2. $a\,x\,(x - x^3) + b\,x^2\,(x^2 - x^3) + \dfrac{x\,(1 - x)}{b + x^2}$

$$a\,x\,(x - x^3) + b\,x^2\,(x^2 + x^3) - \frac{x\,(1 + x)}{b + x^2}$$

3. $$(2\,a\,x + b)^2 + 4\,a\,c - b^2 + \frac{x - a\,x}{1 + a^2}\left\{x + a\,y\right\}$$

$$(2\,a\,x - b)^2 - 4\,a\,c - b^2 + \frac{x + a\,x}{1 + a^2}\left\{x - a\,y\right\}$$

4. $$a - b\,x + c\,x^2 - d\,x^3 + \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}$$

$$a + b\,x + c\,x^2 + d\,x^3 - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3}$$

When the signs of two or more letters are changed, the general rule is, every term changes sign in which an *odd* number of factors changes sign. Thus, in

$$a^3 + a\,b\,c - a\,c^2\,b - b^3\,c^3$$

if b and c both change sign, that is, if $-\,b$ be written for b, and $-\,c$ for c, &c., the term which changes sign is $-\,a\,c^2\,b$, or, $a\,c\,c\,b$, (in which an odd number of factors, c, c, b, change their signs) and the expression becomes

$$a^3 + a\,b\,c + a\,c^2\,b - b^3\,c^3$$

In the following examples one *or more* letters have their signs changed in the second and succeeding ones of each. The student must ascertain which letters they are.

1. $$\frac{a + b}{a - b} + \frac{a - a^2}{b - a\,b} + a\,b^2 c - a\,x\,y$$

$$-\frac{b - a}{a + b} - \frac{a + a^2}{b + a\,b} + a\,b^2 c + a\,x\,y$$

2. $$\frac{a - b + c}{a^2 - b^4 + c^2}\,,\quad \frac{a + b - c}{a^2 - b^2 + c^2}\,,\quad \frac{a - b + c}{b^2 - a^4 - c^2}$$

Observe, that no even power of a ($a^2\,a^4\,a^6$ &c.) changes sign when a changes sign; how then can the first expression become the third by changes of sign only?

3. $$(p + q)\ \ (q + r)\ \ (r + p)\ \ ,\ \ (p + q)\ \ (q - r)\ \ (r - p)$$

$$-\,(p + q)\ \ (q + r)\ \ (r + p)\ \ ,\ \ (p - q)\ \ (q + r)\ \ (p - r)$$

4. $$(b^2 - 4\,a\,c)\ \ (a - b)\ \ (b - c)\ \ ,\ \ -\,(b^2 + 4\,a\,c)\ \ (a + b)\ \ (b - c)$$

$$(b^2 + 4\,a\,c)\ \ (a + b)\ \ (c - b)\ \ ,\ \ -\,(b^2 - 4\,a\,c)\ \ (a + b)\ \ (b + c)$$

SECTION 12. *Multiplication and Division of Single terms.*

$$a\,b = b\,a = \frac{a\,b\,c}{c} = \frac{a^2\,b}{a} = \frac{a\,b^2}{b} = \frac{a^2\,b^2}{a\,b}$$

$$a\,b \times a\,c = a^2\,b\,c \qquad 3\,a\,b \times 6\,a\,c = 18\,a^2\,b\,c$$

$$p^2\,q^3 \times q^7 = p^2\,q^{10} \qquad a \times a \times a^2 = a^4$$

$$a^5 \times a^7 = a^{12} \qquad a^m \times a^n = a^{m+n}$$

$$2\,(a + b) = 2\,a + 2\,b \qquad 3(a - b) = 3\,a - 3\,b$$

$$2\,a\,(a + b + c - d) = 2\,a^2 + 2\,a\,b + 2\,a\,c - 2\,a\,d$$

$$3\,a\,b\,(4\,a - a\,b + x) = 12\,a^2\,b - 3\,a^2\,b^2 + 3\,a\,b\,x$$

$$\frac{1}{2}\left(\frac{1}{2}a - 2\,b\right) = \frac{1}{4}\,a - b \qquad \frac{2}{3}\left(\frac{3}{4}a - \frac{1}{5}\ a\,b\right) = \frac{1}{2}a - \frac{2}{15}a\,b$$

$$4\,a\,b\,c\left(2\,a\,b + \frac{3}{2}\,b\,c - 6\,a\,c\right) = 8\,a^2\,b^2 c + 6\,a\,b^2\,c^2 - 24\,a^2\,b\,c^2$$

$$3\,a^2 z\,(z^3 y - a^3 z + z - 1) = 3\,a^2 z^4 y - 3\,a^5 z^2 + 3\,a^2 z^2 - 3\,a^2 z$$

$$\frac{x}{y}\left(\frac{a}{b} + \frac{c}{d} + \frac{e}{f}\right) = \frac{x}{y} \times \frac{a}{b} + \frac{x}{y} \times \frac{c}{d} + \frac{x}{y} \times \frac{e}{f}$$

$$= \frac{a\,x}{b\,y} + \frac{c\,x}{d\,x} + \frac{e\,x}{f\,y}$$

$$\frac{2\,m}{n}\left(3\,a\,b + \frac{m}{n}\right) = \frac{6\,a\,b\,m}{n} + \frac{2\,m^2}{n^2}$$

$$\frac{3\,a^2}{a} = 3\,a \qquad \frac{14\,a^2\,b}{2\,a\,b} = 7\,a \qquad \frac{210\,a^{10}\,b^{15}}{70\,a^5\,b^4} = 3\,a^5\,b^{11}$$

$$\frac{2\,a\,b}{2\,a\,c} = \frac{b}{c} \qquad \frac{x\,y}{z\,z} = \frac{y}{z} \qquad \frac{x^2}{x\,z} = \frac{x}{z} \qquad \frac{y^2}{y^3 x} = \frac{1}{y\,x}$$

$$\frac{4\,a^2\,b}{8\,a^3\,c} = \frac{b}{2\,a\,c} \qquad \frac{12\,a}{8\,b} = \frac{3\,a}{2\,b} \qquad \frac{13\,a}{13\,a^2} = \frac{1}{a} \qquad \frac{a}{a} = 1$$

$$\frac{6\,a\,b\,c}{9\,a^2\,b^2} = \frac{2\,c}{3\,a\,b} \qquad \frac{x\,y^2\,z^3\,v^2}{x^2\,y^2\,z^2\,v^2} = \frac{z}{x} \qquad \frac{2\,a\,b\,c^2\,m}{8\,a^2\,b\,c^3\,m^6} = \frac{1}{4\,a\,c\,m^5}$$

$$\frac{v\,t}{2\,m^2\,v\,t} = \frac{1}{2\,m^2} \qquad \frac{p^7\,q^{14}}{3\,p^4\,q^{10}} = \frac{p^3}{3\,q^4} \qquad \frac{m^2\,a\,b^2}{m\,a\,b^3} = \frac{m}{b}$$

$$\frac{x\,y\,(x + y)^2}{x^2\,(x + y)^3} = \frac{y}{x\,(x + y)} \qquad \frac{(p - q)^4\,z}{f\,x\,(p - q)^2} = \frac{(p - q)^2\,z}{f\,x}$$

$$\frac{m\,a + n\,a}{a} = \frac{m\,a}{a} + \frac{n\,a}{a} = m + n$$

$$\frac{m\,a^2 - n\,a}{a} = \frac{m\,a^2}{a} - \frac{n\,a}{a} = m\,a - n$$

$$\frac{6\,a\,v^2 + 3\,a\,v^3 - 12\,a^3\,v}{3\,a\,v} = 2\,v + v^2 - 4\,a^2$$

$$\frac{12\,x^4 - 9\,x^3 + 3\,x^2 - 3\,a\,x^2}{3\,x^2} = 4\,x^2 - 3\,x + 1 - a$$

$$\frac{2\,m\,n^2 + 8\,a\,m\,n - 6\,a^2\,m^2\,n^2 - 2\,m\,n}{2\,m\,n} = n + 4\,a - 3\,a^2\,m\,n - 1$$

$$\frac{2\,a + 2}{4} = \frac{a + 1}{2} \qquad \frac{3\,x^2 - 6\,x}{3\,a\,x} = \frac{x - 2}{a} \qquad \frac{3}{3 + 3\,a} = \frac{1}{1 + a}$$

$$\frac{p^2 + 2\,p\,x}{p^2 + 2\,p\,y} = \frac{p + 2\,x}{p + 2\,y} \qquad \frac{a^2\,b - a\,b^2}{a^2\,b^2 + a\,b} = \frac{a - b}{a\,b + 1}$$

$$\frac{y^2 - y + 3\,a\,v\,y}{y - v\,y + 2\,a\,m\,y} = \frac{y - 1 + 3\,a\,v}{1 - v + 2\,a\,m} \qquad \frac{x^2}{x\,y + x} = \frac{x}{y + 1}$$

$$\frac{a\,x^4\,y^3 - a^2\,x^3\,y^2 + a^3\,x^2\,y}{2\,a\,x^5\,y^4 + 3\,a^2\,x^4\,y^3 - 2\,a^3\,x^3\,y^2} = \frac{x^3\,y^2 - a\,x^2\,y + a^2\,x}{2\,x^4\,y^3 + 3\,a\,x^3\,y^2 - 2\,a^2\,x^2\,y}$$

$$\frac{\frac{1}{2}\,a + \frac{1}{3}\,b}{2\,a - \frac{1}{4}\,b} = \frac{(\frac{1}{2}\,a + \frac{1}{3}\,b)\,12}{(2\,a - \frac{1}{4}\,b)\,12} = \frac{6\,a + 4\,b}{24\,a - 3\,b}$$

$$\frac{a + \frac{b}{2}}{b + \frac{c}{2}} - \frac{\frac{1}{3}\,a}{\frac{1}{8}\,b} = \frac{2\,a + b}{2\,b + c} - \frac{8\,a}{3\,b}$$

$$\frac{a+\frac{b}{c}}{a-\frac{b}{c}} = \frac{ac+b}{ac-b} \qquad \frac{2ax+\frac{1}{2}}{\frac{x}{a}-\frac{1}{2}} = \frac{4a^2x+a}{2x-a}$$

$$\frac{\frac{1}{1+x}-x}{1-\frac{x}{1+x}} = \frac{1-x-x^2}{1+x-x} = 1-x-x^2$$

SECTION 13. *Separation of Factors.*

$$a+b = a\left(1+\frac{b}{a}\right) = b\left(\frac{a}{b}+1\right) = ab\left(\frac{1}{b}+\frac{1}{a}\right)$$

$$= c\left(\frac{a}{c}+\frac{b}{c}\right) = \frac{1}{c}\left(ac+bc\right) = \frac{a}{b}\left(b+\frac{b^2}{a}\right)$$

When any expression A is to be put in a form of which m shall be a factor, then $\frac{A}{m}$ must be the other factor: for

$$A = m \times \frac{A}{m}$$

$$b^2-4ac = b^2\left(1-\frac{4ac}{b^2}\right) = 4\left(\frac{b^2}{4}-ac\right) = \frac{a}{c}\left(\frac{cb^2}{a}-4c^2\right)$$

$$x^2-2xy = x\left(x-2y\right) = \frac{x}{y}\left(xy-2y^2\right) = \frac{x^2}{y^2}\left(y^2-\frac{2y^3}{x}\right)$$

$$= xy\left(\frac{x}{y}-2\right) = \frac{y^2}{x^2}\left(x^4-\frac{2x^3}{y}\right) = zx\left(\frac{x}{z}-\frac{2y}{z}\right)$$

$$ax+b = a\left(x+\frac{b}{a}\right) = x\left(a+\frac{b}{x}\right) = b\left(\frac{a}{b}x+1\right)$$

$$x^2+xy+y^2 = x^2\left(1+\frac{y}{x}+\frac{y^2}{x^2}\right) = y^2\left(\frac{x^2}{y^2}+\frac{x}{y}+1\right)$$

$$x^3-3x^2y+3xy^2-y^3 = x^3\left\{1-3\frac{y}{x}+3\frac{y^2}{x^2}-\frac{y^3}{x^3}\right\}$$

$$(a+b)^2-c = (a+b)\left(a+b-\frac{c}{a+b}\right) = (a+b)^2\left(1+\frac{c}{(a+b)^2}\right)$$

$$\frac{x+y}{x-y} = \frac{y\left(\frac{x}{y}+1\right)}{x\left(1-\frac{y}{x}\right)} = \frac{y}{x}\,\frac{\frac{x}{y}+1}{1-\frac{y}{x}} = \frac{y}{x}\,\frac{\frac{x^2}{y}+x}{x-y}$$

$$ab+bc+ca = a\left(b+c+\frac{bc}{a}\right) = ab\left(1+\frac{c}{a}+\frac{c}{b}\right)$$

$$= abc\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right) = bc\left(\frac{a}{c}+1+\frac{a}{b}\right)$$

$$x+y-z = \overline{z-y}\left(\frac{x}{z-y}-1\right) = \overline{x+y}\left(1-\frac{z}{x+y}\right)$$

$$= \overline{x+y+z}\left(1-\frac{2z}{x+y+z}\right) = \overline{z-x}\left(\frac{y}{z-x}-1\right)$$

$$p + q = \overline{p - z}\left(1 + \frac{z + q}{p - z}\right) = \overline{p + z}\left(1 - \frac{z - q}{p + z}\right)$$

$$= \overline{p + z}\left(1 + \frac{q - z}{p + z}\right) = \overline{p + q - z}\left(1 + \frac{z}{p + q - z}\right)$$

$$a^2 + 2\,a\,b = \overline{a^2 + b^2}\left(1 + \frac{2\,a - b}{a^2 + b^2}b\right) = \overline{a^2 - b^2}\left(1 + \frac{(2\,a + b)\,b}{a^2 - b^2}\right)$$

$$\begin{array}{ll}
a^2 + 2\,a\,b + b^2 & = a^2 + a\,b + a\,b + b^2 = (a + b)\,(a + b) \\
a^2 - 2\,a\,b + b^2 & = a^2 - a\,b - (a\,b - b^2) = (a - b)\,(a - b) \\
a^2 - b^2 & = a^2 - a\,b + a\,b - b^2 = (a - b)\,(a + b) \\
a^3 + b^3 & = a^3 + a^2\,b - a^2\,b - a b^2 + a\,b^2 + b^3 \\
 & = \overline{a + b}\,(a^2 - a\,b + b^2) \\
a^3 - b^3 & = a^3 - a^2 b + a^2 b - a\,b^2 + a\,b^2 - b^3 \\
 & = \overline{a - b}\,(a^2 + a\,b + b^2)
\end{array}$$

SECTION 14. *Examples of Multiplication.*

$$\begin{array}{r}
x + 1 \\
x - 3 \\
\hline
x^2 + x \qquad \\
- 3\,x - 3 \\
\hline
x^2 - 2\,x - 3
\end{array}
\qquad
\begin{array}{r}
x - 1 \\
x - 3 \\
\hline
x^2 - x \qquad \\
- 3\,x + 3 \\
\hline
x^2 - 4\,x + 3
\end{array}
\qquad
\begin{array}{r}
x - 1 \\
x + 3 \\
\hline
x^2 - x \qquad \\
+ 3\,x - 3 \\
\hline
x^2 + 2\,x - 3
\end{array}$$

$$\begin{array}{l}
x^2 - 2\,a\,x + 3\,a^2 \\
x^2 + 3\,b\,x - 2\,b^2 \\
\hline
x^4 - 2\,a\,x^3 + 3\,a^2\,x^2 \\
\quad + 3\,b\,x^3 - 6\,a\,b\,x^2 + 9\,a^2\,b\,x \\
\qquad\qquad - 2\,b^2\,x^2 + 4\,a\,b^2\,x - 6\,a^2\,b^2 \\
\hline
x^4 - (2\,a - 3\,b)\,x^3 + (3\,a^2 - 6\,a\,b - 2\,b^2)\,x^2 + a\,b\,(9\,a + 4\,b)\,x - 6\,a^2\,b^2
\end{array}$$

$$\begin{array}{r}
a + b \\
a + b \\
\hline
a^2 + a\,b \qquad \\
+ a\,b + b^2 \\
\hline
a^2 + 2\,a b + b^2
\end{array}
\qquad
\begin{array}{r}
a - b \\
a - b \\
\hline
a^2 - a\,b \qquad \\
- a\,b + b^2 \\
\hline
a^2 - 2\,a b + b^2
\end{array}
\qquad
\begin{array}{r}
a + b \\
a - b \\
\hline
a^2 + a\,b \qquad \\
- a\,b - b^2 \\
\hline
a^2 - b^2
\end{array}$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,a\,b + 2\,b\,c + 2\,c\,a$$

$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2\,a\,b - 2\,b\,c - 2\,c\,a$$

$$(a - b - c)^2 = a^2 + b^2 + c^2 - 2\,a\,b + 2\,b\,c - 2\,c\,a$$

$$(7\,a\,x + 12\,b\,x^2)^2 = 49\,a^2\,x^2 + 168\,a\,b\,x^3 + 144\,b^2\,x^4$$

$$\left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2 \qquad \left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2$$

$$(a + b + c)\,(a + b - c) = a^2 + b^2 - c^2 + 2\,a\,b$$

If $P = b^2 - 4\,a\,c$, $Q = b\,d - 2\,a\,e$, $R = d^2 - 4\,a\,f$

$$Q^2 - P\,R = 4\,a\,(b^2 - 4\,a\,c)\left(\frac{c\,d^2 + a\,e^2 - b\,d\,e}{b^2 - 4\,a\,c} + f\right)$$

and $(b\,x + d)^2 - 4\,a\,(c\,x^2 + e\,x + f) = P\,x^2 + 2\,Q\,x + R$

$$(2\,a\,x + b)^2 + 4\,a\,c - b^2 = 4\,a\,(a\,x^2 + b\,x + c)$$

$$(m^2 - n^2)^2 + 4\,m^2 n^2 = (m^2 + n^2)^2$$

$$\frac{(a c - q^2)\,(a b - r^2) - (q r - a p)^2}{a} = a\,bc + 2\,p\,q\,r - a\,p^2 - b\,q^2 - c\,r^2$$

$$\text{If} \quad \left.\begin{array}{l} A = b\,c - p^2 \\ B = c\,a - q^2 \\ C = a\,b - r^2 \end{array}\right\} \text{ and } \left\{\begin{array}{l} P = q\,r - a\,p \\ Q = r\,p - b\,q \\ R = pq - c\,r \end{array}\right.$$

then the following six quantities are equal: show this by multiplication.

$$\frac{BC - P^2}{a} \quad \frac{CA - Q^2}{b} \quad \frac{AB - R^2}{c} \quad \frac{QR - AP}{p} \quad \frac{RP - BQ}{q} \quad \frac{PQ - CR}{r}$$

We give the formation of one of them at length.

$$\begin{array}{ll} Q = r p - b\,q & A = b\,c - p^2 \\ R = p q - c\,r & P = q\,r - a\,p \\ QR = \overline{p^2 q r - p b q^2} - c p r^2 + b c q r & AP = \overline{b c q r - p^2 q r - a b c p + a p^3} \\ AP = b c q r - p^2 q r - a b c p + a p^3 & \end{array}$$

$$QR - AP = \overline{a b c p + 2 p^2 q r - a p^3 - b p}\,q^2 - c\,p\,r^2$$

$$\frac{QR - AP}{p} = a\,b\,c + 2\,p\,q\,r - a\,p^2 - b\,q^2 - c\,r^2$$

$$(a^2 + b^2 + c^2)\,(p^2 + q^2 + r^2) - (a p + b\,q + c\,r)^2$$
$$= (a\,q - b\,p)^2 + (b\,r - c\,q)^2 + (c\,p - a\,r)^2$$

$$(a\,p + b\,q + c\,r)\,(a\,s + b\,t + c\,v) - (a^2 + b^2 + c^2)\,(p\,s + q\,t + r\,v)$$
$$= (c\,q - b\,r)\,(b\,v - c\,t) + (a\,r - c\,p)\,(c\,s - a\,v) + (b\,p - a\,q)\,(a\,t - b\,s)$$

$$\text{If} \quad P = a + b + c \qquad Q = a + b - c$$
$$R = b + c - a \qquad S = c + a - b$$

$$PQRS = 2\,a^2 b^2 + 2\,b^2 c^2 + 2\,c^2 a^2 - a^4 - b^4 - c^4$$

$$QR + RS + SQ = 2\,b\,c + 2\,c\,a + 2\,a\,b - a^2 - b^2 - c^2$$

$$P^2 + Q^2 + R^2 + S^2 = 4\,a^2 + 4\,b^2 + 4\,c^2$$

$$\begin{array}{lll} QR^2 + RS^2 + SQ^2 = & a^3 + 6\,a\,b\,c & - 5\,a\,b^2 + 3\,b\,a^2 \\ & + b^3 & - 5\,b\,c^2 + 3\,c\,b^2 \\ & + c^3 & - 5\,c\,a^2 + 3\,a\,c^2 \end{array}$$

$$(a^2 + a\,b + b^2)\,(a - b) = a^3 - b^3$$
$$(a^3 + a^2 b + a\,b^2 + b^3)\,(a - b) = a^4 - b^4$$
$$(a^4 + a^3 b + a^2 b^2 + a\,b^3 + b^4)\,(a - b) = a^5 - b^5$$
$$(a^2 - a\,b + b^2)\,(a + b) = a^3 + b^3$$
$$(a^3 - a^2 b + a\,b^2 - b^3)\,(a + b) = a^4 - b^4$$
$$(a^4 - a^3 b + a^2 b^2 - a\,b^3 + b^4)\,(a + b) = a^5 + b^5$$
$$(x - a)\,(x - b) = x^2 - (a + b)\,x + a\,b$$
$$(x + a)\,(x + b) = x^2 + (a + b)\,x + a\,b$$

Examples of Division, when the divisor has more than one term, are of little or no use in the elementary part of algebra.

Section 15. *Operations on Fractions.*

The leading rules of arithmetic, which relate to simple fractions, are embodied in the following formulæ :—

$$\frac{a}{b} = \frac{m\,a}{m\,b}, \qquad a + \frac{b}{c} = \frac{a\,c + b}{c}, \qquad a - \frac{b}{c} = \frac{a\,c - b}{c},$$

$$\frac{b}{c} - a = \frac{b - a\,c}{c}, \qquad \frac{a}{x} + \frac{b}{x} = \frac{a + b}{x}, \qquad \frac{a}{x} - \frac{b}{x} = \frac{a - b}{x},$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a\,d + b\,c}{b\,d}, \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{b\,d}, \qquad \frac{a}{b} \times x = \frac{a\,x}{b},$$

$$\frac{a}{b} \div x = \frac{a}{b\,x}, \qquad \frac{a}{b} \times \frac{c}{d} = \frac{a\,c}{b\,d}, \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a\,d}{c\,b},$$

$$\frac{a}{b} \times x = \frac{a}{b \div x}, \qquad \frac{a}{b} \div x = \frac{a \div x}{b}.$$

The reduction of fractions which have fractional numerators and denominators, is exemplified in the following formulæ:—

$$\frac{\frac{a}{b}}{\frac{x}{y}} = \frac{a\,y}{b\,x} \qquad \frac{\frac{a}{b} - \frac{c}{d}}{\frac{a}{d} + \frac{b}{c}} = \frac{(a\,d - b\,c)\,c}{(a\,c + b\,d)\,b} \qquad \frac{p + \frac{a}{q}}{r - \frac{b}{s}} = \frac{p\,q\,s + a\,s}{q\,r\,s - b\,q}$$

$$\frac{\frac{m}{n} - \frac{x}{y}}{\frac{p}{q} - \frac{a}{b}} = \frac{m\,b\,q\,y - b\,n\,q\,x}{b\,n\,p\,y - a\,n\,q\,y} \qquad \frac{\frac{a}{b}}{\frac{p}{q}} \times \frac{\frac{m}{n}}{\frac{x}{y}} = \frac{a\,m\,q\,y}{b\,n\,p\,x}$$

$$\frac{a}{b} + \frac{m}{n} + \frac{p}{q} - \frac{x}{y} = \frac{a\,n\,q\,y + b\,m\,q\,y + b\,n\,p\,y - b\,n\,q\,x}{b\,n\,q\,y}$$

$$x + \frac{1}{x} = \frac{x^2 + 1}{x} \qquad x - \frac{1}{x} = \frac{x^2 - 1}{x} \qquad \frac{1}{x} - \frac{1}{y} = \frac{y - x}{x\,y}$$

$$\frac{a}{q\,b} + \frac{x}{c\,q} = \frac{a\,c + b\,x}{b\,c\,q} \qquad \frac{x}{a} - \frac{a}{x} = \frac{x^2 - a^2}{a\,x} \qquad \frac{1 - x}{x} \times \frac{1 + x}{x^2} = \frac{1 - x^2}{x^3}$$

$$\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} = \frac{x^2 + x - 1}{x^3} \qquad \frac{y}{x} + \frac{y^2}{x^2} - \frac{y^3}{x^3} = \frac{x^2\,y + x\,y^2 - y^3}{x^3}$$

$$\frac{a}{1 - x} + \frac{a}{x} = \frac{a}{x - x^2} \qquad \frac{p + q}{p - q} + \frac{p}{q} = \frac{p^2 + q^2}{p\,q - q^2}$$

$$\frac{a + b}{a - b} + \frac{a - b}{a + b} = \frac{2\,a^2 + 2\,b^2}{a^2 - b^2} \qquad \frac{3\,x}{x - 1} - \frac{2}{x} = \frac{3\,x^2 - 2\,x + 2}{x^2 - x}$$

$$n\frac{n - 1}{2} = \frac{n^2 - n}{2} \qquad n.\frac{n - 1}{2} \,.\, \frac{n - 2}{3} = \frac{n^3 - 3\,n^2 + 2\,n}{6}$$

$$n.\,\frac{n - 1}{2} \,.\, \frac{n - 2}{3}\;\frac{n - 3}{4} = \frac{n^4 - 6\,n^3 + 11\,n^2 - 6\,n}{24}$$

$$n.\,\frac{n + 1}{2} + n + 1 = \overline{n + 1}\,\frac{n + 2}{2}$$

$$n.\,\frac{n + 1}{2} \,.\, \frac{2\,n + 1}{3} + \overline{n + 1}^{\,2} = \overline{n + 1}\,\frac{n + 2}{2} \,.\, \frac{2\,n + 3}{3}$$

$$\frac{x + a}{x + b} - \frac{x + c}{x + e} = \frac{(a + e - c - b)\,x + a\,e - b\,c}{x^2 + (b + e)\,x + e\,b}$$

$$\frac{x + a}{x - b} - \frac{x - a}{x + b} = \frac{2\,(a + b)\,x}{x^2 - b^2}$$

$$\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=\frac{3x^2+2(a+b+c)x+ab+bc+ca}{x^3+(a+b+c)x^2+(ab+bc+ca)x+abc}$$

$$\frac{ax-b}{a-bx}-\frac{x+1}{x-1}=\frac{(a+b)x^2-2ax-(a-b)}{(a+b)x-a-bx^2}$$

$$\frac{1+x}{1+\frac{1}{x}}=x \qquad \frac{\frac{1}{1-x}-\frac{1}{1+x}}{\frac{x}{1-x}+\frac{1}{1+x}}=\frac{2x}{x^2+1} \qquad \frac{3+\frac{1}{x}}{2-\frac{2}{x}}=\frac{3x+1}{2x-2}$$

We shall find further examples of these operations in the next section.

Section 16. *Solution of Equations of the first degree, or Simple Equations.*

Rule 1. Both sides of an equation may be increased, diminished, multiplied, or divided by the same quantity.

Rule 2. Any term of an equation may be removed from one side of the equation to the other, if the sign be changed from + to −, or from − to +.

Let $x-2=7-x$ What is the value of x?

$$x+x=7+2$$
$$2x=9$$
$$x=\frac{9}{2}=4\frac{1}{2}$$

Verification $4\frac{1}{2}-2=2\frac{1}{2}\quad 7-4\frac{1}{2}=2\frac{1}{2}$

Let $x-a=b-x$

$$2x=a+b$$
$$x=\tfrac{1}{2}(a+b)$$

Verification $\tfrac{1}{2}(a+b)-a=\tfrac{1}{2}(b-a)$

$$b-\tfrac{1}{2}(a+b)=\tfrac{1}{2}(b-a)$$

Let $3x-4=12x-9$

$$9-4=12x-3x$$
$$5=9x$$
$$\frac{5}{9}=x$$

Verification $3\times\frac{5}{9}-4=-\frac{21}{9}$

$$12\times\frac{5}{9}-9=-\frac{21}{9}$$

Before reading the theory of the negative sign, the student must consider this negative result as showing that the equation should have been written

$$4-3x=9-12x$$

which gives $x=\frac{5}{9}$ and can be verified arithmetically.

Let $ax-b=cx-d$

$$ax-cx=b-d$$
$$(a-c)x=b-d$$
$$x=\frac{b-d}{a-c}$$

Verification $a\,\dfrac{b-d}{a-c} - b = \dfrac{ab - ad - ab + bc}{a-c}$

$$= \frac{bc - ad}{a-c}$$

$$c\,\frac{b-d}{a-c} - d = \frac{bc - ad}{a-c}$$

Let $\dfrac{x}{2} + \dfrac{x}{3} = 4 - \dfrac{x}{4}$

$$12\,\frac{x}{2} + 12\,\frac{x}{3} = 48 - 12\,\frac{x}{4}$$

$$6x + 4x = 48 - 3x$$

$$6x + 4x + 3x = 48$$

$$13x = 48 \quad x = 3\,\frac{9}{13} \text{ or } \frac{48}{13}$$

Verification $\dfrac{x}{2} + \dfrac{x}{3} = \dfrac{40}{13} = 4 - \dfrac{x}{4}$

Let $\dfrac{x}{a} + \dfrac{x}{b} = c - \dfrac{x}{d}$

$$abd\,\frac{x}{a} + abd\,\frac{x}{b} = abcd - abd\,\frac{x}{d}$$

$$bdx + adx = abcd - abx$$

$$bdx + adx + abx = abcd$$

$$(bd + ad + ab)x = abcd$$

$$x = \frac{abcd}{bd + ad + ab}$$

Verification $\dfrac{x}{a} = \dfrac{bcd}{bd + ad + ab} \qquad \dfrac{x}{b} = \dfrac{acd}{bd + ad + ab}$

$$\frac{x}{a} + \frac{x}{b} = \frac{bcd + acd}{bd + ad + ab} = c - \frac{x}{d}$$

Let $\dfrac{x-1}{3} - \dfrac{x-4}{5} = 1 - \dfrac{x}{6}$

Multiply by 30, the least common multiple of 3, 5, and 6.

$$10x - 10 - (6x - 24) = 30 - 5x$$

$$10x - 10 - 6x + 24 = 30 - 5x$$

$$9x = 16 \quad x = \frac{16}{9}$$

Verification $\dfrac{x-1}{3} = \dfrac{7}{27}$, $\dfrac{x-4}{5} = -\dfrac{4}{9}$, $1 - \dfrac{x}{6} = \dfrac{19}{27}$, $\dfrac{7}{27} - \left(-\dfrac{4}{9}\right)$

$$= \frac{7}{7} + \frac{4}{9} = \frac{19}{27}.$$

The negative sign in the verification shows that the equation should have been written

$$\frac{x-1}{3} + \frac{4-x}{5} = 1 - \frac{x}{6}$$

which gives the same value of x.

Let $\dfrac{x-a}{b} - \dfrac{x-c}{d} = e - \dfrac{x}{f}$

Multiply both sides by $b\,d\,f$.

$$d\,f\,x - a\,d\,f - b\,f\,(x - c) = b\,e\,d\,f - b\,d\,x$$

$$d\,f\,x - a\,d\,f - b\,f\,x + b\,f\,c = b\,e\,d\,f - b\,d\,x$$

$$d\,f\,x - b\,f\,x + b\,d\,x = b\,e\,d\,f + a\,d\,f - b\,f\,c$$

$$x = \frac{b\,e\,d\,f + a\,d\,f - b\,c\,f}{d\,f - b\,f + b\,d}$$

Verification $$\frac{x - a}{b} = \frac{e\,d\,f - c\,f + a\,f - a\,d}{d\,f - b\,f + b\,d}$$

$$\frac{x - c}{d} = \frac{b\,e\,f + a\,f - c\,f - b\,c}{d\,f - b\,f + b\,d}$$

$$\frac{x - a}{b} - \frac{x - c}{d} = \frac{e\,d\,f - b\,e\,f - a\,d + b\,c}{d\,f - b\,f + b\,d}$$

$$= e - \frac{b\,e\,d + a\,d - b\,c}{d\,f - b\,f + b\,d}$$

Let $$\frac{a\,x - b}{a} - \frac{b\,x - a}{b} = 1 + \frac{x}{a}$$

Multiply both sides by $a\,b$.

$$a\,b\,x - b^2 - a\,b\,x + a^2 = a\,b + b\,x$$

$$b\,x = a^2 - b^2 - a\,b \qquad x = \frac{a^2}{b} - b - a$$

$$\frac{a\,x - b}{a} = \frac{a^2}{b} - b - a - \frac{b}{a}$$

$$\frac{b\,x - a}{b} = \frac{a^2}{b} - b - a - \frac{a}{b}$$

$$\frac{a\,x - b}{a} - \frac{b\,x - a}{b} = \frac{a}{b} - \frac{b}{a}$$

$$1 + \frac{x}{a} = 1 + \frac{a}{b} - \frac{b}{a} - 1 = \frac{a}{b} - \frac{b}{a}$$

Let $$\frac{x - a}{b} + \frac{x - b}{a} + \frac{x - a\,b}{a\,b} = 1$$

$$a\,x - a^2 + b\,x - b^2 + x - a\,b = a\,b$$

$$x = \frac{a^2 + 2\,a\,b + b^2}{a + b + 1}$$

$$\frac{x - a}{b} = \frac{a + b - \frac{a}{b}}{a + b + 1} \qquad \frac{x - b}{a} = \frac{a + b - \frac{b}{a}}{a + b + 1}$$

$$\frac{x - a\,b}{a\,b} = \frac{x}{a\,b} - 1 = \frac{\frac{a}{b} + 2 + \frac{b}{a}}{a + b + 1} - 1$$

$$\frac{x - a}{b} + \frac{x - b}{a} + \frac{x - a\,b}{a\,b} = \frac{2\,a + 2\,b + 2}{a + b + 1} - 1 = 1$$

In the succeeding examples, we give only the solution, or value of x, and the value which each side of the equation assumes when the value of x is substituted.

Let $x + \frac{x - a}{b} - \frac{c + x}{a b} = d - \frac{x - e}{a}$

The value of x will be found to be

$$\frac{a^2 + a b d + b e + c}{a b + a + b - 1}$$

which will give

$$\frac{x - a}{b} = \frac{a b d - a^2 b - a b + a + b e + c}{b (a b + a + b - 1)}$$

$$\frac{c + x}{a b} = \frac{a^2 + a b d + a b c + a c + b c + b e}{a b (a b + a + b - 1)}$$

$$\frac{x - e}{a} = \frac{a^2 + a b d - a b e - a e + c + e}{a (a b + a + b - 1)}$$

and the common value of the first and second sides of the equation is

$$\frac{a^2 (b d + d - 1) + a (b e + e - d) - (c + e)}{a (a b + a + b - 1)}$$

$$\text{Let} \quad \frac{\frac{a}{c} x + a}{b} + \frac{x - \frac{2x - a}{b}}{c} = \frac{a x - a^2}{b c}$$

the value of x, and the value of each side of the equation, are as follows:—

$$a \frac{a + c + 1}{2 - b} \qquad \frac{a^2}{bc} \qquad \frac{a + b + c - 1}{2 - b}$$

Let $(a + x)(b + x) = (c + x) d + x)$

This appears at first sight to be an equation of the second degree, but it is not so, owing to the occurrence of x^2 on both sides of the equation.

$$x = \frac{a b - c d}{c + d - a - b}$$

$$(a + x)(b + x) = (c + x)(d + x) = \frac{\overline{c - a}.\ \overline{c - b}.\ \overline{a - d}.\ \overline{b - d}}{c + d - a - b}$$

In verifying equations, and in algebraical operations generally, remember that addition and subtraction seldom or never take place in denominators, so long as they remain denominators, and only occur when, by the rules for fractional operations, denominators have been incorporated with numerators. Also remember that, except for the purpose of incorporating two expressions, the indication of the multiplication is simpler than the actual result. For instance, we form a better idea of $a + b$ taken $a + b$ times, which involves one multiplication, and two additions of the most simple character, than of its equal $a^2 + 2 a b + b^2$ involving two additions and three multiplications. Hence, the most convenient plan will be, to let the indications of multiplication remain in denominators, without performing the operations, until, in the course of the process, those denominators become numerators or factors of numerators. Thus,

$$\frac{(a + b)(c + d)}{(e + f)(g + h)}$$

should be written

$$\frac{a c + a d + b c + b d}{(e + f)(g + h)}$$

When a denominator occurs which will be written several times in the course of a process, the better way will be to substitute a single letter for it, restoring the original denominator only when the chosen letter comes to appear in a numerator. The following example is worked at full length in every

respect, containing everything which the student would find it necessary to write :—

$$\frac{x-b}{a} - \frac{x-a}{b} = x - \frac{(a+b)x}{ab}$$

$$bx - b^2 - a(x-a) = abx - (a+b)x$$

$$bx - b^2 - ax + a^2 = abx - ax - bx$$

$$a^2 - b^2 = abx - 2bx$$

$$x = \frac{a^2 - b^2}{ab - 2b}. \quad \text{Let } ab - 2b = \mathrm{P}$$

$$x - b = \frac{a^2 - b^2}{\mathrm{P}} - b = \frac{a^2 - b^2 - ab^2 + 2b^2}{\mathrm{P}} = \frac{a^2 + b^2 - ab^2}{\mathrm{P}}$$

$$x - a = \frac{a^2 - b^2}{\mathrm{P}} - a = \frac{a^2 - b^2 - a^2 b + 2ab}{\mathrm{P}}$$

$$\frac{x-b}{a} - \frac{x-a}{b} = \frac{a^2 + b^2 - ab^2}{a\mathrm{P}} - \frac{a^2 - b^2 - a^2 b + 2ab}{b\mathrm{P}}$$

$$= \frac{a^2 b + b^3 - ab^3}{ab\mathrm{P}} - \frac{a^3 - ab^2 - a^3 b + 2a^2 b}{ab\mathrm{P}}$$

$$= \frac{a^2 b + b^3 - ab^3 - a^3 + ab^2 + a^3 b - 2a^2 b}{ab\mathrm{P}}$$

$$= \frac{b^3 - ab^3 - a^3 + ab^2 + a^3 b - a^2 b}{ab\mathrm{P}} \quad \text{(1st side)}$$

Again, $$x - \frac{(a+b)x}{ab} = x\left(1 - \frac{a+b}{ab}\right) = x \cdot \frac{ab - a - b}{ab}$$

$$= \frac{a^2 - b^2}{\mathrm{P}} \cdot \frac{ab - a - b}{ab} = \frac{a^3 b - ab^3 - a^3 + ab^2 - a^2 b + b^3}{ab\mathrm{P}} \quad \text{(2nd side)}$$

Multiplications should always, unless where they are more than one and complicated, be performed as in the last line of the above, without arranging the multiplier and multiplicand under each other.

———

When it is evident that the numerator of a result is the product of certain factors, that result should be written both with the multiplications indicated and developed. As in the following example :—

$$\frac{a^2 x}{b - c} - cd = bx - ac$$

$$x = \frac{c(b-c)(d-a)}{a^2 - b^2 + bc} = \frac{cbd - c^2 d - abc + ac^2}{a^2 - b^2 + bc}$$

the sides of the equation become

$$\frac{b^2 cd - bc^2 d - a^2 c}{a^2 - b^2 + bc}$$

———

In the following examples we shall discuss the cases in which are presented what have been called the *critical* values of the solutions, namely, the forms 0, $\frac{\mathrm{A}}{0}$, and $\frac{0}{0}$, which are treated in all algebraical works, and which are here considered only as connected with the following theorems, which the student is to verify, each as it occurs, upon the examples given.

1. $x = 0$ indicates an equation of the form $ax + b = cx + b$, in which a and c are not equal: or an equation which may be reduced to that form.

2. $x = \frac{\mathrm{A}}{0}$, indicates an equation which either has or may be reduced to, the form $ax + b = ax + c$ in which b and c are not equal; and which cannot be true for any value of x.

3. $x = \frac{0}{0}$, indicates an equation which either has or may be reduced to, the form $ax + b = ax + b$, which is true for all values of x.

$$\frac{x-c}{a} - \frac{x-b}{c} = \frac{p\,x}{c} + \frac{q}{a}$$

$$x = \frac{c\,q + c^2 - a\,b}{c - a - a\,p}$$

The sides of the equation become

$$\frac{a\,c^2\,p - a^2\,b\,p + c^2\,q - a\,c\,q}{a\,c\,(c - a - a\,p)}$$

If $c\,q + c^2 - a\,b = 0$, that is, if $q = \frac{a\,b - c^2}{c}$, the solution becomes $x = 0$, unless it happen that we have also $c - a - a\,p = 0$, in which case it takes the form $\frac{0}{0}$. Let us suppose the first only, and not the second. The equation itself is the same as

$$\left(\frac{1}{a} - \frac{1}{c}\right) x + \frac{b}{c} - \frac{c}{a} = \frac{p}{c}\,x + \frac{q}{a}$$

But if $q = \frac{a\,b - c^2}{c} = \frac{a\,b}{c} - c$;

then $\frac{q}{a} = \frac{b}{c} - \frac{c}{a}$,

which call B. Then the equation is

$$\left(\frac{1}{a} - \frac{1}{c}\right) x + \text{B} = \frac{p}{c}\,x + \text{B}$$

which is in the form given in the first theorem.

If at the same time $c - a - a\,p = 0$,

$$p = \frac{a - c}{a} = 1 - \frac{c}{a},$$

then $\frac{p}{c} = \frac{1}{c} - \frac{1}{a}$, which call A.

The equation is $\text{A}\,x + \text{B} = \text{A}\,x + \text{B}$, which has the form given in the third theorem.

If the latter only be true, the solution has the form $\frac{c\,q^2 + c^2 - a\,b}{0}$, and the equation itself corresponding to this case is (A meaning as before)

$$\text{A}\,x + \frac{b}{c} - \frac{c}{a} = \text{A}\,x + \frac{q}{a}$$

which has the form given in the second theorem.

The student should now go through the various examples which have been given, and should show that the same suppositions which reduce the value of x to either of the forms

$$x = 0 \qquad x = \frac{1}{0} \qquad x = \frac{0}{0}.$$

will, when applied to the equation itself, reduce it to the form

$$\text{A}\,x + \text{B} = \text{C}\,x + \text{B} \qquad \text{A}\,x + \text{B} = \text{A}\,x + \text{D} \qquad \text{A}\,x + \text{B} = \text{A}\,x + \text{B}$$

Equations should also be solved, deferring all reductions until the result has been obtained in the form of a complex fraction, as follows. The equation

$$\frac{x}{a} - \frac{1}{c}\left(b - \frac{p\,x}{a}\right) = \frac{a\,c - x}{a + c}$$

or $\frac{x}{a} - \frac{b}{c} + \frac{p\,x}{a\,c} = \frac{a\,c}{a+c} - \frac{x}{a+c}$

or $\frac{x}{a} + \frac{p\,x}{a\,c} + \frac{x}{a+c} = \frac{a\,c}{a+c} + \frac{b}{c}$

gives $\left(\frac{1}{a} + \frac{p}{a\,c} + \frac{1}{a+c}\right) x = \frac{a\,c}{a+c} + \frac{b}{c}$

$$x = \frac{\dfrac{a\,c}{a+c} + \dfrac{b}{c}}{\dfrac{1}{a} + \dfrac{p}{a\,c} + \dfrac{1}{a+c}}$$

This should now be reduced by multiplying both the numerator and denominator by $a\,c\,(a + c)$, which gives

$$x = \frac{a^2\,c^2 + a^2\,b + a\,b\,c}{(c + p)\,(a + c) + ac}$$

Let $\frac{a\,x - b}{c} + \frac{p - \frac{a - b\,x}{a\,b}}{a + b} = 1 - \frac{b - x}{a}$

or $\frac{a\,x}{c} - \frac{b}{c} + \frac{p}{a+b} - \frac{1}{b\,(a+b)} + \frac{x}{a\,(a+b)} = 1 - \frac{b}{a} + \frac{x}{a}$

$$x = \frac{1 - \frac{b}{a} + \frac{b}{c} - \frac{p}{a+b} + \frac{1}{b(a+b)}}{\frac{a}{c} + \frac{1}{a(a+b)} - \frac{1}{a}}$$

$$= \frac{a\,b\,c\,(a+b) - b^2 c\,(a+b) + a\,b^2 (a+b) - a\,b\,c\,p + a\,c}{a^2 b\,(a+b) + b\,c - b\,c\,(a+b)}$$

One of the things which the student must observe is, that an equation may be, in particular cases, intelligible in some forms and not in others, though the intelligible forms are direct deductions from the unintelligible. For instance suppose

$$\frac{x-a}{c} + b\,x = \frac{1-x}{e} \qquad (1)$$

$$e\,x - a\,e + b\,c\,e\,x = c - c\,x \qquad (2)$$

$$x = \frac{c + a\,e}{b\,c\,e + e + c} \qquad (3)$$

If we suppose $c = 0$, the equation assumes the unintelligible form

$$\frac{x-a}{0} + b\,x = \frac{1-x}{e}$$

while (2) assumes the form

$$e\,x - a\,e = 0, \qquad \text{or } x = a$$

which is also the result of the solution (3) in the case where $c = 0$. We have here nothing to do except with the following circumstance, that the rules for the solution of an equation give solutions to unintelligible as well as intelligible cases, which the student must connect together by means of the usual explanation of the former cases. At present, let all the examples be looked at, and let the following theorem be verified in each case.

When such suppositions are made as to the values of the letters representing known quantities as will make one denominator only equal to nothing, then the same suppositions applied to the solution will give such a value of the unknown quantity as makes the numerator of that denominator equal to nothing.

It is usual to accustom the student to the solution of various problems producing equations of the first degree: these are of no use whatever in themselves, but may be made to furnish illustrations of the several algebraical peculiarities of the results. To these we shall therefore proceed.

SECTION 17.—*Interpretation of the cases of a Problem producing Equations of the first degree.*

We shall solve one problem in general terms, that is, by expressing the known quantities by letters whose values are supposed to be given; and shall then proceed to inquire what particular values of the known quantities will give critical or other peculiar solutions, and what is the meaning of the problem in the cases thus formed.

It is supposed that the student is already acquainted with the usual method of treating the negative sign, and the following examples are for exercise, not for primary instruction.

Problem. There are two pieces of stuff, of l and l' yards in length; of these the owner sells the same number of yards at p and p' shillings a yard, and afterwards selling the remainders at q and q' shillings a yard, finds the same receipts from both pieces. What was the number of yards first sold of each?

Let x be that number of yards. Then

$$p\,x + q\,(l - x)$$

is received from the first piece, and

$$p'\,x + q'\,(l' - x)$$

from the second. The equation of these expressions requires that x should be

$$\frac{q\,l - q'\,l'}{(p' - q') - (p - q)}$$

$$p\,x + q\,(l - x) = \frac{q\,l\,(p' - q') - q'\,l'\,(p - q)}{(p' - q') - (p - q)} = p'\,x + q'\,(l' - x)$$

It is usual to write algebraical expressions in some form which shows symmetry, even where cases may occur in which combinations thus introduced are negative. The rules for the negative sign render such cases as manageable as any others.

Let us first suppose that $p' - q' = p - q$, or that the first and second prices of each stuff exceed each other by the same sum. The solution then takes the form

$$\frac{q\,l - q'\,l'}{0}$$

and the equation becomes $A = p' - q' = p - q$

$$A\,x + q\,l = A\,x + q'\,l'$$

which is never true when $q\,l$ differs from $q'\,l'$. Nevertheless, it may be made to approach as near as we please to the truth by taking x sufficiently great. For it must be remembered, that $x + a$ and $x + b$, for instance, are nearly to equality the greater x is taken. Thus, $1000 + 1$ and $1000 + 2$ are more nearly equal * than $4 + 1$ and $4 + 2$. But at first it cannot be supposed that x is greater than the least of l and l'. Consequently, keeping the literal meaning of the problem, it is here impossible. But we will now state another problem, of which the one just given is a particular case, and which leads to the same equation.

Problem. A man has two pieces of stuff, of l and l' yards in length; he engages to furnish the same number of yards of each at p and p' shillings a yard, and having made this bargain, he finds the prices in the market to be q and q' shillings a yard. He makes such new bargains as enable him to fulfil the first, and leave him without any stuff of either sort. He then finds his receipts (or deficits if he have lost) to be the same from both. What number of yards of each did he first engage to supply?

This is the problem in its most general terms. It meets the case that he may first have engaged to supply more than he has got of one or both; and also that in making up the stipulated quantities, he may be obliged to buy at such a price, that on the whole he will lose instead of gain.

Let us first take the case that he has engaged to supply more than he has of either (x yards). Then he has to go out and buy $x - l$ and $x - l'$ yards of the two sorts at q and q' shillings a yard. This costs him $q\,(x - l)$ and $q'\,(x - l')$ shillings, and he then sells his x yards of both sorts for $p\,x$ and $p'\,x$ shillings respectively. If the second be greater than the first he receives

$$p\,x - q\,(x - l) \text{ and } p'\,x - q'\,(x - l')$$

shillings for the two, and

$$p\,x - q\,(x - l) = p'\,x - q'\,(x - l')$$

by the problem: but if the second be less than the first, he loses

$$q\,(x - l) - p\,x \text{ and } q'\,(x - l') - p'\,x$$

shillings by the two, and

$$q\,(x - l) - p\,x = q'\,(x - l') - p'\,x$$

by the problem. Both these last equations give

$$x = \frac{q\,l - q'\,l'}{p' - q' - (p - q)}$$

the same as in the first case.

Let $l > l'$, and suppose he can furnish the stipulated quantity of the first stock, but not of the second, that is, x is less than l, and greater than l'. He has then by the problem to dispose of his remainder $l - x$ and to make up the deficiency $x - l'$, at q and q' shillings a yard. Consequently, he receives from the first $p\,x + q\,(l - x)$, and from the second he receives $p'\,x$ and has to pay $q'\,(x - l')$ which leaves him $p'\,x - q'\,(x - l')$. The first must be greater than the second, for by the problem he has the same (receipts or deficits) from both, and from the first he clearly receives; therefore he does so from the second. And the equation is

$$p\,x + q\,(l - x) = p'\,x - q'\,(x - l')$$

giving the same value of x as before.

Let us take another possible variety of the same question. He does not engage to *furnish*, but to *take*, the same number of yards of both, at p and p' shillings a yard. He then concludes such a bargain as rids him of his whole stock at q and q' shillings a yard, and finds his receipts or deficits the same from both. It is plain that he then first buys x yards of both for $p\,x$ and $p'\,x$ shillings, and sells his $l + x$ yards of the first, and $l' + x$ yards of the second at q and q' shillings a yard. If he gain by this, the equation is

$$q\,(l + x) - p\,x = q'\,(l' + x) - p'\,x;$$

if he lose, it is

$$p\,x - q\,(l + x) = p'\,x - q'\,(l' + x)$$

and both give

* For a complete explanation of this point the student may refer to the first chapter of the Treatise on the Differential Calculus, which contains nothing more than a student might here read.

$$x = \frac{q' l' - q l}{p' - q' - (p - q)} \text{ not as } \frac{q l - q' l}{p' - q' - (p - q)} \text{ before}$$

but these latter only differ in the sign of the numerator, that is, only differ in sign.

We shall now return to the general problem, and the case where $p' - q' = p - q$. Say that the lengths of the two pieces are 20 and 40 yards, the first and second prices of the first 10 and 8 shillings, those of the second 6 and 4 shillings, or

$$l = 20 \quad l' = 40 \quad p = 10 \quad p' = 8 \quad q = 6 \quad q' = 4$$

$$p' - q' = p - q = 4$$

If we suppose x yards (more than either 20 or 40) to be first bought, the equation is

$$10x - 6(x - 20) = 8x - 4(x - 40)$$

$$\text{or,} \quad 4x + 120 = 4x + 160$$

which cannot be, for the second side always exceeds the first by 40 (shillings). But this excess of 40 may evidently be made as small a part of the transaction as we please, by supposing x sufficiently great, and if two quantities be called nearly equal which have a very small difference when compared with their own magnitude, (which is the usual meaning of nearly equal,) then (Study of Mathematics, p. 41) this problem is said to be solved when x is infinite; that is, may be as nearly solved as we please by taking x sufficiently great.

If in the general problem we suppose $q l = q' l'$, and also $p' - q' = p - q$, the value of x takes the form $\frac{0}{0}$, and the equatinn becomes of the form $A x + B = A x + B$. *Any* value may be given to x. If in the preceding instance we suppose l' to be 30 instead of 40, all the rest remaining the same, the equation becomes

$$4x + 120 = 4x + 120$$

which is true of all values of x, or any quantity cut off or added to the stocks mentioned in the problem, satisfies the equation. The following are other cases of this problem:—

	l	l'	p	p'	q	q'	x	Receipts
I.	50	25	6	6	8	9	−175	750
II.	20	20	3	9	10	6	8	144
III.	70	8	3	8	10	5	66	238
IV.	64	36	1	9	9	16	0	576

The stndent must remember that the answers to this problem will not always be whole numbers, but that these cases have been so contrived, in order to avoid fractions, and render the point we are now considering the only difficulty.

I. The negative sign of the answer shows a diametrically opposite meaning to that which was supposed in the equation. We supposed that the owner began by engaging to *furnish* a quantity of each; the answer shows that the problem cannot be solved in that way, but must be solved by supposing him engaged to *buy* 175 yards of each, both at 6 shillings; or, if it seems more clear, the problem proposed is not possible, but the corresponding problem which supposes him to begin by buying *is* possible, and the quantity so bought must be 175 yards; this is an outlay of 175 × 6, or 1050 for each; but the stocks of 175 + 50 and 175 + 25, or 225 and 200 then in hand, sold at 8 and 9 shillings give 1800 and 1800 shillings, so that the receipts for each are 750 shillings.

II. Is altogether within the limits of the first supposition.

III. Shows that he will have to buy 58 yards of the second to make up the 66 yards which the answer to the problem shows he is engaged to furnish. This costs 58 × 5, or 290 shillings, and the whole 66 at 8 shillings yields 528, giving, as the balance of receipt, 238. Of the first stock he sells 66 yards at 3 shillings, yielding 198 shillings, and the remaining 4 at 10 shillings, yielding 238 shillings in all.

IV. Shows that he must not cut off any of either piece in the first bargain, for selling the remainders (which in this case are the wholes) at 9 and 16 shillings, he just gets the same from both.

Let the student reconsider this question again and again, taking additional examples, and explaining them in the preceding manner. In every subsequent question (particularly in geo-

metry) he must always be on the watch to explain the three forms, namely, negative quantity, $\frac{A}{0}$, and $\frac{0}{0}$.

1. When the value of an unknown quantity appears to be negative, look back at the problem, and consider the negative quantity as unmeaning and unexplained, until it has been shown that the problem requires the unknown quantity to be of the kind diametrically opposite to that which it was at first supposed to be.

2. Consider the form $\frac{A}{0}$ as unmeaning until it is shown that the greater the value given to the unknown quantity, the more nearly is a solution produced to the problem; then, and not before, use the abbreviated form of speech that the unknown quantity is infinite.

3. When $\frac{0}{0}$ *is the result of an equation of the first degree*, let it be clearly ascertained that any value of the unknown quantity is a solution of the problem. What it means as to an equation of the second degree, we shall afterwards explain.

Section 18. *On Equations which are required to be solved in whole Numbers.*

This subject, though of very little practical use, will tend to impress on the mind of the student some considerations connected with whole numbers which will make him more expert in common arithmetic.

The following theorems and definitions are necessary :—

1. A *prime* number is one which admits of no divisors, except 1 and itself: such as 7, 29, 31.

2. All italic letters in this section signify *whole* numbers, and Greek letters whole numbers or fractions.

3. All numbers divisible by a are contained in the formula $a\,b$. Thus, all numbers divisible by 6 are contained in the set $6 \times 1, 6 \times 2, 6 \times 3, 6 \times 4$, &c.

All numbers which divided by a leave a remainder c, are contained in $a\,b + c$. Thus, all the numbers which divided by 13 leave a remainder 4, are contained in the set $13 \times 1 + 4, 13 \times 2 + 4, 13 \times 3 + 4$, &c.

5. All numbers are either prime numbers, or are made by multiplying prime numbers together. 20 is 2.2.5 64 is 2.2.2.2.2.2, 2420 is 2.2.5.11.11, 2310 is 2.3.5.7.11, and so on. That is, every whole number may be represented by

$$a^m \times b^n \times c^q \times \ldots\ldots$$

where $a, b, c, \ldots$ are prime numbers, and m, n, q, &c. are whole numbers, prime or not.

6. No number can be *resolved into prime factors* (the process of the last) in two different ways. Thus, if $a\,b\,c = d\,e\,f$, it is impossible that all the six, $a\,b$, &c., can be prime.

7. If a divide b, the prime factors of a are all among the prime factors of b. Thus, 360 is $2^3.3^2.5$, and all the divisors of 360 are represented by a case of one or other of the sets

$$2^m.3^n.5,\ 2^m.5,\ 3^n.5,\ 2^m.3^n,\ 2^m,\ 3^n,\ 5$$

where m is not $>$ 3 n is not $>$ 2.

8. If a number represented by its prime factors, be $a^m\,b^n\,c^p$, the number of divisors does not at all depend upon a, b, and c, but entirely upon m, n, and p. Thus, 600 being $2^3.5^2.3^1$ has the same number of divisors as 360, or $2^3.3^2 5^1$. The number of divisors (unity and the number itself included) is $(m+1)$ $(n+1)$ $(p+1)$. Thus, 360 and 600 both have $(3+1)$ $(2+1)$ $(1+1)$ or 24 divisors. We exhibit those of 360.

1, 3, 3^2, 2, 2.3, 2.3^2, 2^2, $2^2.3$, $2^2.3^2$, 2^3, $2^3.3$, $2^3.3^2$, 5, 5.3, 5.3^2, 5.2, 5.2.3, $5.2.3^2$, 5.2^2, $5.2^2.3$, $5.2^2.3^2$, 5.2^2, $5.2^2.3$, $5.2^3.3^2$.

Give the reason why the number of divisors is here $(3+1)(2+1)(1+1)$, and try to give a general demonstration of the theorem.

It is required to solve the equation $11\,x + 7\,y = 108$ in whole numbers, or to find all the values of x and y? Or it is required to divide 108 into two numbers, one a multiple of 11, the other of 7, in as many different ways as possible. The process is as follows:— Since y is a whole number, or $\frac{1}{7}(108 - 11\,x)$, or $15 - x + \frac{1}{7}(3 - 4\,x)$, of which $15 - x$ is a whole number; it follows that $\frac{1}{7}(3 - 4\,x)$ is a whole number. Let it be t; then $3 - 4\,x = 7\,t$, or $x = -t + \frac{1}{4}(3 - 3\,t)$. This algebraic fraction is to be a whole number, let it be t', then $3 - 3\,t = 4\,t'$, or $t = 1 - t' - \frac{1}{3}t'$, so that $\frac{1}{3}\,t'$ must be a whole number. Let t'' be this whole number; then $t' = 3\,t''$ $t = 1 - 3\,t'' - t'' = 1 - 4\,t''$; $x = -1 + 4\,t'' + \frac{1}{4}(3 - 3 + 12\,t'') = 7\,t'' - 1$

$$y = 15 - 7t'' + 1 + \frac{1}{7}(3 - 28t'' + 4) = 17 - 11t''$$

$$11x + 7y = 77t'' - 11 + 119 - 77t'' = 108$$

This result being independent of t'', it would seem that we have thus an infinite number of answers. And so we have if we consider algebraical answers, that is, positive or negative whole numbers; but if we restrict ourselves to arithmetical whole numbers, that is, to *positive* algebraical answers, we must so assume t'' that $7t''$ is greater than 1, and $11t''$ less than 17. The only value of t'' which satisfies both conditions is 1, which gives $x = 6$, $y = 6$, or $11.6 + 7.6 = 108$, which is the only arithmetical answer.

As these questions are entirely for exercise in numbers, we shall give no rule, but only a method. It is required to find a set of numbers, which being divided by 4, 5, and 6, give remainders 1, 2, and 3. Let us consider the two first conditions: let x and y be the quotients (rejecting remainders) of this number when divided by 4 and 5; hence, since the remainders are to be 1 and 2, we must have $4x + 1$, and $5y + 2$, both equal to the required number. Consequently

$$4x + 1 = 5y + 2, \text{ or } 4x - 5y = 1$$

must be solved in whole numbers. We might find this as follows: we want to find a multiple of four which exceeds a multiple of five by 1. A case is evidently found where $x = 4$, $y = 3$; add to these any multiples of 5 and 4, such as $5t$ and $4t$, and we have

$$x = 4 + 5t \qquad y = 3 + 4t$$

$$4(4 + 5t) - 5(3 + 4t) = 1.$$

But the preceding, though it shows that these forms satisfy the conditions, does not show that they are the *only* forms which do so. To show this, proceed as before: we have $x = y + \frac{1}{4}(y + 1)$, therefore $y + 1$ must be of the form $4t$ or y of the form $4t - 1$. The preceding gave $4t+3$, which does not appear at first to be the same; but it is so in reality: for though $4t - 1$ is not the same as $4t + 3$, for any one value of t, yet it is plain that 1 less than *one* multiple of 4 is the same as 3 more than *another*. And the value of t may be any whole number. Taking $y = 4t + 3$, then the number $5y + 2$ is $20t + 17$, which formula contains all such numbers as, being divided by 4 and 5, will leave remainders 1 and 2. But this divided by 6 is to leave a remainder 3, by the last clause of the problem. Divide by 6 algebraically, giving $3t + 2 + \frac{1}{6}(2t + 5)$, but $3t + 2$ is a whole number, therefore the remainder 3 must come from $2t + 5$, which must therefore be of the form $6t' + 3$. But $2t + 5 = 6t' + 3$ gives $t = 3t' - 1$, and $20t + 17 = 60t' - 3$, which is the form required. For example, let $t' = 1$, then 57 satisfies the conditions; let $t' = 2$, then 117 also satisfies them; and so on. The student may make abundance of examples for himself, the test of correctness being obvious.

Theorem. Show that $ax + by = c$ cannot be solved in whole numbers if either two of the three, a, b, and c, have a common measure which the third has not.

We shall now suppose it required to solve the equation

$$x^2 + y^2 = z^2$$

in whole numbers.

Always in such a case endeavour to write the equation so that both sides may be reducible to a pair of factors. In the present instance write

$$x^2 = z^2 - y^2,$$

or $\quad x.x = (z - y)(z + y)$

1. Suppose neither of the three to be given. Assume

$$z + y = vx, \text{ then } z - y = \frac{x}{v}$$

$$z = \frac{x}{2}\left(v + \frac{1}{v}\right) \quad y = \frac{x}{2}\left(v - \frac{1}{v}\right)$$

Assume* any *even* number for x, and let $2v$ be any divisor of x. Or assume x any odd number, and v any divisor of it. For example, let $x = 9$, $v = 3$, then $z = 15$, $y = 12$, and $9^2 + 12^2 = 81 + 144 = 225 = 15^2$. This also contains the case where x is given, for it must then be assumed as given.

2. If z be given, the preceding equations give

$$x = \frac{2vz}{v^2 + 1} \quad y = \frac{v^2 - 1}{v^2 + 1}z.$$

Write $\frac{m}{n}$ for v, which gives $(m > n)$

$$x = \frac{2mnz}{m^2 + n^2} \quad y = \frac{m^2 - n^2}{m^2 + n^2}z$$

* Any proof that may be wanting is left for the student.

Prove 1. that if m and n be whole numbers with a common measure, the preceding can be reduced, and other numbers, which have no common measure, substituted for them without altering x or y. 2. That in the latter case $m\,n$ and $m^2 + n^2$ cannot have a common measure.

Hence, prove that when x and y are whole numbers, the problem is impossible except when z or one of its factors is the sum of two unequal squares. Thus, when $z = 5$ or $4 + 1$, let $m^2 = 4$ $n^2 = 1$, and $x = 4$ $y = 3$. But when $z = 11$, the problem is impossible. When $z = 10$, or $9 + 1$, then $m = 3$ $n = 1$ gives $x = 6$ $y = 8$. But 5, a factor of 10, is $4 + 1$, and $m = 2$ $n = 1$ gives $x = 8$ $y = 6$, which is only a reversal of the former.

Show the following. I. If m be greater than n, and both be integers, then

$$x = m^2 - n^2 \quad y = 2\,m\,n \quad z = m^2 + n^2$$

satisfies the preceding problem. II. The square of an even number is divisible by four, and that of an odd number must leave a remainder one. III. A prime number divided by 6, must have a remainder 1 or 5. IV. One more than a square may be a prime number, but one more than a cube cannot.

Problem. To find two numbers, of which the sum of the squares added to the product gives a square, or to solve

$$x^2 + x\,y + y^2 = z^2$$

show that, if x, y, and z, satisfy this equation, any of the same multiples of the three also satisfy it; and hence prove that any real fractional solution gives a solution in whole numbers.

Show that, in this problem, either of the following equations is a consequence of the other.

$$z + y = v\,(x + y) \qquad z - y = \frac{x}{v}$$

From these deduce

$$y = \frac{v^2 - 1}{2 - v} \cdot \frac{x}{v} \qquad z = \frac{v^2 - v + 1}{2 - v}\,\frac{x}{v}$$

Show that v must be greater than 1, and less than 2, or that if $v = \dfrac{m + n}{m}$, n must be less than m. Substitute this value of v, and thus obtain the results

$$y = \frac{2\,m\,n + n^2}{m^2 - n^2}\,x \qquad z = \frac{m^2 + m\,n + n^2}{m^2 - n^2}\,x$$

And from a preceding part show that the following are solutions, $x = m^2 - n^2$, $y = 2\,m\,n + n^2$ $z = m^2 + m\,n + n^2$. For instance, let $m = 3$, $n = 2$, then $x = 5$, $y = 16$, $z = 19$. Verify this, and find other instances.

Show that if n be greater than m, then

$$x = n^2 - m^2 \quad y = 2\,m\,n + n^2 \quad z = m^2 + m\,n + n^2$$

satisfy

$$x^2 - x\,y + y^2 = z^2$$

1. By the theory of the negative sign applied to the preceding. 2. By an independent process similar to the above.

Apply a similar process to

$$a\,x^2 \pm b\,x\,y + y^2 = z^2, \quad a\,x^2 \pm b\,x\,y + c\,y^2 = c\,z^2$$

a, b, and c, being given whole numbers. And, finally, apply the same process to

$$a\,x^2 + b\,x\,y + c^2\,y^2 = e^2\,z^2$$

Problem. Required two numbers, of which the sum of the squares shall be a given number of times the sum.

Let m be the given number of times, so that

$$x^2 + y^2 = m\,(x + y)$$

Assume $x = \dfrac{p}{q}\,y$, which gives by substitution

$$y = \frac{m\,q\,(p + q)}{p^2 + q^2} \qquad x = \frac{m\,p\,(p + q)}{p^2 + q^2}$$

If possible, take p and q so that $m = p^2 + q^2$, or $m = c\,(p^2 + q^2)$, where c is a whole number. Then

$$y = c\,q\,(p + q) \quad x = c\,p\,(p + q)$$

for instance, if $m = 25 = 5\;(4 + 1)$ $x = 15$, $y = 30$.

Problem. Required *triangular* numbers which are also squares. A triangular number is any number which results from making x a whole number in $\frac{1}{2}\,x\,(x + 1)$. Show that this must give a whole number.

We have $\quad \frac{1}{2}\,x\,(x + 1) = y^2$

Assume $x = \dfrac{m}{n}\,y$ then $x + 1 = 2\,\dfrac{n}{m}\,y$

or $\quad y = \dfrac{m\,n}{m^2 - 2\,n^2}, \quad x = \dfrac{2\,n^2}{m^2 - 2\,n^2}$

If we can now get m and n two whole numbers such that $m^2 - 2n^2 = 1$, we have a solution in $y = m\,n$, $x = 2n^2$. Let one set of values of $m^2 - 2n^2 = 1$ be found, say p and q, we shall show how by this one to find another.

Assume $m + 1 = \frac{p}{q}\,n$

whence $m - 1 = 2\,\frac{q}{p}\,n$

giving $n = \frac{2\,p\,q}{p^2 - 2q^2}$ $m = \frac{p^2 + 2\,q^2}{p^2 - 2\,q^2}$

let p and q be the instances which satisfy the preceding: hence it follows that $p^2 - 2\,q^2 = 1$, and $n = 2\,p\,q$, $m = p^2 + 2\,q^2$, which give $m^2 - 2\,n^2 = p^4 - 2\,p^2\,q^2 + 4\,q^4 = (p^2 - 2\,q^2)^2 = 1$: and $x = 8\,p^2\,q^2$ $y = 2p\,q\,(p^2 + 2\,q^2)$

For instance, if $p = 3$ $q = 2$, and $p^2 - 2\,q^2 = 1$, which gives from the first method $x = 8$ $y = 6$, and we have $\frac{1}{2}.8.9 = 6 \times 6$: and from the second $x = 288$ $y = 204$, and $\frac{1}{2}$ 288.289 = 204.204.

Observe, that we are not sure of thus getting all solutions; for it is not necessary that $m^2 - 2\,n^2$ should be 1, it is sufficient that it divide $m\,n$ and $2\,n^2$.

Section 19. *Permutations and Combinations.*

Let $[n,p]$ be the abbreviation of the product of all numbers between n and p, both inclusive. Thus

$[4,9]$ means $4\times5\times6\times7\times8\times9$

Let $\Gamma\,n$ stand for the product of all numbers from 1 up to n *exclusive:* thus

$\Gamma\,8$ means $7\times6\times5\times4\times3\times2\times1$

Now show the following:

$$[n, n + m] = \frac{\Gamma\,(n + m + 1)}{\Gamma\,n}$$

$$\frac{[n + m, n]}{[m + 1, 1]} = \frac{\Gamma\;(n + m + 1)}{\Gamma\,n \times \Gamma\,(m + 2)}$$

Show that $\dfrac{[m, m + p]}{[n, n + q]}$ must be a whole number, if p be greater than q, and $m + p$ greater than $n + q$. Take instances, and show this proposition: for example,

$$\frac{6.7.8.9.10.11}{2.3.4.5.6}$$

and from the instances endeavour to collect a general proof.

If there be n counters marked C_1 C_2 C_3 C_4 .... C_n, and if p be less than n, the number of different ways in which p counters can be drawn, one after the other, counting every two orders, however slightly they differ, as different, is $[n, n - p + 1]$. These are *permutations* of p out of n. But the number of different ways in which p can be taken out of n at once, is $[n, n - p + 1]$ divided by $[1, n]$. These are *combinations* or *selections* of p out of n.

Question. How many different hands can be held at the game of whist, or how many combinations are there of 13 out of 52?

Answer.

$\dfrac{[52, 40]}{[\;1, 13]}$ or 635,013,559,600

How many different choices of 5 may be made out of twelve persons?

Answer.

$\dfrac{12.11.10.9.8}{1.\ 2.\ 3.4.5}$ or $\dfrac{1.11.1.9.8}{1.\ 1.1.1.1}$ or 792

this method of abbreviation must be learned by practice: the 3.4 in the denominator is equivalent to the 12 in the numerator, and the 2.5 in the denominator to the 10 in the numerator.

In how many orders may four be drawn out of 20?

Ans. 20.19.18.17, or 116280.

Question. There are four boxes, containing a, b, c, and d balls. In how many ways may 4 balls be drawn, one from each?

Ans. From every ball in the first arises b methods of drawing from the first two; there are c times as many ways of drawing from the first three; for every way of drawing from the first two may be followed by any of the c ways of drawing from the third. Therefore $a\,b\,c$ is the number of possible drawings from the first three; and by similar reasoning, $a\,b\,c\,d$ is the number of drawings from all four.

In the preceding question, how many ways are there of drawing three from three of the four boxes?

Ans. The first thing to be done is the selection of the three boxes to be drawn from: this may be done in $(4.3.2) \div (1.2.3)$ or four different ways (an abbreviation of this sort might be used: in so many ways as three can be taken, in so many ways can one be left; that is, the number required must be four.)

Calling A B C and D the four boxes, the selections may be

A B C, A B D, A C D, or B C D,

and the drawings from each set may be done in abc, abd, acd, or bcd ways; whence the total number is

$$abc + abd + acd + bcd.$$

There are 3 boxes, with four, five, and six counters. In how many ways may 4 balls be drawn, not taking more than 2 from either? Firstly, consider how many ways 4 can be composed out of three numbers not greater than 2 (0 being included); this gives only 0 + 2 + 2 and 1 + 1 + 2.

This gives as follows, since there are three different ways of varying the cases:—

A (4)	B (5)	C (6)	Number of ways of taking them.
	Balls are taken out of		
0	2	2	$\frac{5.4}{2} \times \frac{6.5}{2}$ or 150
2	0	2	$\frac{4.3}{2} \times \frac{6.5}{2}$ or 90
2	2	0	$\frac{4.3}{2} \times \frac{5.4}{2}$ or 60
1	1	2	$4 \times 5 \times \frac{6.5}{2}$ or 300
1	2	1	$4 \times \frac{5.4}{2} \times 6$ or 240
2	1	1	$\frac{4.3}{2} \times 5 \times 6$ or 180
			In all 1020

If there be n counters, all of a different mark, the total number of different orders in which they can be arranged is $[n, 1]$. What case is this of a preceding theorem? If there be n_1 counters marked C_1, n_2 marked C_2, and n_3 marked C_3, then the total number of different arrangements is

$$\frac{[n_1 + n_2 + n_3, 1]}{[n_1, 1]\,[n_2, 1]\,[n_3, 1]}.$$

Deduce this from what precedes.

Section 20. *On Expressions of the first and second Degrees.*

The following points contain the summary of the theory of expressions of the first and second degrees, which is one of the most important parts of algebra, and without which the mere solution of equations is of little use.

First degree. Let $mx + n$ be the expression of the first degree with respect to x. Let R be its root, or the value of x which makes $mx + n = 0$; then

1. R the root is $-\frac{n}{m}$.

2. $mx + n = m(x - \text{R})$ for all values of x.

3. When x is *greater* than R, $mx + n$ and m have the *same* signs; when x is *less* than R, $mx + n$ and m have different signs.

Example. To verify these rules in the case of $2x - 7$ and $-\frac{3}{2}x - \frac{1}{5}$.

In $2x - 7$ $\quad \text{R} = \frac{7}{2}$, $m = 2$ $n = -7$

$$2x - 7 = 2\left(x - \frac{7}{2}\right) = 2(x - \text{R})$$

Let x be greater than $\frac{7}{2}$ say $= 5$: then $2x - 7 = 3$ and is of the same sign as 2. Let x be less than $\frac{7}{2}$, say* -1: then $2x - 7 = -9$, and is of a different sign from 2. Again, let x be positive, but less than $\frac{7}{2}$, say $= 3$, then $2x - 7 = -1$, and differs in sign from 2.

* Study of Mathematics, page 49.

In $-\frac{3}{2}x - \frac{1}{5}$, $R = -\frac{2}{15}$, $m = -\frac{3}{2}$ $n = -\frac{1}{5}$

$$-\frac{3}{2}x - \frac{1}{5} = -\frac{3}{2}\left(x + \frac{2}{15}\right) = -\frac{3}{2}\left(x - \left(-\frac{2}{15}\right)\right)$$

Let x be greater than $-\frac{2}{15}$.

1. Let $x = -\frac{1}{15}$, $-\frac{3}{2}x - \frac{1}{5} = -\frac{1}{10}$, $m = -\frac{3}{2}$
2. Let $x = 0$ .... $= -\frac{1}{5}$ .....
3. Let $x = 1$ .... $= -\frac{17}{10}$ .....

Let x be less than $-\frac{2}{15}$.

1. Let $x = -\frac{3}{15}$ $-\frac{3}{2}x - \frac{1}{5} = +\frac{1}{10}$, $m = -\frac{3}{2}$
2. Let $x = -\frac{1}{6}$ .... $= \frac{1}{20}$ .....

Verify these assertions upon other expressions, such as $x + 1$, $\frac{1}{2}x - 3$, &c.

Show that the root of $mx + n + m'x + n'$ must lie between the roots of $mx + n$ and $m'x + n'$, by several instances, and by algebraical reasoning. Show also that the root of the first is greater than, equal to, or less than, the average of the roots of the second and third, according as $nm'^2 + n'm^2$ is greater than, equal to, or less than $mm'(n + n')$ if $mm'(m + m')$ be positive: or according as $nm'^2 + n'm^2$ is less than, equal to, or greater than, $mm'(n + n')$, if $mm'(m + m')$ be negative.

Second degree. Let $ax^2 + bx + c$ be the given expression of the second degree, where a, b, and c may be severally positive, negative, or nothing: and let m n and r be the absolute numbers in a b and c, without signs; so that if a and b be positive, and c negative, we mean that a is $+m$, b is $+n$, and c is $-r$.

In making a summary of the properties of the expression above given which are most useful, we distinguish 1, those which are common to all cases; 2, those which distinguish the eight following cases from each other:

$$\left.\begin{array}{r} mx^2 + nx + r \\ -nx^2 - nx - r \end{array}\right\} \qquad \left.\begin{array}{r} mx^2 + nx - r \\ -mx^2 - nx + r \end{array}\right\}$$

$$\left.\begin{array}{r} mx^2 - nx + r \\ -mx^2 + nx - r \end{array}\right\} \qquad \left.\begin{array}{r} mx^2 - nx - r \\ -mx^2 + nx + r \end{array}\right\}$$

General properties. 1. If $b^2 - 4ac$ be positive, there are two different roots: if $b^2 - 4ac = 0$, these two roots are both the same: if $b^2 - 4ac$ be negative, there are no possible roots whatsoever.

2. The roots of the expression $ax^2 + bx + c$, when they exist (call them R_1 and R_2) are

$$R_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad R_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

either of these substituted for x makes

$$ax^2 + bx + c = 0.$$

3. Between the roots and the co-efficients the following equations exist :—

$$R_1 + R_2 = -\frac{b}{a} \qquad R_1 R_2 = \frac{c}{a}.$$

4. The following equation is true for every value of x (when there are roots).

$$ax^2 + bx + c = a(x - R_1)(x - R_2).$$

5. When the two roots are equal, that is, when $b^2 - 4ac = 0$; then

$$R_1 = R_2 = -\frac{b}{2a} \qquad ax^2 + bx + c = a(x - R_1)^2$$

and $ax^2 + bx + c$ is then a perfect square with respect to x, giving

$$\sqrt{ax^2 + bx + c} = \pm\sqrt{a}\left(x + \frac{b}{2a}\right)$$

6. The following equation is always true, and shows (see 1, preceding), that $(ax^2 + bx + c)a$ is the sum of two positive quantities when $b^2 - 4ac$ is negative.

$$ax^2 + bx + c = \frac{(2ax + b)^2 + 4ac - b^2}{4a}.$$

7. $ax^2 + bx + c$ and a *never* differ in sign, *except* when the two roots of the former are *possible* and *different*, and x is taken so as to lie between them.

8. When there are no roots, the least *numerical* value of $ax^2 + bx + c$ happens when

$$2ax + b = 0, \quad \text{or } x = -\frac{b}{2a}$$

$$\text{and is } c - \frac{b^2}{4a}.$$

The preceding properties occur so continually in all applications of algebra, and particularly in geometry, that the student should know them perfectly.

Example 1. What are the properties of

$$-2x^2 + 4x + 3$$

$$a = -2 \qquad b = +4 \qquad c = +3$$

1. $b^2 - 4ac = 40$; there are two different roots.
2. These roots are

$$R_1 = \frac{-4 + \sqrt{40}}{-4} = 1 - \frac{1}{2}\sqrt{10} \quad \text{(neg.)}$$

$$R_2 = \frac{-4 - \sqrt{40}}{-4} = 1 + \frac{1}{2}\sqrt{10} \quad \text{(pos.)}$$

3. $R_1 + R_2 = 2 \qquad R_1 R_2 = -\frac{3}{2}$

4. $-2x^2 + 4x + 3 = -2(x - R_1)(x - R_2)$

$$= -2(x - 1 + \tfrac{1}{2}\sqrt{10})(x - 1 - \tfrac{1}{2}\sqrt{10})$$

5. Does not apply to this case.

6. $$-2x^2 + 4x + 3 = \frac{(-4x + 4)^2 - 40}{-8}$$

$$= -2(x - 1)^2 + 5.$$

7. $-2x^2 + 4x + 3$ is negative for every value of x, except those which lie between

$1 - \frac{1}{2}\sqrt{10}$ or $-1{\cdot}5811389$ nearly
$1 + \frac{1}{2}\sqrt{10}$ or $2{\cdot}5811389$ „

in which cases it is positive. Show that it is positive when $x = 1,\ 1{\cdot}5$, or $2{\cdot}5$.

8. Does not apply.

Example 2. $3x^2 - 12x + 12$.

$a = 3 \qquad b = -12 \qquad c = 12.$

1. $b^2 - 4ac = 0$; there are two equal roots.
2. These roots are both $= 2$.
5. $3x^2 - 12x + 12 = 3(x - 2)^2$
7. The expression is always positive, except only when $x = 2$ (0 is neither + nor −).

Example 3. $2x^2 + x + 1$

$a = 2 \qquad b = 1 \qquad c = 1$

1. $b^2 - 4ac = -7$; there are no roots.

6. $2x^2 + x + 1 = \frac{(4x+1)^2 + 7}{8}$

7. This expression is always positive.

8. Its least value is $\frac{7}{8}$, which is when $x = -\frac{1}{4}$.

When the co-efficients are fractional, reduce the whole to a common denominator; for instance, reduce

$$\frac{1}{2}x^2 + \frac{2}{3}x - \frac{3}{4} \quad \text{to} \quad \frac{6x^2 + 8x - 9}{12}$$

then from the properties of the numerator, those of the fraction may be obtained.

Particular properties. The pairs bracketted together in page 81, present no distinction whatever except difference of sign. Each one is the other of its pair with all the signs changed.

1. When a and c have the same sign, the roots *may* be impossible, and we have

$b^2 - 4ac$ is less than b^2

$\sqrt{b^2 - 4ac}$ „ „ „ b, numerically.

2. When a and c have different signs, the roots cannot be impossible, and we have

$b^2 - 4ac$ is greater than b^2

$\sqrt{b^2 - 4ac}$ „ „ b, numerically.

3. In $\left.\begin{array}{l} +mx^2 + nx + n \\ \text{and } -mx^2 - nx - r \end{array}\right\} \begin{array}{l} + + + \\ - - - \end{array}$

the particular distinction is, that the roots, *if any*, are both negative.

4. In $\left.\begin{array}{l} +mx^2 - nx + r \\ \text{and } -mx^2 + nx - r \end{array}\right\} \left.\begin{array}{l} + - + \\ - + - \end{array}\right\}$

the roots, *if any*, are both positive.

5. In $\left.\begin{array}{l} +mx^2 + nx - r \\ \text{and } -mx^2 - nx + r \end{array}\right\} \left.\begin{array}{l} + + - \\ - - + \end{array}\right\}$

There must be two roots of different signs, the numerically greater root being negative.

6. In $\left.\begin{array}{l} +mx^2 - nx - r \\ \text{and } -mx^2 + nx + r \end{array}\right\} \left.\begin{array}{l} + - - \\ - + + \end{array}\right\}$

There must be two roots of different signs, the numerically greater being positive.

These may all be contained in one rule, as follows: — there may be as many positive roots as there are changes of sign from term to term of the expression, and as many negative roots as there are continuations of sign; and according as the first term and the second present change or continuation, the greater root, numerically, is positive or negative.

Show that $$\sqrt{b^2 - 4ac} - b = \frac{-4ac}{\sqrt{b^2 - 4ac} + b}$$

and from hence that

$$R_1 = \frac{-2c}{b + \sqrt{b^2 - 4ac}} \qquad R_2 = \frac{-2c}{b - \sqrt{b^2 - 4ac}}$$

and also that if a be supposed to vary, becoming smaller and smaller,

R_1 continually approaches to $-\frac{c}{b}$

R_2 increases without limit.

Why cannot this be made clear *in both cases*, when the first forms are used? Also show from the preceding that when a is very small,

$$\sqrt{b^2 - 4ac} = b - \frac{2ac}{b} \quad \text{nearly.}$$

If the roots of $ax^2 + bx + c$ be R_1 and R_2, those of $cx^2 + bx + a$ are $\frac{1}{R_1}$ and $\frac{1}{R_2}$.

Show this by actually forming the roots.

Show that a change of sign in b changes only the sign of the roots, and not their numerical magnitude.

Example. A number $2p$ is divided into two parts, whose product is q^2. What are those parts?

Ans. The roots of the expression $x^2 - 2px + q^2$, or the solutions of the equation

$$x^2 - 2px + q^2 = 0$$

namely, $p + \sqrt{p^2 - q^2}$ and $p - \sqrt{p^2 - q^2}$

Prove from this that it is impossible the product of the two parts into which any number is divided should exceed the square of the half: for instance, that no two numbers or fractions which added together make 10, can have a product exceeding 25.

Example. What are the solutions of

$$(x - a)(x - b) = c?$$

Ans. $\dfrac{a + b \pm \sqrt{(a - b)^2 + 4c}}{2}$.

What are the *conditions of possibility* of this equation? Of the two following, which is possible?

$$(x - 4)(x - 1) = 12 \quad (x - 1)(x - 2) = -100.$$

What do the roots become when $c = 0$? How does this appear from the equation itself?

Example. When is the following expression possible?

$$\sqrt{(b^2 - 4ac)x^2 + (2bd - 4ae)x + d^2 - 4af} \;.\;.\;.\;. \quad (1)$$

The roots of the expression where square root is shown are

$$\frac{2ae - bd \pm \sqrt{4a\{(cd^2 + ae^2 - bde) + (b^2 - 4ac)f\}}}{b^2 - 4ac}.$$

For what values of x will the following equation allow of being solved by possible values of y?

$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

Show by solving the equation on the supposition that y is to be found, that this question reduces itself to the preceding.

Show that the roots of $ax^2 + 2bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - ac}}{a}.$$

As this form often occurs it should be remembered. Compare it with that for the roots of $ax^2 + bx + c = 0$, and explain the difference.

What are the roots of

$$(1 + a)x^2 - 2(1 + 2a)x + 1 + 3a?$$

Ans. 1 and $\dfrac{1 + 3a}{1 + a}$.

For what values of a is the second root negative? Find this out from the expression, without looking at the root; and from the root without looking at the expression.

Ans. When a lies between $-\dfrac{1}{3}$ and -1.

Question. If the difference of the two roots of $ax^2 + bx + c = 0$ be D, what are the roots, and what equation must exist between a, b, c, and D?

Ans.

$$R_1 = \frac{-b + aD}{2a} \quad R_2 = \frac{-b - aD}{2a}$$

$$b^2 - a^2D^2 = 4ac.$$

Prove this in two different ways.

Question. There are two numbers whose sum is p, and the sum of their squares is q: what are the numbers?

Ans.

$$\frac{p + \sqrt{2q - p^2}}{2} \text{ and } \frac{p - \sqrt{2q - p^2}}{2}$$

Show that $x + \dfrac{1}{x}$ cannot be less than 2, or that

$$x + \frac{1}{x} = 2a \quad (a < 1)$$

has no possible roots.

If the sum of two numbers be p, and the sum of their cubes q, then those two numbers are the roots of the expression

$$x^2 - px + \frac{p^3 - q}{3p}.$$

If the roots of $ax^2 + bx + c$ be R_1 and R_2, what is the expression whose roots are $R_1 + k$ and $R_2 + k$?

Ans. $ax^2 + (b - 2ak)x + ak^2 - bk + c$.

What is the expression where roots are $m R_1$ and $m R_2$?.

Ans. $a x^2 + m b x + m^2 c$.

What number is that which exceeds its square root by a? Let v be the square root, and v^2 the number; then

$$v = \frac{1 + \sqrt{1 + 4a}}{2} \quad \text{or} \quad \frac{1 - \sqrt{1 + 4a}}{2}$$

$$v^2 = \frac{1 + 2a + \sqrt{1 + 4a}}{2} \quad \text{or} \quad \frac{1 + 2a - \sqrt{1 + 4a}}{2}$$

verify and explain both cases. Show that v and v^2 are rational when a is a number or fraction added to its square, and that only one square root of v^2 will satisfy the condition. Find also the number, which added to its square root is equal to a, and show that the square root of v^2 which does not satisfy one question, satisfies the other.

What are the roots of $a x^4 + b x^2 + c$?

Ans. $\pm \sqrt{\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}}$ and $\pm \sqrt{\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}}$

Point out the cases in which two roots only are possible, and in which all are possible or none possible. Show that there cannot be one or three possible, and the others or other impossible.

What are the roots of $a x^{2n} + b x^n + c$?

Ans. $\sqrt[n]{\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}}$ and $\sqrt[n]{\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}}$

which may be taken with either sign if n be even. We do not here enter on the impossible roots.

What two numbers are there whose sum is m times their difference, and whose product is n times their sum?

If x and y be the numbers, then we have the equations

$$x + y = m(x - y) \qquad x y = n(x + y)$$

From the first, $y = \dfrac{m - 1}{m + 1} x$,

which substituted in the second, gives

$$x = \frac{2mn}{m - 1} \text{ and thence } y = \frac{2mn}{m + 1}.$$

What numbers are there, the sum of whose squares is m times their product, and the difference of whose squares is n times their product. Examine the equations and prove that these two conditions are contradictory unless $m^2 - n^2 = 4$. Thence prove that they are never true when m is a whole number, unless $m = 2$ $n = 0$, and never true when n is a whole number. Show, as in page 77, that when

$$m = k + \frac{1}{k} \qquad n = k - \frac{1}{k},$$

any values of x and y which satisfy the first of the equations also satisfy the second.

Solve the equations

$$x^2 + y^2 = m(x + y) \qquad x + y = n(x - y).$$

From the second, $\qquad y = \dfrac{n - 1}{n + 1} x$,

which, substituted in the first, gives, after reduction,

$$2(n^2 + 1) x^2 = 2mn(n + 1) x.$$

Ans. Either $x = 0$ $\qquad y = 0$;

$$\text{or } x = mn\frac{n + 1}{n^2 + 1} \qquad y = mn\frac{n - 1}{n^2 + 1}.$$

Solve the equations

$$x^2 - y^2 = m x y \ldots . \text{(A)} \qquad x - y = n x y \ldots . \text{(B)}.$$

First deduce $x + y = \dfrac{m}{n}$ .... (C).

Substitute y obtained from this in both of the first equations, and show that they produce different results, namely,

$$n^2(2+m)x^2 + mn(2-m)x - m^2 = 0 \quad \ldots\ldots \text{(D)}$$

$$nx^2 + (2-m)nx - m = 0 \quad \ldots\ldots \text{(E)}$$

the roots of either of these are values of x, which with $y = \frac{m}{n} - x$, solve the equation; namely,

$$\text{(D)} \qquad x = \frac{-m(2-m) \pm m\sqrt{12+m^2}}{2n(2+m)}$$

$$\text{(E)} \qquad x = \frac{-(2-m)n \pm \sqrt{n^2(2-m)^2 + 4mn}}{2n}.$$

The truth is, that by the artifice of division, which produced the equation (C), we have obtained equations of the *second* degree; whereas, had we simply substituted in (A) the value of y from (B), namely,

$$y = \frac{x}{nx+1},$$

we should, after reduction, have obtained an equation of the *fourth* degree, which may have four possible roots.

We must here remind the student that there is a degree of connexion between algebraical equations, more than is actually and logically contained in the forms of speech by which they are connected. Let us take the following assertions:

(A) All right angles are equal.
(B) P and Q are right angles.
(C) P and Q are equal angles.

If (A) and (B) be both true, (C) must be true; but if (A) and (C) be both true, it does not necessarily follow that (B) must be true, as will easily be seen. But take for (A) (B) and (C) three algebraical equations, of which (C) necessarily follows, or is true, when (A) and (B) are true; for instance,

$$\text{(A)} \qquad x + y = 7$$
$$\text{(B)} \qquad x - y = 5$$
$$\text{(C)} \qquad x^2 - y^2 = 35,$$

of which (C) must be true when (A) and (B) are true. It follows that when (A) and (C) are true, either (B) is true, or (B) is one of *two* equations *in this case* (it might be three, four, &c., in problems of a higher degree) one of which must be true. Assume equations (A) and (C) or

$$x + y = 7 \qquad x^2 - y^2 = 35.$$

If we divide the second by the first, we have $x - y = 5$, which it appears at first must follow absolutely; and this is true, finite numbers only being considered. But we have (*Study of Mathematics*, p. 41) to consider the possibility of an *infinite* solution, or of this problem being one particular case of a general problem, which admits of more solutions than one, in general, but of which solutions one or other increases without limit, as the general problem is made to approach to the particular case in question. We have seen that a problem of the second degree must generally have two solutions, but this seems to have only one; for since equals divided by equals must give equals, it follows that *if* $x + y = 7$, and $x^2 - y^2 = 35$, we *must* have $x - y = 5$, and $x + y = 7$, together with $x - y = 5$, give $x = 6$ $y = 1$, and nothing else whatsoever. The question is, what is become of the second solution which the general problem $ax + by = m$ $px^2 - qy^2 = n$ will be found to have? To solve this question, we must first see whether we really have only one solution. Instead of dividing the second equation by the first, substitute in the second the value of y from the first, which gives

$$x^2 - (7-x)^2 = 35,$$

an equation of the second degree at first sight, and therefore with two roots or none. But on looking further, we see that this equation is no more of the second degree, than $x = 1$ is of the hundredth degree (being $x + x^{100} = 1 + x^{100}$), as the terms of the second degree destroy each other, and leave

$$14x - 49 = 35 \qquad x = 6.$$

But let us now alter the problem by proposing

$$x + y = 7 \qquad (1+k)x^2 - y^2 = 35.$$

We here present a problem which may be made as near as we please to the former, by taking k sufficiently small, and which absolutely becomes the former, when $k = 0$. Now substitute as before, and we have

$$(1+k)\,x^2-(7-x)^2=35, \quad \text{or}$$

$$k\,x^2+14\,x=84$$

$$x=\frac{-14\pm\sqrt{196+336\,k}}{2\,k}$$

$$=\frac{-7+\sqrt{49+84\,k}}{k} \quad \text{or} \quad \frac{-7-\sqrt{49+84\,k}}{k}$$

$$(\text{p. } 83)\quad =\frac{84}{\sqrt{49+84\,k}+7} \qquad \text{or} -\frac{84}{\sqrt{49+84\,k}-7}$$

If we now suppose k to diminish continually, the first root approaches continually to

$$\frac{84}{\sqrt{49}+7} \quad \text{or } 6,$$

while the second is always negative, and has a denominator which diminishes without limit; that is, the root increases numerically without limit. When k is small, the first expression for the second root shows that it is

$$\frac{-7-\sqrt{49}}{k}, \quad \text{or} -\frac{14}{k} \text{ nearly;}$$

so that if k were $\frac{1}{1000}$, the root would be nearly -14000.

Let us suppose k to be very small, and let the great and negative value of x be taken. Then $x+y=7$ shows that y or $7-x$ is a little greater and positive. We now ask, what is the substitute for the equation $x-y=5$, which is necessarily true in the first problem? Is it nearly true in the second problem? To investigate this, take the second equation in the form

$$(x^2-y^2)+k\,x^2=35,$$

divide the three terms by the equals $x+y$, 7, and 7, which gives

$$x-y+\frac{k\,x^2}{7} \qquad =5.$$

This equation will be true, as might be proved by actual substitution: but it will not be true, because $x-y=5$ nearly, but because $x-y$ is a negative quantity, and $\frac{1}{7}\,k\,x^2$ is a positive quantity greater by 5. And though k may be small, x is so large that $k\,x^2$ is considerable. With respect to $x=6$, which will give $y=1$ very nearly, which is one solution of the second problem when k is small, we may say that

$$x-y+\frac{1}{7}\,k\,x^2=5$$

means $x-y=5$ very nearly, because the term rejected is only about 36 sevenths of k, and is small: but we may not call the latter equation nearly true of the larger root.

Question. If $a\,x^2+b\,x+c$, and $a'\,x^2+b'\,x+c'$ have a root in common, what equation must exist between a, b, c, a', b', c'?

Let P be the common root, and let Q be the other root of the first, and Q′ the other root of the second. Then we have

$$\text{P}+\text{Q}=-\frac{b}{a}\;.\;.\;.\;(1) \qquad \text{P}\,\text{Q}=\frac{c}{a}\;.\;.\;.\;(3)$$

$$\text{P}+\text{Q}'=-\frac{b'}{a'}\;.\;.\;.\;(2) \qquad \text{P}\,\text{Q}'=\frac{c'}{a'}\;.\;.\;.\;(4)$$

From these equations deduce the following:

$$\text{Q}'-\text{Q}=\frac{b\,a'-b'\,a}{a\,a'} \qquad \frac{\text{Q}'}{\text{Q}}=\frac{a\,c'}{a'\,c},$$

from which deduce $\quad \text{Q}=\dfrac{c}{a}\,\dfrac{b\,a'-b'\,a}{a\,c'-a'\,c}.$

Now, from (1) and (3) deduce

$$a\,\text{Q}^2+b\,\text{Q}+c=0,$$

and thence (from the preceding)

$$c\,(b\,a'-b'\,a)^2+b\,(b\,a'-b'\,a)\,(a\,c'-a'\,c)+a\,(a\,c'-a'\,c)^2=0,$$

which show to be the same as

$$(a\,c'-a'\,c)^2=(c\,b'-c'\,b)\,(b\,a'-b'\,a)\;.\;.\;.\;.\;(5).$$

Now deduce the same as follows :— since P is a root of both, we have

$$a\,P^2 + b\,P + c = 0$$
$$a'\,P^2 + b'\,P + c' = 0$$

Hence, deduce

$$(a\,c' - a'\,c)\,P^2 - (c\,b' - c'\,b)\,P = 0$$

and also

$$(b\,a' - b'\,a)\,P - (a\,c' - a'\,c) = 0,$$

and from the two last deduce (5).

Question. Given the values of a and c; required the method of finding such values of b as will make the roots of $a\,x^2 + b\,x + c$ rational. Show, from the method explained in page 77, that $b^2 - 4\,a\,c$ is a square when

$$b = a\,m + \frac{c}{m},$$

m being any number or fraction whatsoever. Show that the roots in this case are

$$-m \qquad \text{and} - \frac{c}{a\,m}.$$

Question. In how many different ways can whole and positive rational roots be given to the expression

$$3\,x^2 + b\,x + 36?$$

Show, from the preceding, that m must be either -1, -2, -3, -4, -6, or -12, and that the values of b are -39, -24, -21, -21, -24 and -39. Explain the recurrence of the values of b.

Miscellaneous questions. Deduce from the general equation, the roots of $a\,x^2 + c = 0$, and of $a\,x^2 + b\,x = 0$. On what does it depend whether the roots of the first are possible or not? Show that when b is very small compared with a and c, the roots (if any, on what does this depend?) are one positive and the other negative, but numerically very nearly equal. Show that when c is very small compared with a and b, that one of the roots must be very small, but not the other, necessarily. What are the necessary conditions that both the roots must be very small? If c and a be equal, what relation must exist between the roots? Show that if the sum of the squares of the roots be unity, we must have $b^2 - 2\,a\,c = a^2$. In what relation do the roots of $a\,x^2 + m\,b\,x + m^2\,c$ stand to those of $a\,x^2 + b\,x + c$. Show from the method already given, that if $a\,x^2 + b\,x + c$ and $a'\,x^2 + b'\,x + c'$ have a common root,

$$(a + p\,a')\,x^2 + (b + p\,b')\,x + c + p\,c'$$

has the same for all values of p.

SECTION 21. *On Equations of the first degree, with more unknown quantities than one.*

As few cases of these will occur in the future reading of the student, which present any very complicated operations, we shall here only describe the best method of proceeding when only one of the unknown quantities (three in number) is wanted.

Suppose the equations to be

$$2\,x + 3\,y + 4\,z = 20$$
$$3\,x - 2\,y - 2\,z = 4$$
$$4\,x - y + z = 12.$$

Suppose the value of z is required. Multiply the second equation by p, and the third by q, then add the three together, and suppose p and q to be such that

$$2 + 3\,p + 4\,q = 0$$
$$3 - 2\,p - q = 0.$$

Show that we must then have

$$z = \frac{20 + 4\,p + 12\,q}{4 - 2\,p + q}$$

that from the preceding equations,

$$p = \frac{14}{5}\,q = -\frac{13}{5}, \text{ whence } z = 0.$$

Again, suppose the value of x is wanted in

$$a\,x + b\,y - c\,z = 1$$
$$b\,x + c\,y - a\,z = 1$$
$$c\,x + a\,y - b\,z = 1.$$

Show that if

$$b + c\,p + a\,q = 0$$
$$c + a\,p + b\,q = 0$$

then $\quad x = \dfrac{1 + p + q}{a + b\,p + c\,q}$

and that

$$p = \frac{b^2 - a\,c}{a^2 - b\,c} \qquad q = \frac{c^2 - a\,b}{a^2 - b\,c}$$

$$z = \frac{a^2 + b^2 + c^2 - a\,b - b\,c - c\,a}{a^3 + b^3 + c^3 - 3\,a\,b\,c}$$

Determine the values of x and y by a similar method.

Section 22. *On Exponents.*

Explain the following expressions:

$$x^4,\quad x^{-4},\quad x^{\frac{1}{4}},\quad x^{-\frac{1}{4}},\quad x^{\frac{2}{3}},\quad x^{-\frac{2}{3}},\quad x^{\frac{a}{2}}$$

$$\left(x^{\frac{m}{n}}\right)^{\frac{a}{b}},\qquad \left\{(x+1)^{\frac{2}{3}}\right\}^{\frac{3}{2}}$$

$$x^m \times x^n = x^{m+n},\qquad x^m \times x^{-n} = x^{m-n}$$

$$x^m \div x^n = x^{m-n},\qquad x^m \div x^{-n} = x^{m+n}$$

$$x^{-m} \times x^{-n} = x^{-m-n} = x^{-(m+n)}$$

$$x^{-m} \div x^{-n} = x^{n-m} = x^{-(m-n)}$$

$$\left(x^m\right)^n = x^{mn},\ \left(x^{-m}\right)^n = x^{-mn},\ \left(x^{-m}\right)^{-n} = x^{mn}$$

$$x^2 \times x^3 \times x^{-4} = x \qquad x \times x^{-2} \div x^{-4} = x^3$$

What is the value of p in the following equations?

$$x^7 \times x^6 = x^4 \times x^{-p},\qquad x^{-p} \times x^{-2p} = x^{12}.$$

Reduce the following expressions to fractional exponents and find the results:

$$\sqrt{x} \times \sqrt[3]{x} = \sqrt[6]{x^5} \qquad \sqrt[a]{x^b} \times \sqrt[b]{x^a} = \sqrt[ab]{x^{a^2+b^2}}$$

$$\sqrt{x\sqrt{x}} = \sqrt[4]{x^3} \qquad \sqrt[3]{x^2\sqrt[4]{x}} = \sqrt[4]{x^3}$$

$$\left(a^4 \times \left(a^3 \times a^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} = a^{\frac{23}{12}} \qquad \sqrt[p]{x\sqrt[q]{x}} = \sqrt[pq]{x^{q+1}}$$

The qth root of the pth power of x having been taken, the reciprocal of the result is raised to the rth power, and its sth root is taken: this result having been first multiplied by the pth root of the qth power of x, is squared, and the result being then multiplied by the cube of x, the fourth root of the whole is taken. What is the algebraical method of stating this process, and what power of x is the result?

Expression, $$\left\{x^3\left(x^{\frac{q}{p}}\left(x^{-\frac{p}{q}}\right)^{\frac{r}{s}}\right)^2\right\}^{\frac{1}{4}}$$

Result, $$x^{\frac{3pqs+2q^2s-2p^2r}{4pqs}}$$

or, $$\frac{x^{\frac{3}{4}} \times x^{\frac{q}{2p}}}{x^{\frac{pr}{2qs}}}$$

$$\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2 = a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b$$

$$\left(a^{\frac{1}{3}} + b^{\frac{1}{3}}\right)^3 = a + 3a^{\frac{2}{3}}b^{\frac{1}{3}} + 3a^{\frac{1}{3}}b^{\frac{2}{3}} + b$$

$$\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = a - b$$

$$\left(a^{\frac{m}{n}} + b^{-\frac{m}{n}}\right)^2 = a^{\frac{2m}{n}} + 2a^{\frac{m}{n}}b^{-\frac{m}{n}} + b^{-\frac{2m}{n}}$$

$$a + b = a^{\frac{1}{2}}\left(a^{\frac{1}{2}} + a^{-\frac{1}{2}} b\right) = a^{\frac{1}{3}}\left(a^{\frac{2}{3}} + a^{-\frac{1}{3}} b\right)$$

$$= a^{-\frac{3}{5}}\left(a^{\frac{8}{5}} + a^{\frac{3}{5}} b\right) = a b^{-1}\left(b + a^{-1} b^{2}\right) = a^{-1} b^{-1}\left(a^{2} b + b^{2} a\right)$$

$$= a^{\frac{m}{n}}\left(a^{\frac{n-m}{n}} + a^{-\frac{m}{n}} b\right) = a^{m} b^{n} c^{r}\left(a^{-(m-1)} b^{-n} c^{-r} + a^{-m} b^{-(n-1)} c^{-r}\right)$$

$$a^{\frac{1}{2}} - b^{\frac{1}{2}} = a\left(a^{-\frac{1}{2}} - a^{-1} b^{\frac{1}{2}}\right) = a^{\frac{1}{4}}\left(a^{\frac{1}{4}} - a^{-\frac{1}{4}} b^{\frac{1}{2}}\right)$$

$$a^{m} : a^{n} :: a^{p} : a^{p+n-m} \qquad a^{\frac{3}{2}} : a^{\frac{1}{4}} :: a^{\frac{2}{3}} : a^{-\frac{7}{12}}$$

Multiply together the two series

$$a + a' x + a'' x^{2} + a''' x^{3} + a^{\text{iv}} x^{4} + \ldots\ldots$$

and $\quad b + b' x^{-1} + b'' x^{-2} + b''' x^{-3} + b^{\text{iv}} x^{-4} + \ldots$

and give the terms of the product which involve

$$x^{3}, \quad x^{-3}, \quad x^{n}, \text{ and } x^{-n}.$$

Answers.

$$(b a''' + b' a^{\text{iv}} + b'' a^{\text{v}} + \ldots\ldots) x^{3} \qquad \left(b a^{(n)} + b' a^{(n+1)} + \ldots\ldots\right) x^{n}$$

$$(a b''' + a' b^{\text{iv}} + a'' b^{\text{v}} + \ldots\ldots) x^{-3} \qquad \left(a b^{(n)} + a' b^{(n+1)} + \ldots\ldots\right) x^{-n}$$

$$a - b = \left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)$$

$$= \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}} b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{1}{4}} - b^{\frac{1}{4}}\right)\left(a^{\frac{3}{4}} + a^{\frac{1}{2}} b^{\frac{1}{4}} + a^{\frac{1}{4}} b^{\frac{1}{2}} + b^{\frac{3}{4}}\right)$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{a^{\frac{1}{n}} b^{\frac{n-1}{n}}}{b} = \frac{a}{a^{\frac{n-1}{n}} b^{\frac{1}{n}}} = \frac{a^{\frac{mn+1}{n}} b^{\frac{pn+n-1}{n}}}{a^{m} b^{p+1}}$$

$$\sqrt[4]{\left(\frac{a\sqrt{b}}{\sqrt[3]{ab}}\right)} = a^{\frac{1}{6}} b^{\frac{1}{24}} \qquad \sqrt[n]{\left(\frac{a^{m} \sqrt[p]{b}}{\sqrt[q]{ab}}\right)} = a^{\frac{mq-1}{qn}} b^{\frac{q-p}{pqn}}.$$

SECTION 23. *Miscellaneous Questions.*

How much time elapses between two consecutive conjunctions of the minute hand and hour hand of a watch?

Ans. 1 hour, 5 minutes, 27 seconds, and $\frac{3}{11}$ of a second.

If one hand revolved in a hours, and the other in b hours, what time would elapse between two conjunctions?

Ans. If b be greater than a, $\frac{ab}{b-a}$ hours.

In the last question, how long will it be before the quicker hand has gained $\frac{p}{q}$ of a whole revolution upon the other?

Ans. $\frac{p}{q} \; \frac{ab}{b-a}$ hours.

In $\frac{a+bx}{1+x}$ determine a and b so that if in the expression the expression itself be substituted for x, the result will be $= x$.

Ans. a may have any value, provided $b = -1$, or the expression must be $\frac{a-x}{1+x}$.

Show that in the series

$$1 + nx + n\frac{n-1}{2}x^2 + n\frac{n-1}{2}\frac{n-2}{3}x^3 + \ldots\ldots$$

the same result is produced by writing $n + 1$ instead of n, as would be produced by multiplying the series by $1 + x$. Write the values of the preceding series when $n = 1$ $n = 2$ $n = 3$ $n = 4$, and show that they are the same as $1 + x$, $(1 + x)^2$, $(1 + x)^3$, and $(1 + x)^4$.

Multiply together the two following series:

$$1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \text{ \&c.}$$

$$1 + y + \frac{y^2}{2} + \frac{y^3}{2.3} + \frac{y^4}{2.3.4} + \text{ \&c.}$$

and show that the product is the expression obtained by writing $x + y$ instead of x in the first.

Multiply together the series

$$a + a'x + a''x^2 + a'''x^3 + a^{\text{iv}}x^4 + \ldots.$$

$$\text{and } b + b'x + b''x^2 + b'''x^3 + b^{\text{iv}}x^4 + \ldots.$$

What is the term of the product involving x^n?

Ans.

$$\left(a^{(n)}\ b + a^{(n-1)}b' + a^{(n-2)}b'' + \ldots.. + a'b^{(n-1)} + a\,b^{(n)}\right)x^n$$

In the preceding question, let the second series be

$$1 + x + x^2 + x^3 + x^4 + \ldots..$$

or let $\qquad b = 1 \qquad b' = 1 \qquad b'' = 1$ &c.

Find from the result an easy method of multiplying any series by the last mentioned, and make use of it to find the first five terms of the fifth power of

$$1 + x + x^2 + x^3 + \text{\&c.}$$

Ans. $\quad 1 + 5x + 15x^2 + 35x^3 + 70x^4.$

Show that in the product of the two series

$$a + a'x + a''x^2 + a'''x^3 + a^{\text{iv}}x^4 + \ldots.$$

$$a - a'x + a''x^2 - a'''x^3 + a^{\text{iv}}x^4 - \ldots.:$$

there can be no terms involving odd powers of x.

Show that the following are all equal to each other and to $\dfrac{1}{1-x}$.

$$1 + \frac{x}{1-x} \qquad 1 + x + \frac{x^2}{1-x} \qquad 1 + x + x^2 + \frac{x^3}{1-x} \quad \text{\&c.}$$

Three men, A, B and C, could complete a work as follows: A and B in c days, B and C in a days, A and C in b days. In what time could each complete it, and in what time could they all do it together?

Ans. A B and C could severally do it in

$$\frac{2abc}{ab+ac-bc} \qquad \frac{2abc}{ab+bc-ac} \qquad \frac{2abc}{bc+ac-ab} \quad \text{days,}$$

and all three together in $\dfrac{2abc}{ab+bc+ca}$ days.

Explain the case in which $a = 1$, $b = 4$, and $c = 6$, and also that in which $a = 1$, $b = 2$, $c = 2$.

Every whole number is either one of the series of powers of 2 contained in 1, 2, 4, 8, 16, &c., or may be made by adding together terms of this series without repeating any term twice. And every whole number is either one of the series

of powers of 3 contained in 1, 3, 9, 27, 81, &c., or may be made by addition and subtraction of terms of this series without using any one twice.

Prove the following formulæ:

$$\frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}$$

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4};$$

and having proved these, deduce from them the following theorems: 1. That the sum of all whole numbers up to n is $\frac{1}{2}n(n+1)$. 2. That the sum of the squares of all whole numbers up to n^2 is $\frac{1}{6}n(n+1)(2n+1)$. 3. That the sum of the cubes of all whole numbers up to n^3, is the square of the sum of all whole numbers up to n.

From what immediately precedes, prove that the sum of n terms of the series

$$a, \quad a+b \quad a+2b \dots \text{ is } na + n\frac{n-1}{2}b$$

that the sum of n terms of

$$a^2 \quad (a+b)^2 \quad (a+2b)^2 \dots \text{ is } na^2 + n(n-1)ab + \frac{1}{6}n(n-1)(2n-1)b^2$$

and that the sum of n terms of a^3, $(a+b)^3$ &c., is

$$na^3 + \frac{3}{2}n(n-1)a^2b + \frac{1}{2}n(n-1)(2n-1)ab^2 + \frac{1}{4}n^2(n-1)^2b^3$$

What is the inverse operation to adding one nth part of the whole, and subtracting one nth part of the whole?

Answers. The subtraction of the $(n+1)$th part, and the addition of the $(n-1)$th part. (The principal difficulty is in the correct understanding of the words of the question.)

There is a number to which I add its fourth part; from the sum I take 3, and to the difference I add its fifth part. The result is 10. What was the number?

Ans. $9\frac{1}{15}$.

There is a number to which a is added, and the result is divided by b. To the quotient a' is added, and the result divided by b'. To the quotient a'' is added, and the result divided by b''. The result of the last process is found to be h. What was the number?

Ans. $hb''b'b - a''b'b - a'b - a$.

In the preceding, let the addition be changed into a subtraction, and the division into multiplication. What is the number?

Ans.

$$x = a + \frac{a'}{b} + \frac{a''}{bb'} + \frac{h}{bb'b''}.$$

Multiply the expression $\sqrt{a} + \sqrt{b} + \sqrt{c}$ by $\sqrt{a} + \sqrt{b} - \sqrt{c}$, give the product the form $P + \sqrt{Q}$, and multiply by $P - \sqrt{Q}$. What is the result?

Ans. $a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$.

How many different cases are there of $\pm a \pm b$, and what is the product of all?

Ans. $\left(a^2 - b^2\right)^2$.

How many different cases are there of $\pm a \pm b \pm c$, and what is the product of all?

Ans. $\left(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2\right)^2$.

Show that one of the three, $a - b$, $b - c$, $c - a$, must be negative, and one must be positive; and that $a^2 + b^2 + c^2$ always exceeds $ab + bc + ca$.

If m be a given number, then x can always be taken so great that $(x+m)^2$ shall exceed x^2 as much as we please; and at the same time by as small a fraction of x as we please.

Show that the number of different ways (counting different orders as different ways) in which p numbers, no one of which exceeds m, can be put together so as to make q, must be the co-efficient of x^q in the development of

$$(x + x^2 + x^3 + \ldots\ldots + x^{m-1} + x^m)^p.$$

What are the roots of the following equation

$$a\,(p\,x^2 + q\,x + r)^2 + b\,(p\,x^2 + q\,x + r) + c = 0\,?$$

Ans. The four cases of

$$\frac{-q\sqrt{a} \pm \sqrt{a q^2 - 4 a p r - 2 b p \pm 2 p \sqrt{b^2 - 4 a c}}}{2\sqrt{a}.\ p}.$$

Show that the sum of these four roots is $-\dfrac{2q}{p}$.

Prove the following formula by verification:

$$\sqrt{2}\ \sqrt{a \pm b\sqrt{c}} = \sqrt{a + \sqrt{a^2 - b^2 c}} \pm \sqrt{a - \sqrt{a^2 - b^2 c}}.$$

For what numbers or fractions is $x^2 - c\,y^2$ a square?

Ans. m, n, and p, being any whole numbers or fractions, let

$$x = p\,(c\,m^2 + n^2) \qquad y = 2\,p\,m\,n.$$

How must m, n, and p, be taken, so that c being a fraction, $x^2 - c\,y^2$ may be a square whole number?

Assuming the following notation

$$V_0 = 1,\quad 2V_1 = x + \frac{1}{x},\quad 2V_2 = x^2 + \frac{1}{x^2},\quad 2V_3 = x^3 + \frac{1}{x^3},\ \text{\&c.}$$

show that $V_{n+1} + V_{n-1} = 2\,V_n\,V_1$.

Let p be a given whole number, and show that the following equation is satisfied by one value of n and m, and by one only;

$$m\,2^{n+1} + 2^n - 1 = 2p + 1.$$

Form a series of terms beginning with 1, and such that each term exceeds the preceding by the cube of the units figure in the sum of all the preceding.

Ans. 1, 2, 29, 37, 766, 891, &c.

Show that the following equation, $x^3 = a\,x + b$ is verified by

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}}$$

and show that the product of the two terms in the value of x just given is $\dfrac{a}{3}$.

Find a value of P from the equations $P = Q + x^n$ $P = 1 + Q\,x$, and show how this may be applied to deduce the following equation:

$$\frac{1 - x^n}{1 - x} = 1 + x + x^2 + x^3 + \ldots\ldots + x^{n-1}$$

from which deduce the following:

$$\frac{y^n - x^n}{y - x} = y^{n-1} + y^{n-2}\,x + y^{n-3}\,x^2 + \ldots\ldots + x^{n-1}.$$

Detect the mistake in the following process:

Let $a = b$; then $a^2 = a\,b$, or $a^2 - a\,b = 0$, and $a^2 = b^2$ or $a^2 - b^2 = 0$, thence $a^2 - a\,b = a^2 - b^2$, or $a\,(a-b) = (a+b)\ (a-b)$, or $a = a + b$. But $b = a$, then $a = a + a = 2\,a$.

If a and b be very nearly equal, then

$$\frac{\sqrt{a} - \sqrt{b}}{a - b} = \frac{1}{2\sqrt{a}} \text{ very nearly.}$$

In the equations $a\,x + b\,y = a\,b$. $x^2 + y^2 = c^2$, what relation must exist between a, b, and c, in order that the resulting values of x may be equal?

Ans. $c = \dfrac{a\,b}{\sqrt{a^2 + b^2}}$.

Two circles may cut each other in two points; two straight lines in one point, and a straight line and circle in two points. How many different points

of intersection may there be where there are 12 circles and 10 straight lines?

Ans. 427.

What is the answer to the preceding, when there are m circles, and n straight lines?

Ans. $(m+n)^2 - m - n\frac{n+1}{2}$.

Prove that the preceding expression is always positive when m and n are not less than 1.

There are three species of curves, marked A, B, and C. Two of the sort A may cut each other in a points, and two of the sorts B and C in b and c points. Again, A may cut B in c' points, B may cut C in a' points, and C may cut A in b' points. There are m, n, and p curves of the three sorts: how many points of intersection may there be in the figure?

Ans.

$$\frac{1}{2}\Big(a m^2 + b n^2 + c p^2 + 2 a' n p + 2 b' p m + 2 c' m n - a m - b n - c p\Big)$$

Verify the following equations:

$$1 = (x+1) - x$$
$$1.2 = (x+2)^2 - 2(x+1)^2 + x^2$$
$$1.2.3 = (x+3)^3 - 3(x+2)^3 + 3(x+1)^3 - x^3$$
$$1.2.3.4 = (x+4)^4 - 4(x+3)^4 + 6(x+2)^4 - 4(x+1)^4 + x^4$$

And also the following,

$$0 = (x+2) - 2(x+1) + x$$
$$0 = (x+3)^2 - 3(x+2)^2 + 3(x+1)^2 - x^2$$
$$0 = (x+4)^2 - 4(x+3)^2 + 6(x+2)^2 - 4(x+1)^2 + x^2$$
$$0 = (x+4)^3 - 4(x+3)^3 + 6(x+2)^3 - 4(x+1)^3 + x^3.$$

And also the following:

$$x^2 = x + 2x\frac{x-1}{2}$$

$$x^3 = x + 6x\frac{x-1}{2} + 6x\frac{x-1}{2}\,\frac{x-2}{3}$$

$$x^4 = x + 14x\frac{x-1}{2} + 36x\frac{x-1}{2}\,\frac{x-2}{3} + 24x\frac{x-1}{2}\,\frac{x-2}{3}\,\frac{x-3}{4}.$$

If there be a series of terms $a + a'x + a''x^2 +$ &c., of which the coefficients a, a', a'' &c. follow this law, namely, that each one, after the second, is the sum of the two preceding, then if V represent the sum of the series *ad infinitum*, we must have

$$V = \frac{a + (a'-a)x}{1 - x - x^2};$$

and if V_n represent the sum of the series as far as $a^{(n)}x^n$ inclusive, we must have

$$V_n = \frac{a + (a'-a)x - a^{(n+1)}x^{n+1} - a^{(n)}x^{n+2}}{1 - x - x^2}$$

Show that $a - b + c - e + \ldots.$ must be less than a, and greater than $a - b$, if a, b, c, &c., be a series of decreasing positive quantities.

Reduce the binomial theorem

$$(1+x)^n = 1 + nx + n\frac{n-1}{2}x^2 + n\frac{n-1}{2}\,\frac{n-2}{3}x^3 + \ldots.$$

to the following form:

$$(1+ax)^{\frac{1}{a}} = 1 + x + \frac{x^2}{2}(1-a) + \frac{x^3}{2.3}(1-a)(1-2a) + \ldots.$$

What expressions are those, which, substituted instead of x in the following,

1. $\frac{1-x}{1+x}$ 2. $a x + x^2$ 3. $\frac{x - x^2}{1+x}$ 4. $a(x+m) - m$

will reduce them severally to x.

Answers.

1. $\frac{1-x}{1+x}$ 2. $-\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + x}$ 3. $\frac{1 - x \pm \sqrt{1 - 6x + x^2}}{2}$

4. $\frac{1}{a}(x+m) - m$.

What expression must be substituted for x in $\frac{a + b x + c x^2}{b + x}$ in order that it may become $b + x$?

Ans. $\frac{x \pm \sqrt{x^2 + 4bcx - 4c(a - b^2)}}{2}$

If the series $a + a' x + a'' x^2 + \ldots$, be reduced to the form $a(1 + px + pqx^2 + pqrx^3 + \ldots)$, what are p, q, r, &c.?

If the expression $x^2 + xy + y^2$ change from 2 to 10 when x changes from 1 to 2, what are the changes of y?

What is the least number or fraction by which 7 more than a square number or fraction can exceed 5 times the number or fraction itself?

Ans. $\frac{3}{4}$.

Find the sum of the squares of the roots of $x^2 - (1 + a)x + \frac{1 + a + a^2}{2}$, without finding the roots. *Ans.* a.

Verify the last by finding and squaring the roots. What is the expression which has for its roots the squares of the roots of $a x^2 + b x + c$?

Ans. $a^2 x + (2ac - b^2) x + c^2$.

Divide the number a into two such parts, that the first shall be the square of the second.

Ans. $\frac{2a + 1 - \sqrt{1 + 4a}}{2}$ and $\frac{\sqrt{1 + 4a} - 1}{2}$.

Show that if a be a whole square number, the answer of the last must be irrational, and also that the answer cannot be rational unless a be of the form $b(b+1)$ where b is rational.

Divide the number a into two parts, the product of which shall be a square.

Ans. $\frac{m^2 a}{m^2 + n^2}$ and $\frac{n^2 a}{m^2 + n^2}$

where m and n are any whole numbers. Show that these parts cannot be whole numbers unless a or one of its factors be the sum of two square numbers.

Divide the number a into two such parts that the sum of their squares shall be a square.

Ans. $\frac{(n^2 - m^2) a}{n^2 + 2mn - m^2}$ and $\frac{2mna}{n^2 + 2mn - m^2}$

where m and n are any numbers, n being the greater. Show that these parts may be made whole numbers when a is a whole number, itself or one of its factors being the difference between one square number and the double of another.

Solve the equations

$$\frac{1}{x} + \frac{1}{y} = a \qquad \frac{1}{y} + \frac{1}{z} = b \qquad \frac{1}{z} + \frac{1}{x} = c;$$

and explain the solution when $a + b = c$.

There are n numbers, the sum of all but the first is a_1, of all but the second, a_2, &c. &c., and of all but the last, a_n. What are the numbers?

Ans. The first is

$$\frac{1}{n-1}(a_1 + a_2 + a_3 + \ldots + a_n) - a_1 \qquad \text{the second is}$$

$$\frac{1}{n-1}(a_1 + a_2 + a_3 + \ldots + a_n) - a_2 \qquad \text{and so on.}$$

If there be three whole numbers, the product of every two of which is a square, then the numbers themselves must be squares.

Required the least number, which, divided by 4, 6, or 9, leaves a remainder 3, and by 15, a remainder 12. *Ans.* 147.

Of the four numbers, $x \quad x y \quad x y^2 \quad x y^3$, in continued proportion, the sum of the first and last is b, of the second and third a. Required x and y.

$$Ans.\ y = \frac{a + b \pm \sqrt{b^2 + 2ab - 3a^2}}{2a}$$

$$x = \frac{4a^3}{(a + b \pm \sqrt{b^2 + 2ab - 3a^2})\ (3a + b \pm \sqrt{b^2 + 2ab - 3a^2}}$$

The upper or lower sign being used throughout. Explain the two solutions, show that of the two values of y, each is unity divided by the other. Show that the preceding results are rational when, m and n being any whole numbers,

$$b = \frac{3n^2 + m^2}{n(n-m)}\ \frac{a}{2}.$$

In x, xy, xy^2, xy^3, let the sum of the first and third be p, and that of the second and fourth q. Required x and y.

$$Ans. \quad y = \frac{q}{p} \qquad x = \frac{p^3}{p^2 + q^2}.$$

$$\text{Let } x = \frac{a}{b+p} \qquad p = \frac{c}{d+q} \qquad q = \frac{e}{f+r} \qquad r = \frac{g}{h}.$$

Required the value of x, so that p, q, and r shall not appear in it.

$$Ans. \quad \cfrac{a}{b + \cfrac{c}{d + \cfrac{e}{f + \cfrac{g}{h}}}} = \frac{(dfh + dg + eh)\,a}{bdfh + bdg + beh + cfh + cg}$$

Explain this in the case where $dfh + dg + eh = 0$.

Show that the following equations may all be satisfied by a value of x which is less than unity.

$$(a + h)^2 = a^2 + 2(a + x).h$$
$$(a + h)^3 = a^3 + 3(a + x)^2.h$$
$$(a + h)^3 = a^3 + 3a^2 h + 3(a + x)h^2$$
$$(a + h)^4 = a^4 + 3(a + x)^3.h$$
$$\sqrt{a + h} = \sqrt{a} + \frac{1}{2}\frac{h}{\sqrt{a + x}}$$
$$\sqrt{(a + h)^2 + 1} = \sqrt{a^2 + 1} + \frac{a + x}{\sqrt{(a + x)^2 + 1}}\ h.$$

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